Chapter 4

The Jordan-von Neumann Constant and Fixed Points for Multivalued Nonexpansive Mappings

The purpose of this chapter is to study the existence of fixed points for nonexpansive multivalued mappings in a particular class of Banach spaces. Furthermore, we demonstrate a relationship between the weakly convergent sequence coefficient $WCS(X)$ and the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ of a Banach space X. Using this fact, we prove that if $C_{\mathrm{NJ}}(X)$ is less than an appropriate positive number, then every multivalued nonexpansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a Banach space X and $KC(E)$ is the class of all nonempty compact convex subsets of E.

4.1 Introduction

In 1969, Nadler [54] established the multivalued version of Banach's contraction principle. Since then the metric fixed point theory of multivalued mappings has been rapidly developed. Some classical fixed point theorems for single-valued nonexpansive mappings have been extended to multivalued nonexpansive mappings.

In 1974, Lim [47], using Edelstein's method of asymptotic center, proved the existence of a fixed point for a multivalued nonexpansive self-mapping $T : E \to K(E)$ where E is a nonempty bounded closed convex subset of a uniformly convex Banach space X. In 1990, Kirk and Massa [43] extended Lim's theorem. They proved that every multivalued nonexpansive self-mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a Banach space X for which the asymptotic center in E of each bounded sequence of X is nonempty and compact. In 2001, Xu [66] extended Kirk-Massa's theorem to nonself-mapping $T : E \to KC(X)$ which satisfies the inwardness condition.

In 2004, Dominguez and Lorenzo [24] proved that every multivalued nonexpansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a Banach space X with $\varepsilon_\beta(X) < 1$. Consequently, they can give an affirmative answer of a problem in [65] proving that every nonexpansive self-mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a nearly uniformly convex Banach space. In chapter 3, we gave an existence of a fixed point for a multivalued nonexpansive and $1-\chi-$contractive mapping $T : E \to KC(X)$ which satisfies the inwardness condition, where E is a nonempty bounded closed convex separable subset of a reflexive Banach space which satisfies the Dominguez-Lorenzo condition, i.e., an inequality concerning the asymptotic radius and the Chebyshev radius of the asymptotic center for some types of sequences. Consequently, they could show that if X is a uniformly nonsquare Banach space satisfying property WORTH and $T : E \to KC(X)$ is a nonexpansive mapping which satisfies the inwardness condition, where E is a nonempty bounded closed convex separable subset of X, then T has a fixed point. Furthermore, we also ask : Does $C_{\mathrm{NJ}}(X) < \frac{1+\sqrt{3}}{2}$ imply the existence of a fixed point for multivalued nonexpansive mappings ?

In this study, we organize as follows. We define a property for a Banach spaces which we call property (D) (see definition in Section 4.3), which is weaker than the Dominguez-Lorenzo condition and stronger than weak normal structure and we prove that if X is a Banach space satisfying property (D) and E is a nonempty weakly compact convex subset of X, then every nonexpansive mapping $T : E \to KC(E)$ has a fixed point. Then we state a relationship between the weakly convergent sequence coefficient $WCS(X)$ and the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ of a Banach space X. Finally, using this fact, we prove that if $C_{\mathrm{NJ}}(X)$ is less than an appropriate positive number, then every multivalued nonexpansive mapping $T : E \to KC(E)$ has a fixed point. In particular, we give a partial answer to the question which has been asked in [15].

4.2 Preliminaries

Throughout this study we let X^* stand for the dual space of a Banach space X. By B_X and S_X we denote the closed unit ball and the unit sphere of X, respectively. Let A be a nonempty bounded subset of X.

The number

$$r(A) := \inf \big\{ \sup_{y \in A} \|x - y\| : x \in A \big\}$$

is called the Chebyshev radius of A and the number

$$\operatorname{diam}(A) := \sup\{\|x-y\| : x, y \in A\}$$

is called the diameter of A.

A Banach space X has normal structure (resp. weak normal structure) if

$$r(A) < \operatorname{diam}(A)$$

for every bounded closed (resp. weakly compact) convex subset A of X with $\operatorname{diam}(A) > 0$.

