Chapter 3

The Dominguez-Lorenzo Condition and Multivalued Nonexpansive Mappings

Let E be a nonempty bounded closed convex separable subset of a reflexive Banach space X which satisfies the Dominguez-Lorenzo condition. The objective of this study is to show that every multivalued nonexpansive and $1-\chi-$contractive nonself mapping $T : E \to KC(X)$ which satisfies the inwardness condition ($Tx \subset I_E(x)$ for all $x \in E$) has a fixed point. All uniformly nonsquare Banach spaces with property WORTH as well as all spaces X with $\varepsilon_\beta(X) < 1$ satisfy the Dominguez-Lorenzo condition, and each Banach space that satisfies the Dominguez-Lorenzo condition always has weak normal structure. Thus the main result extends a corresponding result obtained recently by Dominguez and Lorenzo. Following the proof of the above result when a domain of mappings under consideration is a nonexpansive retract, we obtain a common fixed point of nonexpansive commuting mappings $t : E \to E$ and $T : E \to KC(E)$ where E is a nonempty bounded closed convex subset of a uniformly convex Banach space X.

3.1 Introduction

One of the most celebrated results about multivalued mappings was given by T.C. Lim [47] in 1974. By using Edelstein's method of asymptotic centers, he proved that every multivalued nonexpansive self mapping $T : E \to K(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a uniformly convex Banach space X. In 1990, W.A. Kirk and S. Massa [43] proved that if a nonempty bounded closed convex subset E of a Banach space X has the property that the asymptotic center relative to E of each bounded sequence of X is nonempty and compact, then every multivalued nonexpansive self mapping $T : E \to KC(E)$ has a fixed point. In 2001, H.K. Xu [66] extended Kirk and Massa's theorem to a nonself mapping $T : E \to KC(X)$ which satisfies the inwardness condition.

Recently, Dominguez and Lorenzo [24] proved that every nonexpansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a Banach space X with $\varepsilon_\beta(X) < 1$. Consequently, they give an affirmative answer to the problem 6 in [65, Xu] which states that every multivalued nonexpansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a nearly uniformly convex Banach space X. Furthermore, they [23] proved that if $T : E \to KC(X)$ is a nonexpansive and $1-\chi-$contractive mapping such that $T(E)$ is a bounded set and which satisfies the inwardness condition where E is a nonempty bounded closed convex separable subset of a Banach space X with $\varepsilon_\beta(X) < 1$, then T has a fixed point.

By investigating the proofs in [23] and [24], we observe that the main tool that is used in their proofs is a relationship between the Chebyshev radius of the asymptotic center of a bounded sequence relative to E and the modulus of noncompact convexity of a Banach space associated to the measure of noncompactness. In this thesis, we define the Dominguez-Lorenzo condition and prove that every reflexive Banach space X satisfying the Dominguez-Lorenzo condition and every nonempty bounded closed convex separable subset E of X, every nonexpansive and $1 - \chi-$contractive mapping $T : E \to KC(X)$ which satisfies the inwardness condition has a fixed point. The main idea of the proof comes from the proofs of Theorem 3.4 and Theorem 3.6 in [23]. We also prove that a uniformly nonsquare Banach space X satisfying property WORTH is one of the examples of Banach spaces that satisfy the Dominguez-Lorenzo condition. Moreover, we show that every Banach space which satisfies the Dominguez-Lorenzo condition has weak normal structure.

Finally, we use the celebrated theorem of Deimling [19] to obtain a common fixed point for nonexpansive commuting mappings $t : E \to E$ and $T : E \to KC(E)$ where E is a nonempty bounded closed convex subset of a uniformly convex Banach space X.

3.2 Preliminaries

We first recall the property WORTH and the non-strict Opial condition.

Definition 3.2.1.

(a) X is said to satisfy property WORTH [62, Sims] if for any $x \in X$ and any weakly null sequence $\{x_n\}$ in X,

$$\limsup_{n\to\infty} \|x_n + x\| = \limsup_{n\to\infty} \|x_n - x\|.$$

(b) X is said to satisfy the Opial condition [55, Opial] if, whenever a sequence $\{x_n\}$ in X converges weakly to x, then for $y \neq x$,

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

If the inequality is non-strict, we say that X satisfies the non-strict Opial condition.

