

Chapter 2

Basic Concepts

In this chapter we collect information that will be needed for an understanding of the research work. Although details are included in some cases, many of the fundamental principles of functional analysis are merely stated without proof.

2.1 Banach space theory

Let X denote any nonempty set that contains with each of its elements x and each real number α a unique element $\alpha \cdot x$, written as αx , called a scalar multiple of x . (One could also include complex numbers α as well, but we restrict ourselves here to the real case.) Also assume that for each two elements $x, y \in X$ there exists a unique element $x + y \in X$ called the sum of x and y . The system $(X, \cdot, +)$ is called a linear space (over \mathbb{R}) if the following conditions are satisfied. Here $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$.

- (1) $x + y = y + x$;
- (2) $x + (y + z) = (x + y) + z$;
- (3) $\alpha(x + y) = \alpha x + \alpha y$;
- (4) $x + y = x + z$ implies $y = z$;
- (5) $(\alpha + \beta)x = \alpha x + \beta x$;
- (6) $(\alpha\beta)x = \alpha(\beta x)$;
- (7) $1x = x$.

A finite subset $\{x_1, \dots, x_n\}$ of a linear space X is said to be linearly dependent if for any $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ with $\alpha_1 x_1 + \dots + \alpha_n x_n = 0$ implies $\alpha_1 = \dots = \alpha_n = 0$. If, in addition, every $x \in X$ is a linear combination of x_1, \dots, x_n , that is $x = \alpha_1 x_1 + \dots + \alpha_n x_n$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, then we say that X has the dimension n .

A function $\|\cdot\|$ from a (real) linear space X into \mathbb{R} is called a norm if it satisfies the following properties for all $x, y \in X$ and $\alpha \in \mathbb{R}$:

- (1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;
- (2) $\|\alpha x\| = |\alpha|\|x\|$;
- (3) $\|x + y\| \leq \|x\| + \|y\|$.

From this norm we can define a metric, induced by the norm $\|\cdot\|$, by

$$d(x, y) = \|x - y\| \quad (x, y \in X).$$

A linear space X equipped with the norm $\|\cdot\|$ is called a normed linear space. A normed linear space $(X, \|\cdot\|)$ (or simply X) which is complete, i.e. every Cauchy sequence in X is convergent, is called a Banach space.

A subset K of a Banach space is said to be convex if $\alpha x + (1 - \alpha)y \in K$ for each $x, y \in K$ and $\alpha \in [0, 1]$.

If $A \subset X$, the set

$$\overline{\text{conv}}(A) = \bigcap \{K \subset X : K \text{ is closed, convex and } K \supset A\}$$

is called the closed convex hull of A . It is not difficult to see that $\overline{\text{conv}}(A)$ is closed and convex.

The modulus of convexity of a Banach space X is the function $\delta : [0, 2] \rightarrow [0, 1]$ defined by

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}.$$

The characteristic (or coefficient) of convexity of a Banach space X is the number

$$\varepsilon_0 = \varepsilon_0(X) = \sup\{\varepsilon \geq 0 : \delta(\varepsilon) = 0\}.$$

Definition 2.1.1. A Banach space X with modulus of convexity δ is said to be uniformly convex if $\delta(\varepsilon) > 0$ for each $\varepsilon \in (0, 2]$, or equivalently, if $\varepsilon_0(X) = 0$. If $\varepsilon_0(X) < 2$, then X is said to be uniformly nonsquare.

Definition 2.1.2. A Banach space X is said to be strictly convex if

$$\left\| \frac{1}{2}(x + y) \right\| < 1,$$

whenever x and y are different points of the unit sphere of X .