A Banach space X is said to have uniform normal structure (resp. weak uniform normal structure) if

$$\inf\left\{\frac{\operatorname{diam} A}{r(A)}\right\} > 1,$$

where the infimum is taken over all bounded closed (resp. weakly compact) convex subsets A of X with $\operatorname{diam} A > 0$.

Let X be a Banach space without Schur property, that is, there is weakly convergent sequence which is not norm convergent. The weakly convergent sequence coefficient $WCS(X)$ [12] of X is the number

$$WCS(X) := \inf\left\{\frac{A(\{x_n\})}{r_a(\{x_n\})}\right\},$$

where the infimum is taken over all sequences $\{x_n\}$ in X which are weakly (not strongly) convergent, $A(\{x_n\}) := \limsup_{n\to\infty}\{\|x_i - x_j\| : i, j \geq n\}$ is the asymptotic diameter of $\{x_n\}$, and $r_a(\{x_n\}) := \inf\{\limsup_{n\to\infty}\|x_n - y\| : y \in \overline{\operatorname{conv}}(\{x_n\})\}$ is the asymptotic radius of $\{x_n\}$.

Some equivalent definitions of the weakly convergent sequence coefficient can be found in [3, p. 120] as follows :

$$WCS(X) = \inf\Big\{\frac{\lim_{n,m;n\neq m}\|x_n - x_m\|}{\lim_{n\to\infty}\|x_n\|} : \{x_n\} \text{ converges weakly to zero },$$

$$\lim_{n,m;n\neq m}\|x_n - x_m\| \text{ and } \lim_{n\to\infty}\|x_n\| \text{ exist}\Big\},$$

$$WCS(X) = \inf\Big\{\lim_{n,m;n\neq m}\|x_n - x_m\| : \{x_n\} \text{ converges weakly to zero},$$

$$\|x_n\| = 1 \text{ and } \lim_{n,m;n\neq m}\|x_n - x_m\| \text{ exists}\Big\},$$

$$WCS(X) = \inf\left\{\frac{A(\{x_n\})}{\limsup_{n\to\infty}\|x_n\|} : \{x_n\} \text{ converges weakly to zero}\right\}$$

and

$$WCS(X) = \inf\left\{\frac{a}{\limsup_{n\to\infty}\|x_n\|} : \{x_n\} \text{ converges weakly to zero and } \lim_{n,m;n\neq m}\|x_n - x_m\| = a\right\}.$$

It is easy to see, from the definitions of $WCS(X)$, that $1 \leq WCS(X) \leq 2$, and it is known that $WCS(X) > 1$ implies X has weak uniform normal structure [12].

For a Banach space X, the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ of X, introduced by Clarkson [13], is defined by

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2} : x, y \in X \text{ not both zero}\right\}.$$

The constant $R(a, X)$, which is a generalized Garcia-Falset coefficient [29], is introduced by Dominguez [20] : For a given nonnegative real number a,

$$R(a, X) := \sup\{\liminf_n \|x + x_n\|\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequences $\{x_n\}$ in the unit ball of X such that $\lim_{n,m;n\neq m}\|x_n - x_m\| \leq 1$.

A relationship between the constant $R(1, X)$ and the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ can be found in [57] :

$$R(1, X) \leq \sqrt{2C_{\mathrm{NJ}}(X)}.$$

4.3 Main results

Definition 4.3.1. A Banach space X is said to satisfy property (D) if there exists $\lambda \in [0, 1)$ such that for any nonempty weakly compact convex subset E of X, any sequence $\{x_n\} \subset E$ which is regular asymptotically uniform relative to E, and any sequence $\{y_n\} \subset A(E, \{x_n\})$ which is regular asymptotically uniform relative to E we have

$$r(E, \{y_n\}) \leq \lambda r(E, \{x_n\}). \tag{4.8}$$

Recall the Dominguez-Lorenzo condition introduced in [15] as follow : A Banach space X is said to satisfy the Dominguez-Lorenzo condition if there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset E of X and for every bounded sequence $\{x_n\}$ in E which is regular relative to E,

$$r_E(A(E,\{x_n\})) \leq \lambda r(E,\{x_n\}).$$

It is clear from the definition that property (D) is weaker than the Dominguez-Lorenzo condition. In fact, property (D) is strictly weaker than the Dominguez-Lorenzo condition as shown in [21]. The next result shows that property (D) is stronger than weak normal structure.