It is known that if X satisfies property WORTH, then X satisfies the non-strict Opial condition (Garcia-Falset and Sims [30]).

A multivalued mapping $T : E \to 2^X$ is said to be upper semicontinuous on E if $\{x \in E : Tx \subset V\}$ is open in E whenever $V \subset X$ is open; T is said to be lower semicontinuous on E if $T^{-1}(V) = \{x \in E : Tx \cap V \neq \emptyset\}$ is open in E whenever $V \subset X$ is open; and T is said to be continuous if it is both upper and lower semicontinuous.

If Tx is compact for every $x \in X$, the above definition of continuity of T is equivalent to $H(Tx_n, Tx) \to 0$ whenever $x_n \to x$ as $n \to \infty$ where H is the Hausdorff distance.

In our proofs, we rely heavily on the following result.

Theorem 3.2.2. [19, Deimling] *Let E be a nonempty bounded closed convex subset of a Banach space X and $T : E \to FC(X)$ an upper semicontinuous $\chi-$condensing mapping. Assume $Tx \cap \overline{I_E(x)} \neq \varnothing$ for all $x \in E$. Then T has a fixed point.*

If C is a bounded subset of X, the Chebyshev radius of C relative to E is defined by

$$r_E(C) = \inf\{r_x(C) : x \in E\},$$

where $r_x(C) = \sup\{\|x - y\| : y \in C\}$.

Theorem 3.2.3. [22, Dominguez] *Let E be a bounded closed convex subset of a reflexive Banach space X and let $\{x_n\}$ be a bounded sequence in E which is regular relative to E. Then*

$$r_E(A(E, \{x_n\})) \leq (1 - \triangle_{X,\beta}(1^-))r(E, \{x_n\}).$$

Moreover, if X satisfies the non-strict Opial condition, then

$$r_E(A(E, \{x_n\})) \leq (1 - \triangle_{X,\chi}(1^-))r(E, \{x_n\}).$$

Using Theorem 3.2.3 as the main tool, Dominguez and Lorenzo [23] proved the following theorem :

Theorem 3.2.4. [23, Theorem 3.6] *Let X be a Banach space with $\varepsilon_\beta(X) < 1$. Assume that $T : E \to KC(X)$ is a nonexpansive and $1 - \chi-$contractive mapping such that $T(E)$ is a bounded set, and which satisfies the inwardness condition where E is a nonempty bounded closed convex separable subset of X. Then T has a fixed point.*

Moreover, they [24] used the same tool to solve the open problem in [65] on the existence of a fixed point of a multivalued nonexpansive self mapping $T : E \to KC(E)$ where E is a nonempty bounded closed convex subset of a nearly uniformly convex Banach space X.

3.3 Fixed point theorems

Definition 3.3.1. A Banach space X is said to satisfy the Dominguez-Lorenzo condition if there exists $\lambda \in [0, 1)$ such that for every weakly compact convex subset E of X and for every bounded sequence $\{x_n\}$ in E which is regular relative to E,

$$r_E(A(E, \{x_n\})) \le \lambda r(E, \{x_n\}). \tag{3.3}$$

We are going to show that every Banach space that satisfies the Dominguez-Lorenzo condition also enjoys the weak normal structure. A Banach space X is said to have weak normal structure if any weakly compact convex subset E of X for which $\mathrm{diam}(E) > 0$ contains a point x_0 for which

$$r_{x_0}(E) < \mathrm{diam}(E).$$

Theorem 3.3.2. *Let X be a Banach space satisfying the Dominguez-Lorenzo condition. Then X has weak normal structure.*

Proof. Suppose on the contrary that X does not have weak normal structure. Thus, there exists a weakly null sequence $\{x_n\}$ in B_X such that for $C := \overline{\mathrm{conv}}(\{x_n\})$,

$$\lim_{n\to\infty} \|x_n - x\| = \mathrm{diam}(C) = 1 \text{ for all } x \in C$$