Proposition 2.1.3. *If E is a nonempty closed convex subset of a strictly convex space X and if $T : E \rightarrow E$ is nonexpansive, then the set $\text{Fix}T$ is closed and convex.*

A function $f : X \rightarrow \mathbb{R}$ is said to be linear if $f(\alpha x + y) = \alpha f(x) + f(y)$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$. In addition, if $|f(x)| \leq M\|x\|$ for all $x \in X$, we say that f is a bounded linear functional. It is not difficult to see that the class of all bounded linear functionals of X , denoted by X^* , is a Banach space equipped with the norm defined by

$$\|f\| = \sup\{|f(x)| : x \in B_X\} = \sup\{|f(x)| : x \in S_X\}$$

where $B_X = \{x \in X : \|x\| \leq 1\}$ is the unit ball of X and $S_X = \{x \in X : \|x\| = 1\}$ is the unit sphere of X .

The most well-known theorem in Banach space theory is the Hahn-Banach theorem: for each $x \in X$ there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

The topology induced by a norm is too strong in the sense that it has many open sets. Indeed, in order that each bounded sequence in X has a norm convergent subsequence, it is necessary and sufficient that X be finite dimensional. This fact leads us to consider other weaker topologies on normed spaces which are related to the linear structure of the spaces to search for subsequential extraction principles. So it is worthwhile to define the weaker topology for a Banach space X . We say that a net (x_α) in X converges weakly to x , denoted by

$$w - \lim_{\alpha} x_{\alpha} = x,$$

if $\lim_{\alpha} f(x_{\alpha}) = f(x)$ for all $f \in X^*$. A subset K of X is weakly closed if it is closed in the weak topology, that is, if it contains the weak limit of each of its weakly convergent nets. The weakly open sets are now taken as those sets whose complements are weakly closed. The resulting topology on X is called the weak topology on X . Sets which are compact in this topology are said to be weakly compact.

It is important to know that the weak topology on a Banach space is a Hausdorff topology, and that weak limits are unique. This is because the functionals in X^* separate points in X , that is, given any two points $x \neq y \in X$ there exists an $f \in X^*$ such that $f(x) \neq f(y)$. This is another consequence of the Hahn-Banach Theorem.

For $x \in X$ and $f \in X^*$ define $i(x)(f) = f(x)$. It is easily seen that $i(x) \in X^{**}$ and that, in fact, the mapping $i : X \rightarrow X^{**}$ is an isometric isomorphism, called the canonical embedding of X into X^{**} . If $i(X) = X^{**}$, then X is said to be reflexive.

Analogously we can also consider the weak convergence in X^* . Moreover, there is an other important mode of convergence: if (x_n^*) is a sequence in the dual space X^* and $x^* \in X^*$, then we say that x_n^* converges $*$ -weakly to x^* , denoted by $x_n^* \xrightarrow{w^*} x^*$, if $x_n^*(x) \rightarrow x^*(x)$ for all $x \in X$. By the definition, we see that norm convergence implies weak convergence and, in turn, implies $*$ -weak convergence in the dual space.

We now collect for later using some well-known properties of the weak and weak $*$ topology. Despite the fact that proofs of these results can be found in any standard functional analysis text we included selected details.

Proposition 2.1.4. *A convex subset K of a Banach space is weakly closed if and only if it is closed.*

The above leads to the following.

Proposition 2.1.5. *If a subset K of a Banach space is weakly compact, then $\overline{\text{conv}}(K)$ is also weakly compact.*

Neither of the above facts holds for the weak $*$ topology. However the following holds for both topologies.

Proposition 2.1.6. *If $K \subset X$ is weakly compact (or weak $*$ compact if X is a dual space), then K is bounded.*

The following fact holds only for the weak $*$ topology (except, of course, in reflexive spaces).

Proposition 2.1.7. (Alaoglu's Theorem) *The unit ball B_{X^*} (hence any ball) in a dual space X^* is always compact in the weak $*$ topology.*

The following says that in the weak topology compactness is equivalent to sequential compactness. This fact holds also for the weak $*$ topology on X^* if X is separable, because in this case the weak $*$ topology is metrizable, but it does not hold in general for the weak $*$ topology.