Theorem 4.3.2. *Let X be a Banach space satisfying property (D). Then X has weak normal structure.*

Proof. Suppose on the contrary, thus there exists a weakly null sequence $\{x_n\} \subset B_X$ such that $\lim\limits_{n\to\infty} \|x_n - x\| = 1$ for all $x \in C = \overline{\text{conv}}(\{x_n\})$ (see [63]).
By passing through a subsequence, we may assume that $\{x_n\}$ is regular relative to C. We see that $r(C,\{x_n\}) = 1$ and $A(C,\{x_n\}) = C$. Moreover $\{x_n\}$ is asymptotically uniform relative to C. Indeed, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ we have

$$A(C,\{x_{n_k}\}) = \big\{x \in C : \limsup_{k\to\infty} \|x_{n_k} - x\| = r(C,\{x_{n_k}\})\big\} = C.$$

Since $\{x_n\} \subset C = A(C,\{x_n\})$ and X satisfies property (D) with a corresponding $\lambda \in [0,1)$, we have

$$r(C,\{x_n\}) \leq \lambda r(C,\{x_n\})$$

which leads to a contradiction. □

The following results will be very useful in order to prove our main theorem.

Theorem 4.3.3 (Dominguez and Lorenzo [22]). *Let E be a nonempty weakly compact separable subset of a Banach space X, $T : E \to K(E)$ a nonexpansive mapping, and $\{x_n\}$ a sequence in E such that $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$. Then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that*

$$Tx \cap A \neq \emptyset, \quad \forall x \in A := A(E,\{z_n\}).$$

Theorem 4.3.4 (Dominguez and Lorenzo [24]). *Let E be a nonempty weakly compact convex separable subset of a Banach space X. Assume that $T : E \to$*

$KC(E)$ is a contraction mapping. If A is a closed convex subset of E such that $Tx \cap A \neq \emptyset$ for all $x \in A$, then T has a fixed point in A.

We can now state our main theorem.

Theorem 4.3.5. *Let E be a nonempty weakly compact convex subset of a Banach space X which satisfies property (D). Assume that $T : E \to KC(E)$ is a nonexpansive mapping. Then T has a fixed point.*

Proof. The first part of the proof is similar to the proof of Theorem 4.2 in [22]. Therefore, we only sketch this part of the proof. From [45] we can assume that E is separable. Fix $z_0 \in E$ and define a contraction $T_n : E \to KC(E)$ by

$$T_n(x) = \frac{1}{n}z_0 + (1 - \frac{1}{n})Tx, \quad x \in E.$$

By Nadler's theorem [54], for any $n \in \mathbb{N}$, T_n has a fixed point, say x_n^1. It is easy to prove that $\lim_{n\to\infty} \text{dist}(x_n^1, Tx_n^1) = 0$. By Lemma 2.2.3, we can assume that sequence $\{x_n^1\} \subset E$ is regular asymptotically uniform relative to E. Denote $A_1 = A(E, \{x_n^1\})$. By Theorem 4.3.3 we can assume that $Tx \cap A_1 \neq \emptyset$ for all $x \in A_1$. Fix $z_1 \in A_1$ and define a contraction $T_n : E \to KC(E)$ by

$$T_n(x) = \frac{1}{n}z_1 + (1 - \frac{1}{n})Tx, \quad x \in E.$$

Convexity of A_1 implies $T_n(x) \cap A_1 \neq \emptyset$ for all $x \in A_1$. By Theorem 4.3.4, T_n has a fixed point in A_1, say x_n^2. Consequently, we can get a sequence $\{x_n^2\} \subset A_1$ which is regular asymptotically uniform relative to E and $\lim_{n\to\infty} \text{dist}(x_n^2, Tx_n^2) = 0$. Since X satisfies the property (D) with a corresponding $\lambda \in [0, 1)$, we have

$$r(E, \{x_n^2\}) \leq \lambda r(E, \{x_n^1\}).$$

By induction, we can find a sequence $\{x_n^k\} \subset A_{k-1} = A(E, \{x_n^{k-1}\})$ which is regular asymptotically uniform relative to E,

$$\lim_{n\to\infty} \text{dist}(x_n^k, Tx_n^k) = 0,$$

and

$$r(E, \{x_n^k\}) \leq \lambda r(E, \{x_n^{k-1}\}) \ \text{ for all } k \in \mathbb{N}.$$