(cf. [63]). By passing through a subsequence, we may assume that $\{x_n\}$ is regular. It is easy to see that $r(C, \{x_n\}) = 1$, $A(C, \{x_n\}) = C$, and $r_C(A(C, \{x_n\})) =$

$r_C(C) = 1$. Since X satisfies the Dominguez-Lorenzo condition with a corresponding $\lambda \in [0,1)$, it must be the case that

$$1 = r_C(C) \leq \lambda r(C, \{x_n\}) < 1.$$

This leads to a contradiction. □

In view of the above theorem and the famous Kirk's fixed point theorem [41], we can conclude that every Banach space X which satisfies the Dominguez-Lorenzo condition has the weak fixed point property, i.e., for every weakly compact convex subset E of X, every nonexpansive mapping $T : E \to E$ has a fixed point. Moreover, the next theorem shows that every reflexive Banach space that satisfies the Dominguez-Lorenzo condition has the fixed point property for certain multivalued nonexpansive mappings.

Theorem 3.3.3. *Let X be a reflexive Banach space satisfying the Dominguez-Lorenzo condition and let E be a bounded closed convex separable subset of X. If $T : E \to KC(X)$ is a nonexpansive and $1-\chi-$contractive mapping which satisfies the inwardness condition:*

$$Tx \subset I_E(x) \quad \textit{for all } x \in E,$$

then T has a fixed point.

Proof. Let $x_0 \in E$ be fixed and consider for each $n \geq 1$ the contraction $T_n : E \to KC(X)$ defined by

$$T_n x = \frac{1}{n} x_0 + (1 - \frac{1}{n}) Tx, \quad x \in E.$$

Bearing in mind that for each $x \in E$ the set $I_E(x)$ is convex and contains E, it is easily seen that $T_n x \subset I_E(x)$ for all $x \in E$. We can apply Theorem 2.2.8 to obtain a fixed point $x_n \in E$ of T_n. Thus we have a sequence $\{x_n\} \subset E$ such that

$$\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0.$$

By the boundedness of $\{x_n\}$ and the separability of E, we can assume that $\{x_n\}$ is a regular asymptotically uniform sequence relative to E.
Since X satisfies the Dominguez-Lorenzo condition, we obtain

$$r_E(A) \leq \lambda r(E, \{x_n\})$$

for some $\lambda \in [0,1)$, where $A = A(E, \{x_n\})$.

We can show that the mapping $T : A \to KC(X)$ is nonexpansive, $1-\chi-$contractive, and satisfies the condition

$$Tx \cap I_A(x) \neq \emptyset \quad \text{for all } x \in A.$$

Indeed, the compactness of Tx_n implies that for each n, we can take $y_n \in Tx_n$ such that

$$\|x_n - y_n\| = \operatorname{dist}(x_n, Tx_n).$$

Since Tx is compact, for each $x \in A$, we can find $z_n \in Tx$ such that

$$\|y_n - z_n\| = \operatorname{dist}(y_n, Tx) \leq H(Tx_n, Tx) \leq \|x_n - x\|.$$

By passing through a subsequence,if necessary, we can assume that there exists $z \in Tx$ such that $\lim_{n\to\infty} z_n = z$. It should remain to prove $z \in I_A(x)$.
On the other hand, we know that

$$\limsup_{n\to\infty} \|x_n - z\| = \limsup_{n\to\infty} \|y_n - z_n\| \leq \limsup_{n\to\infty} \|x_n - x\| = r(E, \{x_n\}).$$

Since $z \in Tx \subset I_E(x)$ there exists $\lambda \geq 0$ such that $z = x + \lambda(v - x)$ for some $v \in E$. If $\lambda \leq 1$, then it is clear that $z \in E$ by the convexity of E. From the above inequality, $z \in A \subset I_A(x)$. So assume that $\lambda > 1$ and write

$$v = \mu z + (1-\mu)x, \quad \mu = \frac{1}{\lambda} \in (0,1).$$

Therefore we have

$$\limsup_{n\to\infty} \|x_n - v\| \leq \mu \limsup_{n\to\infty} \|x_n - z\| + (1-\mu) \limsup_{n\to\infty} \|x_n - x\| \leq r(E, \{x_n\})$$