Proposition 2.1.8. (Eberlein-Smulian Theorem) *For any weakly closed subset A of a Banach space the following are equivalent.*

- (1) *Each sequence (x_n) in A has a subsequence which converges weakly to a point of A .*
- (2) *Each net (x_α) in A has a subnet which converges weakly to a point of A .*

(3) A is weakly compact.

The following lists are the several properties which characterize reflexivity.

Proposition 2.1.9. *For a Banach space X the following are equivalent.*

- (1) X is reflexive.
- (2) X^* is reflexive.
- (3) B_X is weakly compact in X .
- (4) Any bounded sequence in X has a weakly convergent subsequence.
- (5) For any $f \in X^*$ there exists $x \in B_X$ such that $f(x) = \|f\|$.
- (6) For any bounded closed convex subset K of X and any $f \in X^*$ there exists $x \in K$ such that $f(x) = \sup\{f(y) : y \in K\}$.
- (7) If (K_n) is any descending sequence of a nonempty bounded closed convex subsets of X , then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

We conclude this section by noting that Property (7) above offers a quick way, which we will not prove here, to confirm the following fact.

Theorem 2.1.10. *If X is a uniformly convex Banach space, then X is reflexive.*

A more general result is the following.

Theorem 2.1.11. *If X is a uniformly nonsquare Banach space, then X is reflexive.*

2.2 Multivalued mappings

Let E be a bounded closed convex subset of a Banach space X and $T : E \rightarrow E$ a nonexpansive mapping. The problem of finding suitable geometrical condition on X which assures the existence of a fixed point for T has been widely studied in the last 40 years. In the case of multivalued nonexpansive mappings $T : E \rightarrow K(E)$, a very general problem is the following : Does T have a fixed point under the suitable condition on X which assure the existence of fixed point for single-valued mappings? The answer to this question is unknown, but some papers have appeared showing geometrical properties on X which state fixed point results for multivalued mappings.

Let X be a Banach space and E a nonempty subset of X . We shall denote by $FB(E)$ the family of nonempty bounded closed subsets of E , by $K(E)$ the family of nonempty compact subsets of E , by $FC(E)$ the family of nonempty closed convex subsets of E , and by $KC(E)$ the family of nonempty compact convex subsets of E . Let $H(\cdot, \cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A, B) = \max\left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\}, \quad A, B \in FB(X),$$

where $\text{dist}(x, C) = \inf\{\|x - c\| : c \in C\}$ is the distance from the point x to the subset C . A multivalued mapping $T : E \rightarrow FB(X)$ is said to be a contraction if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq k\|x - y\|, \quad x, y \in E. \quad (2.2)$$

In this case, we also say that T is k -contractive.

If (2.2) is valid when $k = 1$, then T is called nonexpansive. A point x is a fixed point for a multivalued mapping T if $x \in Tx$.

In 1969, Nadler [54] established the multivalued version of Banach's contraction principle.

Theorem 2.2.1. [54] *Let X be a complete metric space and $T : X \rightarrow FB(X)$ a contraction. Then T has a fixed point.*

In contrast to Banach's Contraction Principle for single-valued mapping, the preceding theorem does not assert that the fixed point is unique.

The following method and results deal with the concept of asymptotic centers. For a bounded sequence $\{x_n\}$ in a Banach space X and E a bounded subset of X we associate the number

$$r(E, \{x_n\}) = \inf\{\limsup_{n \rightarrow \infty} \|x_n - x\| : x \in E\}$$

and the set

$$A(E, \{x_n\}) = \{x \in E : \limsup_{n \rightarrow \infty} \|x_n - x\| = r(E, \{x_n\})\}.$$

$r(E, \{x_n\})$ and $A(E, \{x_n\})$ are called, respectively, the asymptotic radius and the asymptotic center of $\{x_n\}$ relative to E .

