Chapter 2

Basic Knowledge

The aim of this chapter is to give some definitions, notations and properties of Fatou and Julia sets of meromorphic functions which will be used in the later chapters.

2.1 Extended complex plane

The *extended complex plane* is the union

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

To obtain a metric on $\overline{\mathbb{C}}$, we identify $\mathbb{C}$ with the horizontal plane

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}$$

in $\mathbb{R}^3$ and proceed to construct the usual model for $\overline{\mathbb{C}}$ as a sphere. Let S be a sphere in $\mathbb{R}^3$ with unit radius and center at the origin, and denote the point $(0, 0, 1)$ (the top point of S) by ζ. We now project each point z in $\mathbb{C}$ linearly towards (or away from) ζ until it meets S at a point z^* distinct from ζ : the map $\pi : z \mapsto z^*$ is called *the stereographic projection* of $\mathbb{C}$ into S. Clearly, if $|z|$ is large, then z^* is near to ζ, and we define the projection $\pi(\infty)$ of ∞ to be ζ. With this definition, π is a bijective map from $\overline{\mathbb{C}}$ to S and this explains why $\overline{\mathbb{C}}$ is also called *the complex* (or *Riemann*) *sphere*: see Figure2.1.

We define *the chordal metric* on $\overline{\mathbb{C}}$ by

$$\begin{aligned} \sigma(z, w) &= |\pi(z) - \pi(w)| \\ &= |z^* - w^*| \\ &= \frac{2|z - w|}{(1 + |z|^2)^{1/2}(1 + |w|^2)^{1/2}} \end{aligned}$$

where z and w are in $\mathbb{C}$, while for z in $\mathbb{C}$

$$\sigma(z, \infty) = \lim_{w \to \infty} \sigma(z, w) = \frac{2}{(1 + |z|^2)^{1/2}}.$$

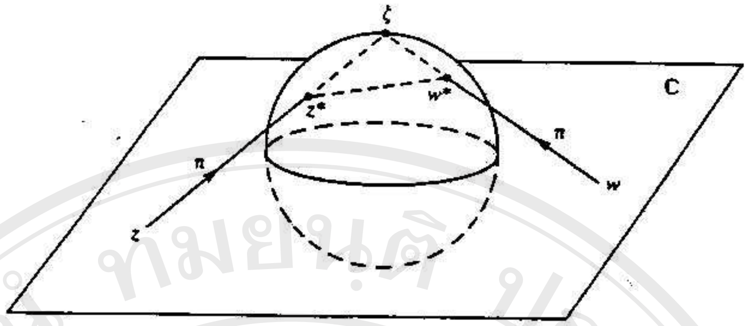

Figure 2.1: Sterographic projection

That is, the chordal metric is the Euclidean length of the chord joining z^* to w^*. There is an alternative metric on $\overline{\mathbb{C}}$, namely the *spherical metric* χ, and this is equivalent to the chordal metric σ. The spherical distance $\chi(z,w)$ between z and w in $\overline{\mathbb{C}}$ is, by definition, the Euclidean length of the shortest path on S (an arc of a great circle) between z^* and w^*. If the chord joining z^* and w^* subtends an angle θ at the origin then, of course,

$$\chi(z,w) = \theta, \quad \sigma(z,w) = 2\sin(\theta/2)$$

so

$$\sigma(z,w) = 2\sin(\frac{\chi(z,w)}{2}).$$

More useful are the inequalities

$$\frac{2}{\pi} \cdot \chi(z,w) \leq \sigma(z,w) \leq \chi(z,w)$$

which follow from the elementary inequalities

$$\frac{2\theta}{\pi} \leq \sin\theta \leq \theta, \quad 0 \leq \theta \leq \pi/2.$$

2.2 Entire and meromorphic functions

Let f be a function whose domain of definition contains a neighborhood of a point z_0. The *derivative* of f at z_0, written $f'(z_0)$, is defined by the equation