Consequently,

$$r(E, \{x_n^k\}) \leq \lambda r(E, \{x_n^{k-1}\}) \leq ... \leq \lambda^{k-1} r(E, \{x_n^1\}).$$

In view of [3, p. 48], we may assume that for each $k \in \mathbb{N}$,

$$\lim_{n,m;n\neq m} \|x_n^k - x_m^k\| \quad \text{exists},$$

and in addition $\|x_n^k - x_m^k\| < \lim_{n,m;n\neq m} \|x_n^k - x_m^k\| + \frac{1}{2^k}$ for all $n, m \in \mathbb{N}$ and $n \neq m$. Let $\{y_n\}$ be the diagonal sequence $\{x_n^n\}$. We claim that $\{y_n\}$ is a Cauchy sequence. For each $n \geq 1$, we have for any positive number m,

$$\begin{aligned} \|y_n - y_{n-1}\| &\leq \|y_n - x_m^{n-1}\| + \|x_m^{n-1} - y_{n-1}\| \\ &= \|y_n - x_m^{n-1}\| + \|x_m^{n-1} - x_{n-1}^{n-1}\| \\ &\leq \|y_n - x_m^{n-1}\| + \lim_{i,j;i\neq j} \|x_i^{n-1} - x_j^{n-1}\| + \frac{1}{2^{n-1}} \end{aligned}$$

Taking upper limit as $m \to \infty$,

$$\begin{aligned} \|y_n - y_{n-1}\| &\leq \limsup_{m\to\infty} \|y_n - x_m^{n-1}\| + \lim_{i,j;i\neq j} \|x_i^{n-1} - x_j^{n-1}\| + \frac{1}{2^{n-1}} \\ &\leq r(E, \{x_n^{n-1}\}) + \limsup_i \|x_i^{n-1} - y_n\| + \limsup_j \|x_j^{n-1} - y_n\| + \frac{1}{2^{n-1}} \\ &\leq 3r(E, \{x_n^{n-1}\}) + \frac{1}{2^{n-1}} \\ &\leq 3\lambda^{n-2} r(E, \{x_n^1\}) + \frac{1}{2^{n-1}}. \end{aligned}$$

Since $\lambda < 1$, we conclude that there exists $y \in E$ such that y_n converges to y. Consequently,

$$\operatorname{dist}(y, Ty) \leq \|y - y_n\| + \operatorname{dist}(y_n, Ty_n) + H(Ty_n, Ty) \to 0 \text{ as } n \to \infty.$$

Hence y is a fixed point of T. □

Theorem 4.3.6. *Let E be a nonempty weakly compact convex subset of a Banach space X with*

$$C_{NJ}(X) < 1 + \frac{WCS(X)^2}{4}.$$

Assume that $T : E \to KC(E)$ is a nonexpansive mapping. Then T has a fixed point.

Proof. We will prove that X satisfies property (D). Since $C_{\mathrm{NJ}}(X) < 1 + \frac{WCS(X)^2}{4}$, we choose $\lambda = \frac{2\sqrt{C_{\mathrm{NJ}}(X)-1}}{WCS(X)} < 1$. Let D be a nonempty weakly compact convex subset of X, $\{x_n\} \subset D$ and $\{y_n\} \subset A(D, \{x_n\})$ be regular asymptotically uniform sequences relative to D. We will show that (4.8) is satisfied. By choosing a subsequence, if necessary, we can assume that $\{y_n\}$ converges weakly to $y \in D$ and

$$\lim_{k,j;k\neq j} \|y_k - y_j\| = l \quad \text{for some } l \geq 0. \tag{4.9}$$

Let $r = r(D, \{x_n\})$. The condition (4.8) easily follows when $r = 0$ or $l = 0$. We assume now that $r > 0$ and $l > 0$. Let $\varepsilon > 0$ so small that $0 < \varepsilon < l \wedge r$. From (4.9) we assume that

$$\big|\|y_k - y_j\| - l\big| < \varepsilon \quad \text{for all } k \neq j. \tag{4.10}$$

.