If X is reflexive and E is closed and convex, then $A(E, \{x_n\})$ is always a nonempty closed convex subset of E for any bounded sequence $\{x_n\}$ in X . To see this observe that for each $\varepsilon > 0$ the set

$$C_\varepsilon = \{x \in E : \limsup_{n \rightarrow \infty} \|x_n - x\| \leq r(E, \{x_n\}) + \varepsilon\}$$

is nonempty by definition of $r(E, \{x_n\})$ and straightforward argument shows that each of the set C_ε is closed and convex. Hence

$$A(E, \{x_n\}) = \bigcap_{\varepsilon > 0} C_\varepsilon,$$

and the latter set is nonempty by weak compactness of E .

Clearly, $A(E, \{x_n\})$ is a nonempty weakly compact convex set as E is [32, Goebel and Kirk].

Definition 2.2.2. A bounded sequence $\{x_n\}$ is said to be regular relative to a bounded subset E of a Banach space X if $r(E, \{x_n\}) = r(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$; further, $\{x_n\}$ is said to be asymptotically uniform relative to E if $A(E, \{x_n\}) = A(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Lemma 2.2.3. [31, Goebel], [47, Lim] *Let $\{x_n\}$ be a bounded sequence in a Banach space X and E a bounded subset of X . Then*

- (i) *there always exists a subsequence of $\{x_n\}$ which is regular relative to E ;*
- (ii) *if E is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform relative to E .*

Lemma 2.2.4. *If E is a nonempty bounded closed convex subset of a uniformly convex Banach space X , then for any bounded sequence $\{x_n\}$ in X , the asymptotic center $A(E, \{x_n\})$ consists of exactly one point.*

By using Edelstein's method of asymptotic centers, T.C. Lim [47] obtained a fixed point theorem for multivalued nonexpansive mappings.

Theorem 2.2.5. *Let E be a nonempty bounded closed convex subset of a uniformly convex Banach space X and $T : E \rightarrow K(E)$ a nonexpansive mapping. Then T has a fixed point.*

In 1990, W.A. Kirk and S. Massa [43] proved the following theorem.

Theorem 2.2.6. [43] *Let E be a nonempty bounded closed convex subset of a Banach space X and $T : E \rightarrow KC(E)$ a nonexpansive mapping. Suppose that the asymptotic center in E of each bounded sequence of X is nonempty and compact. Then T has a fixed point.*

Definition 2.2.7. Let E be a nonempty closed subset of a Banach space X . The inward set of E at $x \in E$ is given by

$$I_E(x) = \{x + \lambda(y - x) : \lambda \geq 1, y \in E\}.$$

In case E is a nonempty closed convex subset of a Banach space X , we have

$$I_E(x) = \{x + \lambda(y - x) : \lambda \geq 0, y \in E\}.$$

A multivalued mapping $T : E \rightarrow 2^X$ is said to be inward (resp. weakly inward) on E if

$$Tx \subset I_E(x) \text{ (resp. } Tx \subset \overline{I_E(x)}) \text{ for all } x \in E.$$

Next theorem is very useful in order to prove the results on fixed points for multivalued mappings

Theorem 2.2.8. [47] *Let E be a nonempty closed subset of a Banach space X and $T : E \rightarrow F(X)$ a contraction which satisfies the weak inwardness condition : $Tx \subset \overline{I_E(x)}$ for all $x \in E$. Then T has a fixed point.*

In 2001, H.K. Xu [66] extended Kirk and Massa's theorem to a nonself mapping $T : E \rightarrow KC(X)$ which satisfies the inwardness condition.

