Chapter 1
Introduction

Let X be a set and $T : X \to X$ a mapping. The solutions we seek are represented by points invariant under T. These are the points satisfying

$$x = Tx. \tag{1.1}$$

Such points are said to be fixed under T or fixed points of T. The set of all solutions of (1.1) is called the fixed point set of T and denoted by Fix T. If the mapping T does not have a fixed point we often say that T is fixed point free.

Fundamental to the study of Fixed Point Theory is the attempt to identify conditions which may be imposed on the set X and/or the mapping T that will assure Fix $T \neq \varnothing$. Usually it is more efficient to study a family $\mathcal{T}$ of mapping satisfying some common conditions rather than an individual mapping. If all the mapping $T \in \mathcal{T}$ have fixed points, then we say that X has the fixed point property with respect to $\mathcal{T}$. The term " fixed point property" is often abbreviated as fpp, and if we are dealing with the fixed specific family $\mathcal{T}$ the words " with respect to $\mathcal{T}$ " are omitted.

Typically, a fixed point theorem has the following form.

Generic Theorem. *Let X be a set having structure* **A** *and let $\mathcal{T}$ be the family of mappings $T : X \to X$ satisfying condition* **B**. *Then each mapping $T \in \mathcal{T}$ has a fixed point.*

The presence or absence of a fixed point is an intrinsic property of T. One of the first and most celebrated results on this matter is the one proved by Brouwer [8] in 1912.

Theorem 1.0.1. ([8], Brouwer) *If B stands for the closed unit ball of $\mathbb{R}^n$, then each continuous mapping $T : B \to B$ has a fixed point.*

An important generalization of Brouwer's theorem was discovered in 1930 by Schauder [60].

Theorem 1.0.2. ([60], Schauder) *Let X be a Banach space. If C is a nonempty compact convex subset of X, then each continuous mapping $T : C \to C$ has a fixed point.*

The fixed point theorem, generally known as the Banach Contraction Principle, appeared in explicit form in Banach's thesis in 1922 where it was used to establish the existence of a solution for an integral equation. Since then, because of its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis.

Theorem 1.0.3. ([5], Banach) *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction, that is, for some $k \in [0, 1)$,*

$$d(Tx, Ty) \leq kd(x, y) \qquad \text{for all } x, y \in X.$$

Then T has a unique fixed point x_0. Moreover, for each $x \in X$, we have that

$$\lim_{n\to\infty} T^n(x) = x_0.$$

In Theorem 1.0.2 a minimal condition is imposed on the mapping while the nature of the domain C is heavily constrained. On the contrary, in Theorem 1.0.3 a stringent form of the continuity is imposed on the mapping T, while the assumption on the domain X is minimal for the existence of a fixed point. The questions which we will be concerned are in a intermediate sense to these two results. More specifically, we will be interested in identifying Banach space X with one or other of the properties listed below.

A mapping $T : C \subset X \to C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.
The fixed point property (fpp): X has the fpp if every nonexpansive self-mapping of a nonempty bounded closed convex subset of X has a fixed point.
The weak fixed point property (wfpp): X has the wfpp if every nonexpansive self-mapping of a nonempty weakly compact convex subset of X has a fixed point.

For a reflexive space both of these properties coincide. In general fpp $\Rightarrow$ wfpp.

Metric fixed point theory has its origin in four papers which appeared in 1965. In the first of these, Browder [9] proved that Hilbert spaces have the fpp. Later, the same author and, independently, Göhde [33], extended this result to uniformly convex Banach spaces.

Recall that, for a given Banach space X, the modulus of convexity(of Clarkson) of X is the function $\delta : [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\right\}.$$

The characteristic (or coefficient) of convexity of a Banach space X is the number

$$\varepsilon_0 = \varepsilon_0(X) = \sup\{\varepsilon \ge 0 : \delta(\varepsilon) = 0\}.$$

A Banach space X is said to be uniformly convex if $\delta(\varepsilon) > 0$ for each $\varepsilon \in (0,2]$, or equivalently, if $\varepsilon_0(X) = 0$. If $\varepsilon_0(X) < 2$, then X is said to be uniformly nonsquare.

At the same time Kirk proves a more general result. Next we describe the concepts we need in order to state Kirk's theorem. A convex subset K of a Banach space X is said to have normal structure if any bounded closed convex subset H of K for which $\operatorname{diam}(H) := \sup\{\|x-y\| : x, y \in H\} > 0$ contains a point x_0 for which

$$\sup\{\|x_0 - x\| : x \in H\} < \operatorname{diam}(H).$$

Such a point x_0 is called a nondiametral point of H.

Kirk's theorem has been stated as the following:

Theorem 1.0.4. [41, Kirk] *Let K be a nonempty bounded closed convex subset of a reflexive Banach space X, and suppose K has normal structure. If $T : K \to K$ is a nonexpansive mapping, then T has a fixed point.*

Corollary 1.0.5. *If X is a reflexive Banach space with normal structure, then X has the fpp.*

If we agree to say that a Banach space has weak normal structure when all of its weakly compact convex subsets have normal structure, the original proof of Kirk can be reformulated to prove the following result.

Theorem 1.0.6. *Any Banach space with weak normal structure has the wfpp.*

Many spaces which have normal structure satisfy an even stronger condition.

Definition 1.0.7. A convex subset K of a Banach space is said to have uniform normal structure if there exists a constant $c < 1$ such that any bounded closed convex subset H of K for which $\operatorname{diam}(H) > 0$ contains a point x_0 for which

$$\sup\{\|x_0 - x\| : x \in H\} < c \operatorname{diam}(H).$$

Maluta [56] (see also [4]) has proved that uniform normal structure implies reflexivity. The following says that every uniformly convex Banach space has uniform normal structure.

Theorem 1.0.8. *Suppose X is a Banach space for which $\varepsilon_0(X) < 1$. Then X has uniform normal structure.*

Recall some constants concerning the existence of a fixed point for nonexpansive mappings.

The James constant, or the nonsquare constant is defined by Gao and Lau [27] as

$$J(X) = \sup\{\|x+y\| \wedge \|x-y\| : x, y \in B_X\}.$$

Another important constant is the Jordan-von Neumann constant defined by Clarkson [13] as

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2} : x, y \in X \text{ not both zero}\right\}.$$

Proposition 1.0.9. *Let X be a Banach space. The following conditions are equivalent :*

(1) *X is uniformly nonsquare.*

(2) *$J(X) < 2$.*

(3) *$C_{NJ}(X) < 2$.*

In 2003, Mazcuňán-Navarro proved that uniformly nonsquare Banach spaces have the fpp.

Theorem 1.0.10. [16] If $J(X) < \frac{1+\sqrt{5}}{2}$, then X has uniform normal structure.

Theorem 1.0.11. [17], [59] If $C_{\mathrm{NJ}}(X) < \frac{1+\sqrt{3}}{2}$, then X has uniform normal structure.

Some classical fixed point theorems for single-valued nonexpansive mappings have been extended to multivalued mappings. Many questions remain open about the existence of fixed points for multivalued nonexpansive mappings when the Banach space satisfies geometric properties which assure the existence of a fixed point in the single-valued case, for instance, if X is a uniformly nonsquare Banach space. In this thesis, we are interested in the existence of a fixed point for a multivalued nonexpansive mapping concerning the James constant $J(X)$ and the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$.