This implies that $v \in A$ and thus $z \in I_A(x)$.
Fix $x_0 \in A$, define $T_n : A \to KC(X)$ by

$$T_n x = \frac{1}{n}x_0 + (1 - \frac{1}{n})Tx, \;\; x \in A.$$

It is easy to see that T_n is $\chi-$condensing (see [22]). Furthermore, since $I_A(x)$ is convex we also obtain

$$T_n x \cap I_A(x) \neq \emptyset \quad \text{for all } x \in A.$$

Hence by Theorem 3.2.2, T_n has a fixed point. Consequently, we obtain a sequence $\{x_n^1\}$ in A satisfying

$$\lim_{n\to\infty} \operatorname{dist}(x_n^1, Tx_n^1) = 0.$$

We proceed as before to obtain that

$$Tx \cap I_{A_1}(x) \neq \emptyset, \quad \forall x \in A_1 = A(E, \{x_n^1\})$$

and since $\{x_n^1\} \subset A$, we also have

$$r_E(A_1) \leq \lambda r(E, \{x_n^1\}) \leq \lambda r_E(A).$$

By induction, for each integer $m \geq 1$ we take a sequence $\{x_n^m\}_n \subset A_{m-1}$ such that $\lim_{n\to\infty} \operatorname{dist}(x_n^m, Tx_n^m) = 0$ and

$$r_E(A_m) \leq \lambda^m r_E(A),$$

where $A_m = A(E, \{x_n^m\})$.

Choose $x_m \in A_m$. We will prove that $\{x_m\}_m$ is a Cauchy sequence. For each $m \geq 1$, we have for any positive n

$$\begin{aligned} \|x_{m-1} - x_m\| &\leq \|x_{m-1} - x_n^m\| + \|x_n^m - x_m\| \\ &\leq \text{diam } A_{m-1} + \|x_n^m - x_m\|. \end{aligned}$$

Taking upper limit as $n \to \infty$,

$$\begin{aligned} \|x_{m-1} - x_m\| &\leq \text{diam } A_{m-1} + \limsup_{n\to\infty} \|x_n^m - x_m\| \\ &= \text{diam } A_{m-1} + r(E, \{x_n^m\}) \\ &\leq \text{diam } A_{m-1} + r_E(A_{m-1}) \\ &\leq 2r_E(A_{m-1}) + r_E(A_{m-1}) \\ &= 3r_E(A_{m-1}) \\ &\leq 3\lambda^{m-1} r_E(A). \end{aligned}$$

Since $\lambda < 1$, we conclude that there exists $x \in E$ such that x_m converges to x. For each $m \geq 1$,

$$\begin{aligned} \operatorname{dist}(x_m, Tx_m) &\leq \|x_m - x_n^m\| + \operatorname{dist}(x_n^m, Tx_n^m) + H(Tx_n^m, Tx_m) \\ &\leq 2\|x_m - x_n^m\| + \operatorname{dist}(x_n^m, Tx_n^m). \end{aligned}$$

Taking upper limit as $n \to \infty$,

$$\operatorname{dist}(x_m, Tx_m) \leq 2 \limsup_{n\to\infty} \|x_m - x_n^m\| \leq 2\lambda^{m-1} r_E(A).$$

Finally, taking limit m in both sides we obtain $\limsup_{n\to\infty} \operatorname{dist}(x_m, Tx_m) = 0$, and the continuity of T implies that $\operatorname{dist}(x, Tx) = 0$, that is $x \in Tx$. □

From Theorem 3.2.3 it can be seen that every Banach space X with $\varepsilon_\beta(X) < 1$ satisfies the Dominguez-Lorenzo condition. We now present other Banach spaces which satisfy the Dominguez-Lorenzo condition. We consider here the James constant or the nonsquare constant $J(X)$.

For a Banach space X, the James constant, or the nonsquare constant is defined by Gao and Lau [27] as

$$J(X) = \sup\{\|x+y\| \wedge \|x-y\| : x, y \in B_X\}.$$

Clearly, X is uniformly nonsquare if and only if $J(X) < 2$.