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided this limit exists. The function f is said to be *differentiable* at z_0 when its derivative at z_0 exists. A function $f : D \to \mathbb{C}$ defined on the plane domain D is *holomorphic* (or *analytic*) in D if it has a derivative at each point in that set. If we should speak of a function f that is holomorphic in a set S which is not open, it is to be understood that f is holomorphic in an open set containing S. In particular, f is *holomorphic at a point* z_0 if it is holomorphic in a neighborhood of z_0. An *entire function* is a function that is holomorphic at each point in the entire complex plane. A function $f : D \to \overline{\mathbb{C}}$ is *meromorphic* in D if each point of D has a neighborhood on which either f or $1/f$ is holomorphic. A function f is said to be *defined near* ∞ if it is defined on some set $\{|z| > r\} \cup \{\infty\}$, and in this sense, f is holomorphic (or meromorphic) at ∞ if the map $z \mapsto f(1/z)$ is holomorphic (or meromorphic) near the origin. A *rational function* is a function $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the form

$$R(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are polynomials, not both being the zero polynomial, and have no common factors. If P is zero, then R is the constant function zero, and if Q is zero, then R is the constant function ∞. If $Q(z) = 0$, then $R(z)$ is defined to be ∞, and we define $R(\infty)$ as the limit of $R(z)$ as $z \to \infty$. The *degree* of R, $deg(R)$, is defined by

$$deg(R) = \max\{deg(P), deg(Q)\}.$$

If R is a constant function with value α, where $\alpha \neq 0, \infty$ we have $deg(R) = 0$, and it is convenient to define $deg(R) = 0$ even when α is 0 or ∞. A function f is *transcendental meromorphic function* in $\mathbb{C}$ if f is meromorphic in $\mathbb{C}$ and not rational. Next, we distinguish three cases:

(i) $f \in \mathcal{M} \Leftrightarrow f$ is transcendental meromorphic in $\mathbb{C}$;

(ii) $f \in \mathcal{E} \Leftrightarrow f$ is transcendental entire in $\mathbb{C}$;

(iii) $f \in \mathcal{R} \Leftrightarrow f$ is rational with $deg(f) \geq 2$.

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function. A value a is called *a Picard exceptional value of* f if f does not assume the value a in $\mathbb{C}$. We denote by $PV(f)$ the set of the finite Picard exceptional values of f. According to the classical Picard's theorem, f has at most one or two finite exceptional values according to f is entire or meromorphic, respectively. Thus the set $PV(f)$ contains at most two points. If f has exactly one pole, which is a Picard exceptional value of f, then f has the form

$$f(z) = z_0 + (z - z_0)^{-m} e^{g(z)}$$

for some positive integer m and some entire function $g(z)$, and f is a holomorphic self-map of the punctured plane $\mathbb{C} \backslash \{z_0\}$. We denote the set of such functions by $\mathcal{P}$, that is, $f \in \mathcal{P}$ if f has exactly one pole, which is a Picard exceptional value.

The reader should keep in mind that f always denotes a given meromorphic function, unless specified otherwise.

2.3 Normal family

Definition 2.3.1 *A sequence of functions* $\{f_n\}$ converges spherically uniformly *to* f *on a set* $E \subset \mathbb{C}$ *if, for any* $\epsilon > 0$, *there is a number* N *such that* $n \geq N$ *implies*

$$\chi(f(z), f_n(z)) < \epsilon,$$

for all $z \in E$.

Normality of meromorphic functions is an important concept, whose definition is the following:

Definition 2.3.2 *A family* $\mathcal{F}$ *of functions meromorphic in a domain* D *is* normal *in* D *if every sequence* $\{f_n\} \subset \mathcal{F}$ *contains a subsequence which converges spherically uniformly on compact subsets of* D. *The limit function is meromorphic or the constant* ∞. *For* $z_0 \in D$, *if there exists a neighborhood* $D(z_0) \subset D$ *of* z_0 *such that* $\mathcal{F}$ *is normal in* $D(z_0)$, *then we call that* $\mathcal{F}$ *is normal at* z_0.

