Chapter 1

Introduction

Iteration occurs in many parts of mathematics. For example, several algorithms in numerical analysis are based on it. Given a set D and a function $f : D \to D$ the iterated f^n of f are defined by $f^0 = id$ and, inductively, $f^1 = f, \ldots, f^n = f \circ f^{n-1}, \ldots$ for $n \geq 2$ where the composition is understood as the multiplication in the semigroup $\{f^n : n \in \mathbb{Z}, n \geq 0\}$. The main problem in iteration theory is to study the behavior of the sequence $\{f^n\}$ as $n \to \infty$. For example, one may ask for which $z \in D$ the sequence $\{f^n(z)\}$ is convergent or periodic. Of course, the theory developed heavily depends on the assumptions that are made about D and f.

In this thesis, we will study the case that D is a domain on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and f is meromorphic in D. It turns out that the limit behavior of $\{f^n\}$ is relatively simple if $\overline{\mathbb{C}} \backslash D$ contains more than two points. We will mainly consider the remaining cases which, after the normalization, are $D = \overline{\mathbb{C}}$, $D = \mathbb{C}$ and $D = \mathbb{C}^* = \mathbb{C} \backslash \{0\}$. We will see that here, after exclusion of certain trivial cases, the domain D is divided into two sets, the Fatou set $F(f)$ and the Julia set $J(f)$, such that the limiting behavior of $\{f^n\}$ is relatively tame in $F(f)$ while it is chaotic in $J(f)$. These sets turn out to be quite complicated, with beauty also appealing to the non-mathematician. Computer graphics of the Julia sets can be founded in [73] and many other places.

The iteration of rational functions originated in the work of Gaston Julia and Piere Fatou, who wrote long memoirs on the subject between 1918 and 1920. Earlier work on iteration had mainly been concerned with the behavior of the iterates near fixed points. In 1926, Fatou [42] extended some of the results to entire functions. The case of holomorphic self-maps of $\mathbb{C}^*$ was first considered by H. Radström [74] in 1953.

Consequently, there was not much activity in this field for about sixty years. Around 1980, a renewed interest in the complex dynamics was generated due to the beautiful computer graphics introduced into this subject. I. N. Baker extended much of the work of Fatou and Julia to the class of entire functions, showing along the way that new type of stable behavior (wandering domain) could occur for transcendental entire functions. The dynamics of entire functions is quite different from the dynamics of polynomials or rational functions, mainly because of the essential singularity at ∞. By Picard's theorem, any neighborhood of ∞ is mapped over the entire plane infinitely often, missing at most one point which means that an entire function exibits a tremendous amount of hyperbolicity near ∞.

For more details of the concepts and properties in the iteration theory, we refer to the books by A. F. Beardon [20], L. Carleson and T. W. Gamelin [33], C. McMullen [61] as well as J. Milnor's lecture notes [62] and N. Steinmetz [77] for rational functions and the survey articles of I. N. Baker [12], W. Bergweiler [22], A. E. Eremenko and M. Y*ü* Lyubich [39], G. P. Kapoor and others [51], for rational and entire functions and W. Bergweiler [21] and the book by X. H. Hua and C. C. Yang [48] for transcendental meromorphic functions.

The class **M**={ f: there is a compact totally disconnected set $E = E(f)$ such that f is meromorphic in $E^c = \overline{\mathbb{C}} \backslash E$ and the cluster set of f at any point $z_0 \in E$ with respect to E^c, that is, the set $C(f, E^c, z_0)$= $\{w \in \overline{\mathbb{C}} : w = \lim_{n\to\infty} f(z_n)$ for some sequence $z_n \in E^c$ with $z_n \to z_0\}$ is equal to $\overline{\mathbb{C}}$. If $E = \emptyset$ we make the further assumption that f is neither constant nor univalent in $\overline{\mathbb{C}}$} was first investigated in [16] and the basic concepts such as the Fatou and Julia sets and the basic properties of the dynamics of functions in class M was established here. We also mention the other articles, [17] and [85].

This thesis is organized as follows: In chapter 2, we first give some definitions, notations and some important results in complex dynamics which will be used in this thesis mostly about the Fatou and Julia sets. We also introduce a class M and give some basic concepts and basic properties of functions in this class which will be used in chapter 5.

In chapter 3, we propose to study meromorphic solutions of the functional equation

$$f \circ S = S^k \circ f$$

where $k \geq 2$ and S is a Möbius transformation which has two fixed points, say a and b in $\mathbb{C}$. Without loss of generality we may assume that a is an attracting fixed point and b is a repelling fixed point of S. We will show that for a given complex number α distinct from a and b, there exists a unique solution of this functional equation which fixes α, a, and b. Under some certain conditions, we propose to give the explicit form of f. We also show that the Julia sets of rational solutions of this functional equation are circles on the sphere.

Moreover, we also study meromorphic solutions f of the following functional equation

$$f \circ R = R^k \circ f$$

where $k \geq 2$ and R is a Möbius transformation which has only one fixed point, say $a \in \mathbb{C}$ (so a is a global attractor of R).

In chapter 4, we find sufficient conditions for the boundedness of components of Fatou sets of composite entire and meromorphic functions. We will prove that if $f_i, i = 1, 2, \ldots, m$ are transcendental meromorphic functions with at most finitely many poles of growth order less than $\frac{1}{2}$ of which at least one of them has positive lower order and $g = f_m \circ f_{m-1} \circ \cdots \circ f_1$, then either g has no unbounded Fatou components or at least one unbounded Fatou component is multiply connected.

In chapter 5, we consider the class $\mathbf{M}$=$\{$ f: there is a compact totally disconnected set $E = E(f)$ such that f is meromorphic in $E^c = \overline{\mathbb{C}} \backslash E$ and the cluster set of f at any point $z_0 \in E$ with respect to E^c, that is, the set $C(f, E^c, z_0)$= $\{w \in \overline{\mathbb{C}} : w = \lim_{n\to\infty} f(z_n)$ for some sequence $z_n \in E^c$ with $z_n \to z_0\}$ is equal to $\overline{\mathbb{C}}$. If $E = \emptyset$ we make the further assumption that f is neither constant nor univalent in $\overline{\mathbb{C}}\}$ which is closed under composition. We study relationship between dynamics of $f \circ g$ and $g \circ f$ where f and g are two functions in class $\mathbf{M}$. Let U be a component of $F(f \circ g)$ and V be a component of $F(g \circ f)$ which contains $g(U)$. We show that under certain conditions U is a wandering domain if and only if V is a wandering domain; if U is periodic, then so is V and moreover, V is of the same type according to the classification of periodic components as U unless U is a Siegel disc or Herman ring.