Theorem 3.3.4. *Let X be a Banach space satisfying property WORTH and let E be a weakly compact convex subset of X. Assume that $\{x_n\}$ is a bounded sequence in E which is regular relative to E. Then*

$$r_E(A(E,\{x_n\})) \leq \frac{J(X)}{2} r(E,\{x_n\}).$$

Proof. Denote $r = r(E,\{x_n\})$ and $A = A(E,\{x_n\})$. Since $\{x_n\} \subset E$ is bounded and E is a weakly compact set, we can assume, by passing through a subsequence if necessary, that x_n converges weakly to some element in E, say x.
It should be noted that passing through a subsequence of $\{x_n\}$ does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$ because $\{x_n\}$ is regular. Let observe here that for any subsequence $\{y_n\}$ of $\{x_n\}$, $r_E(A(E,\{x_n\})) \leq r_E(A(E,\{y_n\}))$. This observation will be needed at the end of the proof. Since X satisfies property WORTH, it satisfies the non-strict Opial condition, and thus it must be the case that $x \in A$, that is

$$\limsup_{n\to\infty} \|x_n - x\| = r. \tag{3.4}$$

Now let $z \in A$. Thus $\limsup_{n\to\infty} \|x_n - z\| = r$.
By regularity of $\{x_n\}$, we can choose a subsequence $\{x_{n'}\}$ of $\{x_n\}$ so that

$$\lim_{n'\to\infty} \|x_{n'} - x\| = r = \lim_{n'\to\infty} \|x_{n'} - z\|.$$

Property WORTH and the fact that $x_{n'} - x \xrightarrow{w} 0$ yield the following:

$$\begin{aligned} r &= \lim_{n'\to\infty} \|x_{n'} - z\| \\ &= \lim_{n'\to\infty} \|(x_{n'} - x) + (x - z)\| \\ &= \lim_{n'\to\infty} \|(x_{n'} - x) - (x - z)\| \\ &= \lim_{n'\to\infty} \|x_{n'} - 2x + z\|. \end{aligned}$$

Thus we have

$$\lim_{n'\to\infty} \|\frac{x_{n'} - z}{r}\| = 1 = \lim_{n'\to\infty} \|\frac{x_{n'} - 2x + z}{r}\|. \quad (3.5)$$

Let consider an ultrapower $\widetilde{X}$ of X. Put

$$\tilde{u} = \frac{1}{r}\{x_{n'} - z\}_{\mathcal{U}} \text{ and } \tilde{v} = \frac{1}{r}\{x_{n'} - 2x + z\}_{\mathcal{U}}.$$

(3.5) guarantees that $\tilde{u}, \tilde{v} \in S_{\widetilde{X}}$. We see that

$$\begin{aligned} \|\tilde{u} + \tilde{v}\| &= \lim_{\mathcal{U}} \|\frac{1}{r}(x_{n'} - z) + \frac{1}{r}(x_{n'} - 2x + z)\| \\ &= \lim_{\mathcal{U}} \|\frac{2}{r}(x_{n'} - x)\| \\ &= \frac{2}{r} \lim_{\mathcal{U}} \|(x_{n'} - x)\| \\ &= \frac{2}{r}(r) = 2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\tilde{u} - \tilde{v}\| &= \lim_{\mathcal{U}} \|\frac{1}{r}(x_{n'} - z) - \frac{1}{r}(x_{n'} - 2x + z)\| \\ &= \frac{2}{r}\|x - z\|. \end{aligned}$$

Thus by the definition of $J(\widetilde{X})$ and lower semi continuity of the norm, we have

$$\begin{aligned} J(\widetilde{X}) &\geq \|\tilde{u} + \tilde{v}\| \wedge \|\tilde{u} - \tilde{v}\| \\ &= 2 \wedge \frac{2}{r}\|x - z\| \\ &= \frac{2}{r}\|x - z\|. \end{aligned}$$