Fix $k \neq j$. Since $y_k, y_j \in A(D, \{x_n\})$ and using the convexity of $A(D, \{x_n\})$, we can assume, passing through a subsequence, that

$$\|x_n - y_k\| < r + \varepsilon,\ \|x_n - y_j\| < r + \varepsilon, \tag{4.11}$$

and

$$\Big\|x_n - \frac{y_k + y_j}{2}\Big\| > r - \varepsilon \quad \text{for all large } n. \tag{4.12}$$

From the definition of $C_{\mathrm{NJ}}(X)$, by (4.10), (4.11), and (4.12) we have for n large enough,

$$\begin{aligned} C_{\mathrm{NJ}}(X) &\geq \frac{\|2x_n - (y_k + y_j)\|^2 + \|y_k - y_j\|^2}{2\|x_n - y_k\|^2 + 2\|x_n - y_j\|^2} \\ &\geq \frac{4(r-\varepsilon)^2 + (l-\varepsilon)^2}{4(r+\varepsilon)^2}. \end{aligned}$$

Since ε is arbitrary small, it follows that

$$C_{\mathrm{NJ}}(X) \geq \frac{4r^2 + l^2}{4r^2}.$$

Since

$$WCS(X) = \inf\Big\{\frac{\lim_{j,k;j\neq k}\|u_j - u_k\|}{\limsup_j \|u_j\|} : u_j \xrightarrow{w} 0, \lim_{j,k;j\neq k}\|u_j - u_k\| \text{ exists}\Big\},$$

we can deduce that

$$\begin{aligned} C_{\mathrm{NJ}}(X) &\geq 1 + \frac{WCS(X)^2(\limsup_n \|y_n - y\|)^2}{4r^2} \\ &\geq 1 + \frac{WCS(X)^2 r(D, \{y_n\})^2}{4r^2}. \end{aligned}$$

Consequently,

$$r(D, \{y_n\}) \leq \frac{2\sqrt{C_{\mathrm{NJ}}(X) - 1}}{WCS(X)} r = \lambda r(D, \{x_n\})$$

as desired. $\square$

In order to prove our next result, we need the following theorem which states a relationship between the weakly convergent sequence coefficient and the Jordan-von Neumann constant of a Banach space X.

Theorem 4.3.7. *For a Banach space X,*

$$[WCS(X)]^2 \geq \frac{2C_{\mathrm{NJ}}(X)+1}{2[C_{\mathrm{NJ}}(X)]^2}.$$

Proof. Since $C_{\mathrm{NJ}}(X) \leq 2$ and the result is obvious if $C_{\mathrm{NJ}}(X) = 2$, we can assume that $C_{\mathrm{NJ}}(X) < 2$. It is known that $C_{\mathrm{NJ}}(X) < 2$ implies X and X^* are reflexive. Put $\alpha = \sqrt{2C_{\mathrm{NJ}}(X)}$. Let $\{x_n\}$ be a normalized weakly null sequence in X and $d := \lim_{n,m;n\neq m} \|x_n - x_m\|$. Consider a sequence $\{f_n\}$ of norm one functionals for which $f_n(x_n) = 1$. Since X^* is reflexive we can assume that $\{f_n\}$ converges weakly to some f in X^*. Let ε be an arbitrary positive number and choose $K \in \mathbb{N}$ large enough so that $|f(x_n)| < \varepsilon$ and $d-\varepsilon \leq \|x_n - x_m\| \leq d+\varepsilon$ for any $m \neq n;\ m, n \geq K$. Then we have

$$\lim_n (f_n - f)(x_K) = 0 \quad \text{and } \lim_n f_K(x_n) = 0.$$

Since $\lim\limits_{n,m;n\neq m} \|\frac{x_n - x_m}{d+\varepsilon}\| < 1$ and $\|\frac{x_K}{d+\varepsilon}\| \leq 1$, we have, by the definition of $R(1, X)$,

$$\limsup_n \|x_n + x_K\| \leq (d+\varepsilon)R(1,X) \leq (d+\varepsilon)\sqrt{2C_{\mathrm{NJ}}(X)} = (d+\varepsilon)\alpha.$$