A lot of normality criteria have been given. Based on Arzelá-Ascoli Theorem, Montel in 1907 observed that for a family of holomorphic functions, locally boundedness implies equicontinuity.

Theorem 2.3.3 *(Montel) If $\mathcal{F}$ is a locally bounded family of holomorphic functions on a domain D, then $\mathcal{F}$ is normal in D; Conversely, if $\mathcal{F}$ is normal in D and is bounded on a subset of D, then $\mathcal{F}$ is locally bounded.*

Theorem 2.3.4 *(Montel) A family $\mathcal{F}$ of meromorphic functions in a domain D is normal if and only if $\mathcal{F}$ is spherically equicontinuous in D.*

The following fundamental normality test is very useful.

Theorem 2.3.5 *(Montel) The family of all meromorphic (holomorphic) functions of a domain D into the three (two)-punctured plane is normal there.*

Note that the above theorem, the three (two) constants do not depend on functions in the family. The condition can be improved as follows.

Theorem 2.3.6 *[63] Let $\{f(z)\}$ be a family of holomorphic functions in a domain D. If each function $f(z)$ of the family does not take values $a(f)$ and $b(f)$ such that*

$$|a(f)| < M, \quad |b(f)| < M, \; |a(f) - b(f)| > d,$$

where $M > 0, d > 0$ are two constants independent of f, then $\{f(z)\}$is normal in D.

For a family of meromorphic functions, we have

Theorem 2.3.7 *(**Marty's criterion**)[60] A family $\mathcal{F}$ of meromorphic functions on a domain D is normal if and only if the spherical derivatives*

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

are locally uniformly bounded in D for all $f \in \mathcal{F}$.

An improvement is the following Lappan's criterion.

Theorem 2.3.8 *[58] Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. If there exist five distinct values $a_j (j = 1, 2, \ldots, 5)$ such that*

$$sup\{f^{\#}(z) : f(z) = a_j, z \in D, j = 1, 2, \ldots, 5\} < \infty,$$

then $\mathcal{F}$ is normal in D.

The following lemma due to Zalcman is very important for the study of normal families.

Lemma 2.3.9 *(***Zalman***)[87] Let $\mathcal{F}$ be a family of meromorphic functions on D : $|z| < 1$ and α be a real number satisfying $-1 < \alpha < 1$. Then $\mathcal{F}$ is not normal in D if and only if there exist*
(i) a number r, $0 < r < 1$;
(ii) a sequence of points z_k, $|z_k| < r$;
(iii) a positive sequence ρ_k, $\rho_k \to 0$ and
(iv) a sequence $\{f^n\}_{n\in\mathbb{N}} \subset \mathcal{F}$,
such that $\rho_k^\alpha f_k(z_k + \rho_k\zeta) \to g(\zeta)$ spherically uniformly on compact subset of $\mathbb{C}$, where g is a non-constant meromorphic function.

Remark 2.3.10 *One can choose z_k and ρ_k properly such that,*

$$\rho_k \leq \frac{2}{f_k^{\#}(z_k)^{\frac{1}{1+|\alpha|}}},\ f_k^{\#}(z_k) \geq f_k^{\#}(0).$$

2.4 Nevanlinna theory

In 1925, R. Nevanlinna established first and second fundamental theorems, initiating the new study of value distributions. In this section, we collect basic results of Nevanlinna theory. We refer reader to [45] for more detail. We denote by $n(r, \frac{1}{f-a})$ the number of roots of $f(z) = a$ on $\{|z| \leq r\}$ counting multiplicity and by $n(r, f)$ the number of poles of $f(z)$ on $\{|z| \leq r\}$ counting multiplicity. The *counting function* of f are defined as follows:

$$N\left(r, \frac{1}{f-a}\right) = \int_0^r \frac{n(t, \frac{1}{f-a}) - n(0, \frac{1}{f-a})}{t} dt + n\left(0, \frac{1}{f-a}\right) \log r, a \in \mathbb{C}$$

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

Furthermore, define $m(r, f)$ and $m(r, \frac{1}{f-a})(a \neq \infty)$ as

$$m\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\theta}) - a}\right| d\theta,$$

$$m(r, f) = \int_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| d\theta,$$

where $\log^+ x = \max\{\log x, 0\}$. The *Nevanlinna characteristic function* of a meromorphic function f is defined as

$$T(r, f) = m(r, f) + N(r, f).$$

By applying Jensen formula, it is easy to deduce the First fundamental theorem.