Since the James constants of X and of $\widetilde{X}$ are the same, we obtain

$$J(X) \geq \frac{2}{r}\|x - z\|.$$

This holds for arbitrary $z \in A$. Hence we have

$$r_x(A) \leq \frac{J(X)}{2} r,$$

and therefore, by the previous observation, $r_E(A) \leq \frac{J(X)}{2} r$. □

From the above theorem we immediately have

Corollary 3.3.5. *Let X be a uniformly nonsquare Banach space satisfying property WORTH. Then X satisfies the Dominguez-Lorenzo condition.*

Proof. Uniform nonsquareness of X is equivalent to $J(X) < 2$. Put $\lambda = \frac{J(X)}{2}$. Then $\lambda < 1$ and by Theorem 3.3.4 the result follows. □

Theorem 3.3.3 and Corollary 3.3.5 give

Corollary 3.3.6. *Let X be a uniformly nonsquare Banach space satisfying property WORTH and let E be a nonempty bounded closed convex separable subset of X. If $T : E \to KC(X)$ is a nonexpansive mapping which satisfies the inwardness condition:*

$$Tx \subset I_E(x) \quad \text{for all } x \in E,$$

then T has a fixed point.

Proof. By Corollary 3.3.5, X satisfies the Dominguez-Lorenzo condition. It is known that uniform nonsquareness implies reflexivity of X. Since X has the nonstrict opial condition, we can conclude that the nonexpansive mapping $T : E \to K(X)$ with bounded range is $1-\chi-$contractive (see [23]). Now Theorem 3.3.3 can be applied to obtain a fixed point. □

Questions.

(1) It has been shown in [16, Theorem 3.1] that a Banach space X has uniform normal structure whenever $J(X) < \frac{1+\sqrt{5}}{2}$. It is natural to ask if the condition of being uniform nonsquareness and having property WORTH can be replaced by the condition "$J(X) < \frac{1+\sqrt{5}}{2}$" or some other upper bounds.

(2) A similar question about the Jordan-von Neumann constants can be asked in the sense of (1). Here we ask if we can replace the condition of being uniform nonsquareness and having property WORTH by the condition $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$ or some other upper bounds. Note that it has shown in [17, Theorem 3.16] that a Banach space X has uniform normal structure whenever $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$.

3.4 The common fixed points in uniformly convex Banach spaces

In this section, we assure the existence of the common fixed points in uniformly convex Banach spaces by using the theorem proved by Bruck [11].

Definition 3.4.1. Let E be a nonempty bounded closed convex subset of a Banach space X, $t : E \to X$, and $T : E \to FB(X)$. Then t and T are said to be commuting if for every $x, y \in E$ such that $x \in Ty$ and $ty \in E$, there holds

$$tx \in Tty.$$

Theorem 3.4.2. *Let E be a nonempty bounded closed convex subset of a uniformly convex Banach space X, $t : E \to E$, $T : E \to KC(E)$ a single valued and a multi-valued nonexpansive mapping, respectively. Assume that t and T are commuting. Then t and T have a common fixed point, i.e., there exists a point x in E such that $x = tx \in Tx$.*

Proof. It is known that the fixed point set of t, denoted by Fix(t), is nonempty closed and convex. Let $x \in \text{Fix}(t)$.
Since t and T are commuting, we have $ty \in Tx$ for each $y \in Tx$.
We see that, for $x \in \text{Fix}(t)$, $Tx \cap \text{Fix}(t) \neq \emptyset$.
For a fixed element $x_0 \in \text{Fix}(t)$, define a contraction $T_n : \text{Fix}(t) \to \text{KC}(E)$ by

$$T_n(x) = \frac{1}{n}x_0 + (1 - \frac{1}{n})T(x), \quad x \in \text{Fix}(t).$$

It is easy to see that for each $x \in \text{Fix}(t)$, $T_n x \cap \text{Fix}(t) \neq \emptyset$ as T does.
Since Fix(t) is a nonexpansive retract of E (Bruck [11]), we can show that T_n : Fix(t) $\to$KC(E) is $\chi-$condensing. Indeed, let B be a bounded subset of Fix(t) and $\chi(B) > 0$. Given $d > 0$ be such that