Theorem 2.2.9. [66] *Let E be a nonempty bounded closed convex subset of a Banach space X and $T : E \rightarrow KC(X)$ a nonexpansive mapping with satisfies the inwardness condition : $Tx \subset I_E(x)$ for all $x \in E$. Suppose that the asymptotic center in E of each bounded sequence of X is nonempty and compact. Then T has a fixed point.*

2.3 Measures of noncompactness in metric fixed point theory

Recall that the Kuratowski, separation, and Hausdorff measures of noncompactness of a nonempty bounded subset B of a complete metric space X are respectively defined as the numbers:

$$\alpha(B) = \inf\{d > 0 : B \text{ can be covered by finitely many sets of diameters } \leq d\},$$

$\beta(B) = \sup\{\varepsilon > 0 : \text{there exists a sequence } \{x_n\} \text{ in } B \text{ such that } \text{sep}(\{x_n\}) \geq \varepsilon\}$,

where $\text{sep}(\{x_n\}) = \inf\{d(x_n, x_m) : n \neq m\}$,

and

$$\chi(B) = \inf\{d > 0 : B \text{ can be covered by finitely many balls of radii } \leq d\}.$$

Then we have

$$\chi(B) \leq \beta(B) \leq \alpha(B) \leq 2\chi(B)$$

for all bounded subsets B of X .

Proposition 2.3.1. *Let ϕ denote α , β or χ and \mathcal{B} be a family of bounded subsets of X . Then the following properties are satisfied in any complete metric space X :*

- (1) *Regularity: $\phi(B) = 0$ if and only if B is precompact.*
- (2) *Invariant under passage to the closure : $\phi(\overline{B}) = \phi(B)$ for all $B \in \mathcal{B}$.*
- (3) *Semi-additivity : $\phi(B_1 \cup B_2) = \max\{\phi(B_1), \phi(B_2)\}$, $\forall B_1 \in \mathcal{B}, \forall B_2 \in \mathcal{B}$.*
- (4) *Monotonicity : $B_1 \subset B_2$ implies $\phi(B_1) \leq \phi(B_2)$, $\forall B_1 \in \mathcal{B}, \forall B_2 \in \mathcal{B}$.*
- (5) *$\phi(B_1 \cap B_2) \leq \min\{\phi(B_1), \phi(B_2)\}$, $\forall B_1 \in \mathcal{B}, \forall B_2 \in \mathcal{B}$.*
- (6) *Generalized Cantor's intersection theorem : If $\{B_n\}$ is a decreasing sequence of nonempty closed convex subsets of X and $\lim_{n \rightarrow \infty} \phi(B_n) = 0$, then the intersection B_∞ of all B_n is nonempty and compact.*

If X is a Banach space, then we also have :

- (7) *Semi-homogeneity : $\phi(tB) = |t|\phi(B)$ for any real number t and $B \in \mathcal{B}$.*
- (8) *Algebraic semi-additivity : $\phi(B_1 + B_2) \leq \phi(B_1) + \phi(B_2)$, $\forall B_1 \in \mathcal{B}, \forall B_2 \in \mathcal{B}$.*
- (9) *Permanence under translations : $\phi(x_0 + B) = \phi(B)$ for any $x_0 \in X$ and $B \in \mathcal{B}$.*

Definition 2.3.2. Let X be a Banach space and $\phi = \alpha, \beta$ or χ . The modulus of noncompact convexity associated to ϕ is defined in the following way:

$$\Delta_{X,\phi}(\varepsilon) = \inf\{1 - \text{dist}(0, A) : A \subset B_X \text{ is convex, } \phi(A) \geq \varepsilon\},$$

where B_X is the unit ball of X .

The characteristic of noncompact convexity of X associated with the measure of noncompactness ϕ is defined by

$$\varepsilon_\phi(X) = \sup\{\varepsilon \geq 0 : \Delta_{X,\phi}(\varepsilon) = 0\}.$$

The relationships among the different moduli are

$$\Delta_{X,\alpha}(\varepsilon) \leq \Delta_{X,\beta}(\varepsilon) \leq \Delta_{X,\chi}(\varepsilon),$$

and consequently,

$$\varepsilon_\alpha(X) \geq \varepsilon_\beta(X) \geq \varepsilon_\chi(X).$$

See [3, Ayerbe et al.] for all these and more details.