Theorem 2.4.1 (First fundamental theorem) *[45] Let $f(z)$ be meromorphic in $|z| < R(\leq \infty)$. If a is an arbitrary complex number and $0 < r < R$, and*

$$f(z) - a = \sum_{i=m}^{\infty} c_i z^i, \;\; c_m \neq 0, \;\; m \in \mathbb{Z},$$

is the Laurent expansion of $f - a$ at the origin, then we have

$$T(r, f) = T\left(r, \frac{1}{f-a}\right) + \log|c_m| + \varepsilon(r, a),$$

where $|\varepsilon(r, a)| \leq \log 2 + \log^+ a$.

The *order* and the *lower order* of a meromorphic function f are defined by

$$\rho = \rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r},$$

and

$$\mu = \mu(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r},$$

respectively.

Denote the *maximum modulus* of an entire function f on $\{|z| \leq r\}$ by $M(r, f) = \max_{|z|\leq r} |f(z)|$. Since for any entire function f,

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R, f),$$

whenever $0 < r < R$, we obtained

$$\rho = \rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r},$$

and

$$\mu = \mu(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r} = \liminf_{r\to\infty} \frac{\log\log M(r, f)}{\log r}.$$

Now we state Nevanlinna's Second fundamental theorem.

Theorem 2.4.2 (Second fundamental theorem) *[45] Let f be a non-constant meromorphic function, let $q \geq 2$ and let $a_1, a_2, \ldots, a_q$ be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_\nu| > \delta$ for $1 \leq \mu < \nu \leq q$. Then*

$$m(r,f) + \sum_{i=1}^{q} m\left(r, \frac{1}{f-a_i}\right) \leq 2T(r,f) - N_1(r) + S(r,f)$$

where

$$N_1(r) = 2N(r,f) - N(r,f') + N\left(r, \frac{1}{f'}\right),$$

and

$$S(r,f) = O(\log(rT(r,f))), \quad (r \to \infty)$$

except for a set E with a finite linear measure.

Denote the *deficiency* of a by

$$\delta(a,f) = \limsup_{r\to\infty} \frac{m(r,\frac{1}{f-a})}{T(r,f)} = 1 - \liminf_{r\to\infty} \frac{N(r,\frac{1}{f-a})}{T(r,f)},$$

for a non-constant meromorphic function f and for $a \in \overline{\mathbb{C}}$. Then it follows from the Second fundamental theorem that

$$\sum_{a\in\overline{\mathbb{C}}} \delta(a,f) \leq 2.$$

Now we denote by $\bar{n}(r, \frac{1}{f-a})$ the number of zeros of $f(z) - a$ in $\{|z| \leq r\}$, each zero being counted only once, and $\bar{n}(r,f)$ the number of poles of $f(z)$ in $\{|z| \leq r\}$, each pole counted only once. Moreover, denote

$$\overline{N}\left(r, \frac{1}{f-a}\right) = \int_0^r \frac{\bar{n}(t,\frac{1}{f-a}) - \bar{n}(0,\frac{1}{f-a})}{t} dt + \bar{n}\left(0, \frac{1}{f-a}\right)\log r, a \in \mathbb{C}$$

$$\overline{N}(r,f) = \int_0^r \frac{\bar{n}(t,f) - \bar{n}(0,f)}{t} dt + \bar{n}(0,f)\log r.$$