We construct elements of $\widetilde{X}$ and $\widetilde{X^*}$:

$$\tilde{x} = \left\{\frac{x_n - x_K}{d+\varepsilon}\right\}_{\mathcal{U}} \quad , \quad \tilde{y} = \left\{\frac{x_n + x_K}{(d+\varepsilon)\alpha}\right\}_{\mathcal{U}} \quad ,$$

$$\tilde{f} = \{f_n\}_{\mathcal{U}} \ \text{, and} \quad \tilde{g} = \dot{f_K}.$$

Here $\dot{h}$ denotes an equivalence class of the sequence $\{h_n\}$ such that $h_n \equiv h$ for all $n \in \mathbb{N}$. Clearly $\tilde{x}, \tilde{y} \in B_{\widetilde{X}}$ and $\tilde{f}, \tilde{g} \in S_{\widetilde{X^*}}$. Moreover,

$$\tilde{f}(\{x_n\}_{\mathcal{U}}) = 1 \quad \text{and} \quad |\tilde{f}(\dot{x_K})| = |\dot{f}(\dot{x_K})| < \varepsilon.$$

On the other hand,

$$\tilde{g}(\{x_n\}_{\mathcal{U}}) = 0 \quad \text{and} \quad \tilde{g}(\dot{x_K}) = 1.$$

Let consider

$$\begin{aligned}\|\tilde{f}-\tilde{g}\| &\geq (\tilde{f}-\tilde{g})(\tilde{x}) = \tilde{f}(\tilde{x}) - \tilde{g}(\tilde{x}) \\ &= \frac{1}{d+\varepsilon}\big(\tilde{f}(\{x_n\}_{\mathcal{U}}) - \tilde{f}(\dot{x_K}) - [\tilde{g}(\{x_n\}_{\mathcal{U}}) - \tilde{g}(\dot{x_K})]\big) \\ &\geq \frac{1}{d+\varepsilon}\big(1-\varepsilon-0+1\big) = \frac{2-\varepsilon}{d+\varepsilon}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\tilde{f}+\tilde{g}\| &\geq (\tilde{f}+\tilde{g})(\tilde{y}) = \tilde{f}(\tilde{y}) + \tilde{g}(\tilde{y}) \\ &= \frac{1}{(d+\varepsilon)\alpha}\big(\tilde{f}(\{x_n\}_{\mathcal{U}}) + \tilde{f}(\dot{x_K}) + \tilde{g}(\{x_n\}_{\mathcal{U}}) + \tilde{g}(\dot{x_K})\big) \\ &\geq \frac{1}{(d+\varepsilon)\alpha}\big(1-\varepsilon+0+1\big) = \frac{2-\varepsilon}{(d+\varepsilon)\alpha}.\end{aligned}$$

Thus we have

$$\begin{aligned}C_{\mathrm{NJ}}(\widetilde{X^*}) &\geq \frac{\|\tilde{f}+\tilde{g}\|^2 + \|\tilde{f}-\tilde{g}\|^2}{2\|\tilde{f}\|^2 + 2\|\tilde{g}\|^2} \\ &\geq \frac{(\frac{2-\varepsilon}{d+\varepsilon})^2 + (\frac{2-\varepsilon}{(d+\varepsilon)\alpha})^2}{4} \\ &= (\frac{1}{d+\varepsilon})^2\big(\frac{(2-\varepsilon)^2}{4} + \frac{(2-\varepsilon)^2}{4\alpha^2}\big).\end{aligned}$$

Since ε is arbitrary and the Jordan-von Neumann constants of $X^*, X, \widetilde{X}$ and $\widetilde{X}^*$ are all equal, we obtain

$$C_{\mathrm{NJ}}(X) \geq \big(\frac{1}{d^2}\big)\big(1 + \frac{1}{2C_{\mathrm{NJ}}(X)}\big).$$

Thus

$$[WCS(X)]^2 \geq \frac{2C_{\mathrm{NJ}}(X)+1}{2[C_{\mathrm{NJ}}(X)]^2}.$$

□

Using Theorem 4.3.7, we obtain the following corollary.