$$B \subset \cup_{i=1}^{n} B(x_i, d), \; x_i \in E.$$

Let R be a nonexpansive retraction of E onto Fix(t).
For each $a \in B(x_i, d) \cap B$, we have

$$\|Rx_i - a\| = \|Rx_i - Ra\| \leq \|x_i - a\| \leq d.$$

Therefore $B(x_i, d) \cap B \subset B(Rx_i, d)$ for each $i \in \{1, ..., n\}$, and hence

$$B \subset \cup_{i=1}^{n} B(Rx_i, d).$$

Since T_n is $(1 - \frac{1}{n})-$contractive,

$$T_n(B) \subset \cup_{i=1}^{n} (T_n Rx_i + (1 - \frac{1}{n})dB(0, 1)).$$

Thus

$$\chi(T_n(B)) \leq (1-\frac{1}{n})\chi(B) < \chi(B),$$

and T_n is $\chi-$condensing.

Now we can apply Theorem 3.2.2 to conclude that T_n has a fixed point, say x_n. Moreover, we can show that

$$\text{dist}(x_n, Tx_n) \to 0.$$

Let $\widetilde{X}$ be a Banach space ultrapower of X and

$$\dot{\text{Fix}}(t) = \{\dot{x} = \{x_n\}_{\mathcal{U}} : x_n \equiv x \in \text{Fix}(t)\}.$$

Then $\dot{\text{Fix}}(t)$ is a nonempty closed convex subset of $\widetilde{X}$.

Now, for each $n \in \mathbb{N}$, let y_n be the unique nearest point of x_n in Tx_n, i.e., $\|x_n - y_n\| = \text{dist}(x_n, Tx_n)$. Consequently, $\{x_n\}_{\mathcal{U}} = \{y_n\}_{\mathcal{U}}$. Nonexpansiveness of t and being a point of Fix(t) of x_n imply

$$\|x_n - ty_n\| = \|tx_n - ty_n\| \leq \|x_n - y_n\|$$

for each $n \in \mathbb{N}$. Since $ty_n \in Tx_n$, we have $y_n = ty_n \in \text{Fix}(t)$ for each $n \in \mathbb{N}$.

Since $\dot{\text{Fix}}(t)$ is a closed convex subset of a uniformly convex Banach space $\tilde{X}$, thus we have $\{x_n\}_{\mathcal{U}}$ has a unique nearest point $\dot{v} \in \dot{\text{Fix}}(t)$, i.e., $\|\{x_n\}_{\mathcal{U}} - \dot{v}\| = \text{dist}(\{x_n\}_{\mathcal{U}}, \dot{\text{Fix}}(t))$.

As Tv is closed and convex, we can find $v_n \in Tv$ satisfying

$$\|y_n - v_n\| = \text{dist}(y_n, Tv) \leq H(Tx_n, Tv).$$

We note here that $v_n \in$Fix(t) for each n. It follows from the nonexpansiveness of T that

$$\|y_n - v_n\| \leq \|x_n - v\|.$$

This means

$$\|\{y_n\}_{\mathcal{U}} - \{v_n\}_{\mathcal{U}}\| \leq \|\{x_n\}_{\mathcal{U}} - \dot{v}\|.$$

Since $\{x_n\}_{\mathcal{U}} = \{y_n\}_{\mathcal{U}}$, we have

$$\|\{x_n\}_{\mathcal{U}} - \{v_n\}_{\mathcal{U}}\| \leq \|\{x_n\}_{\mathcal{U}} - \dot{v}\|. \tag{3.6}$$

Because of the compactness of Tv, there exists $w \in Tv$ such that $w = \lim_{\mathcal{U}} v_n$. It follows that $\{v_n\}_{\mathcal{U}} = \dot{w}$. This fact and (3.6) imply

$$\|\{x_n\}_{\mathcal{U}} - \dot{w}\| \leq \|\{x_n\}_{\mathcal{U}} - \dot{v}\|. \tag{3.7}$$

Moreover, $w \in \text{Fix}(t)$ and then $\dot{w} \in \dot{\text{Fix}}(t)$. Hence $\dot{w} = \dot{v}$ and so $v = w \in Tv$ which then completes the proof. □