A multivalued mapping $T : E \rightarrow 2^X$ is called ϕ -condensing (resp. $1 - \phi$ -contractive) where ϕ is a measure of noncompactness if, for each bounded subset B of E with $\phi(B) > 0$, there holds the inequality

$$\phi(T(B)) < \phi(B) \quad (\text{resp. } \phi(T(B)) \leq \phi(B)).$$

Here $T(B) = \cup_{x \in B} Tx$.

2.4 Ultrapower techniques

Ultrapowers of a Banach space are proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. We recall some basic facts about the ultrapowers. Let \mathcal{F} be a filter on \mathbb{N} , that is $\mathcal{F} \subset 2^{\mathbb{N}}$, satisfying :

- (1) If $A \in \mathcal{F}$ and $A \subset B \subset \mathbb{N}$, then $B \in \mathcal{F}$.
- (2) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

A filter \mathcal{U} on \mathbb{N} is called an ultrafilter if it is maximal with respect to the the ordering of filters on \mathbb{N} given by set-inclusion. That is, if $\mathcal{U} \subset \mathcal{F}$ and \mathcal{F} is a filter on \mathbb{N} , then $\mathcal{F} = \mathcal{U}$. An ultrafilter is called trivial if it is of the form $\{A : A \subset \mathbb{N}, n_0 \in A\}$ for some fixed $n_0 \in \mathbb{N}$, otherwise, it is called nontrivial.

Let $\{x_n\}$ be a sequence in a Hausdorff topological space X and \mathcal{U} a ultrafilter on \mathbb{N} . The sequence $\{x_n\}$ is said to converge to x with respect to \mathcal{U} , denoted by

$$\lim_{\mathcal{U}} x_n = x,$$

if for each neighborhood U of x , $\{n \in \mathbb{N} : x_n \in U\} \in \mathcal{U}$.

Limits along \mathcal{U} are unique and if $\{x_n\}$ is a bounded sequence in \mathbb{R} , then

$$\liminf_{n \rightarrow \infty} x_n \leq \lim_{\mathcal{U}} x_n \leq \limsup_{n \rightarrow \infty} x_n.$$

Moreover, if E is a closed subset of X and $\{x_n\} \subset E$, then $\lim_{\mathcal{U}} x_n$ belongs to E whenever it exists.

We will use the fact that

- (1) \mathcal{U} is an ultrafilter if and only if for any subset $A \subset \mathbb{N}$, either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$.
- (2) If X is compact, then the $\lim_{\mathcal{U}} x_n$ of a sequence $\{x_n\}$ in X always exists and is unique.
- (3) Suppose $\{x_n\}$ converges to x in the topology of the space X . Then $\{x_n\}$ converges to x with respect to any ultrafilter \mathcal{U} .
- (4) Let X be a linear topological vector space. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{\mathcal{U}} x_n$ and $\lim_{\mathcal{U}} y_n$ exist. Then

$$\lim_{\mathcal{U}} (x_n + y_n) = \lim_{\mathcal{U}} x_n + \lim_{\mathcal{U}} y_n$$

and

$$\lim_{\mathcal{U}} \alpha x_n = \alpha \lim_{\mathcal{U}} x_n,$$

for any scalar $\alpha \in \mathbb{R}$.

Let $(X, \|\cdot\|)$ be a Banach space and \mathcal{U} an ultrafilter on \mathbb{N} .

Consider the Banach space

$$l_{\infty}(X) = \{x = \{x_n\} \subset X : \sup_n \|x_n\| < \infty\}.$$

The subset defined by

$$N_{\mathcal{U}} = \{x = \{x_n\} \in l_{\infty}(X) : \lim_{\mathcal{U}} \|x_n\| = 0\}$$

is a closed linear subspace of $l_{\infty}(X)$.