Corollary 4.3.8. [17, Theorem 3.16], [59, Theorem 2] *Let X be a Banach space. If $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$, then X and X^* has uniform normal structure.*

Proof. Let $\widetilde{X}$ be a Banach space ultrapower of X. Since $C_{\mathrm{NJ}}(\widetilde{X}) = C_{\mathrm{NJ}}(X)$, Theorem 4.3.7 can be applied to $\widetilde{X}$. The inequality in Theorem 4.3.7 implies

$WCS(\widetilde{X}) > 1$ if $C_{\mathrm{NJ}}(\widetilde{X}) < \frac{1+\sqrt{3}}{2}$. Since $WCS(\widetilde{X}) > 1$ implies $\widetilde{X}$ has weak normal structure [12] and since $\widetilde{X}$ is reflexive, it must be the case that $\widetilde{X}$ has normal structure. By [27, Theorem 5.2], X has uniform normal structure as desired. □

Recall that the Cardano's formula is a formula for solving the polynomial of degree three ; $ax^3 + bx^2 + cx + d = 0$.

The solution of $ax^3 + bx^2 + cx + d = 0$ is

$$x = \left\{q + \left[q^2 + (r - p^2)^3\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}} + \left\{q - \left[q^2 + (r - p^2)^3\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}} + p$$

where

$$p = \frac{-b}{3a},\ q = p^3 + \frac{bc - 3ad}{6a^2} \text{ and } r = \frac{c}{3a}.$$

Using the inequality appearing in Theorem 4.3.7, we see that $C_{\mathrm{NJ}}(X) < 1 + \frac{WCS(X)^2}{4}$ if $8(C_{\mathrm{NJ}}(X))^3 - 8(C_{\mathrm{NJ}}(X))^2 - 2C_{\mathrm{NJ}}(X) - 1 < 0$. By applying the Cardano's formula to the equation $8(C_{\mathrm{NJ}}(X))^3 - 8(C_{\mathrm{NJ}}(X))^2 - 2C_{\mathrm{NJ}}(X) - 1 = 0$ we have

$$p = \frac{1}{3},\ q = \frac{61}{432} \text{ and } r = -\frac{1}{12}.$$

Thus

$$\begin{aligned} C_{\mathrm{NJ}}(X) &= \left\{\frac{61}{432} + \left[(\frac{61}{432})^2 + \left((-\frac{1}{12}) - (\frac{1}{3})^2\right)^3\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}} \\ &\quad + \left\{\frac{61}{432} - \left[(\frac{61}{432})^2 + \left((-\frac{1}{12}) - (\frac{1}{3})^2\right)^3\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}} + \frac{1}{3} \\ &= \left\{\frac{61}{432} + \left[\frac{3721}{186624} - \frac{343}{46656}\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}} \\ &\quad + \left\{\frac{61}{432} - \left[\frac{3721}{186624} - \frac{343}{46656}\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}} + \frac{1}{3} \\ &= \left\{\frac{61}{432} + \left(\frac{109594944}{8707129344}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}} \\ &\quad + \left\{\frac{61}{432} - \left(\frac{109594944}{8707129344}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}} + \frac{1}{3}. \end{aligned}$$

Therefore the equation $8(C_{\mathrm{NJ}}(X))^3 - 8(C_{\mathrm{NJ}}(X))^2 - 2C_{\mathrm{NJ}}(X) - 1 = 0$ has a unique real solution which is 1.273....

This implies that $C_{\mathrm{NJ}}(X) < 1 + \frac{WCS(X)^2}{4}$ if $C_{\mathrm{NJ}}(X) < c_0 = 1.273....$ Hence we can state :

Corollary 4.3.9. *Let E be a nonempty bounded closed convex subset of a Banach space X with*

$$C_{NJ}(X) < c_0 = 1.273....$$

Assume that $T : E \to KC(E)$ is a nonexpansive mapping. Then T has a fixed point.