The Banach space ultrapower of X (over \mathcal{U}) is defined to be the quotient space

$$X_{\mathcal{U}} = l_{\infty}(X)/N_{\mathcal{U}}$$

equipped with the quotient norm. When it is not necessary to mention the ultrafilter, we write \tilde{X} instead of $X_{\mathcal{U}}$. The equivalence class of an element $x = \{x_n\} \in l_{\infty}(X)$ is denoted by $\tilde{x} = \{x_n\}_{\mathcal{U}}$. It follows from (2) above and the definition of the quotient norm that

$$\|\tilde{x}\| = \|\{x_n\}_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_n\|.$$

The mapping $\mathcal{J}: X \rightarrow \tilde{X}$ defined by

$$\mathcal{J}(x) = \{x\}_{\mathcal{U}} = \{x_n\}_{\mathcal{U}}, \quad \text{where } x_n = x \quad \text{for all } n \in \mathbb{N}$$

is an isometric embedding of X into \tilde{X} . Using the map \mathcal{J} , one may identify X with $\mathcal{J}(X)$ seen as a subspace of \tilde{X} . When it is clear we will omit mention of the map \mathcal{J} and simply regard X as a subspace of \tilde{X} .

If E is a subset of a Banach space X , we associate to it subset \tilde{E} of \tilde{X} defined by

$$\tilde{E} = \{\{x_n\}_{\mathcal{U}} : x_n \in E \text{ for each } n \in \mathbb{N}\}.$$

The following properties hold :

- (1) If E is convex, then \tilde{E} is convex.
- (2) If E is closed, then \tilde{E} is closed.
- (3) If E is bounded, then \tilde{E} is bounded with $\text{diam}(\tilde{E}) = \text{diam}(E)$.

If X is a Banach space and $\{f_n\}$ is a sequence in X^* with $\sup_{n \geq 1} \|f_n\| < \infty$, then we can define the functional $\tilde{f} = \{f_n\}_{\mathcal{U}} : X_{\mathcal{U}} \rightarrow \mathbb{R}$ by

$$\tilde{f}(\{x_n\}_{\mathcal{U}}) = \lim_{\mathcal{U}} f_n(x_n).$$

We have $\tilde{f} \in (X_{\mathcal{U}})^*$ and $\|\tilde{f}\| = \lim_{\mathcal{U}} \|f_n\|$.

Theorem 2.4.1. *Let X be a Banach space and \mathcal{U} a nontrivial ultrafilter on \mathbb{N} . Then the following statements are equivalent :*

- (1) \tilde{X} is strictly convex.
- (2) \tilde{X} is uniformly convex.
- (3) X is uniformly convex.

References for more detailed treatment and proofs of all the result here stated are [2, Aksoy and Khamsi] and [61, Sims].

2.5 Nonexpansive retracts

A nonexpansive retraction of a metric space (X, d) onto one of its subspace E is a nonexpansive mapping R of X onto E (i.e., $d(R(x), R(y)) \leq d(x, y)$ for each $x, y \in X$) which leaves each point of E fixed. In this case E is said to be a nonexpansive retract of X .

Recall that $T : E \rightarrow X$ is said to satisfy the conditional fixed point property (CFP) if either T has no fixed points, or T has a fixed point in each nonempty bounded closed convex set it leaves invariant.

The following theorem has been proved by Bruck [11].

Theorem 2.5.1. *If E is a nonempty closed convex locally weakly compact subset of a Banach space X , $T : E \rightarrow E$ is nonexpansive and T satisfies (CFP), then $\text{Fix}T$ is a nonexpansive retract of E .*

Consequently, we have

Corollary 2.5.2. *If E is a nonempty bounded closed convex subset of a uniformly convex Banach space X and $T : E \rightarrow E$ is nonexpansive, then $\text{Fix}T$ is a nonexpansive retract of E .*