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)},$$

$$\theta(a,f) = \liminf_{r\to\infty} \frac{N(r,\frac{1}{f-a}) - \overline{N}(r,\frac{1}{f-a})}{T(r,f)}.$$

Then the Nevannlinna's Second fundamental theorem can be expressed as in the following: if $f(z)$ is non-constant meromorphic in the finite plane, then the set of values a for which $\Theta(a,f) > 0$ is at most countable and

$$\sum_{a\in\overline{\mathbb{C}}} \{\delta(a,f) + \theta(a,f)\} \leq \sum_{a\in\overline{\mathbb{C}}} \Theta(a,f) \leq 2.$$

2.5 The fixed point theory

In this section, we will discuss the fixed points of meromorphic functions. The study of complex dynamics begins with the description of the local behavior of a function near its fixed points. We are concerned with the number of fixed points and the existence of canonical coordinate systems at fixed points. These results will enable us to establish some basic properties of the Fatou-Julia theory.

2.5.1 Classification of fixed points

Given z_0, the *forward orbit* of z_0 is the set

$$O^+(z_0) = \{z_n = f^n(z_0) : n = 0, 1, \ldots\};$$

and the *backward orbit* of z_0 is the set

$$O^-(z_0) = \{z : f^n(z) = z_0, n = 0, 1, \ldots\},$$

which is the collection of the pre-images of z_0 under $f, f^2, \ldots$. The point z_0 is called *periodic* if $z_n = z_0$ for some n. The minimum n is called its *period.* In particular, if $f(z_0) = z_0$, then z_0 is called *a fixed point* of f. The point z_0 is called *pre-periodic* if $f^k(z_0)$ is periodic for some integer $k > 0$, and *strictly pre-periodic* if it is pre-periodic but not periodic. A periodic point of period n is called a fixed point of exact order n.

For periodic point z_0 with period n, the orbit

$$O^+(z_0) = \{z_0, \ldots, z_n = z_0\}$$

is called *the cycle* and $\lambda = (f^n)'(z_0)$ is called its *multiplier(or eigenvalue)*. By the chain rule,

$$(f^n)'(z_0) = \prod_{j=0}^{n-1} f'(f^j(z_0)).$$

This has to be modified if $z_0 = \infty$: we define λ to be the multiplier of the fixed point of the conjugate $z \mapsto 1/f^n(1/z)$. The cycle $O^+(z_0)$ is called

$$\begin{cases} \text{attracting} & \text{if} \quad |\lambda| < 1 \\ \text{indifferent} & \text{if} \quad |\lambda| = 1 \\ \text{repelling} & \text{if} \quad |\lambda| > 1. \end{cases}$$

In particular, the cycle $O^+(z_0)$ is called super-attracting if and only if $\lambda = 0$. The indifferent cycles are subdivided into two situations:

$$\begin{cases} \text{rationally indifferent} & \text{if} \quad \lambda^m = 1 \text{ for some } m \in \mathbb{N} \\ \text{irrationally indifferent} & \text{if} \quad \lambda = e^{2\pi i\theta}, \theta \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

A rationally indifferent cycle is also called a parabolic cycle.

2.5.2 Fixed points of iterated functions

The first thing is to know whether or not there exists a fixed point for f^n.

Theorem 2.5.1 *(see in [48]) If $f \in \mathcal{E}$, then for each integer $n \geq 2$, $f(z)$ has an infinite number of periodic points of period n. If $f \in \mathcal{P}$, then f always has infinitely many fixed points.*

Note that any fixed point of f is a fixed point of f^n for any positive integer n. The example $e^z + z$ shows that the conclusion does not hold for $n = 1$.

W. Bergweiler proved that for any $n \geq 2$, $f \in \mathcal{E}$, f has infinitely many repelling fixed points of period n.

2.5.3 Attracting and repelling fixed points

Let $f(z)$ be holomorphic on some neighborhood of $z = 0$. We first suppose that $z = 0$ is an attracting (but not super-attracting) or repelling fixed point of f. Then near $z = 0$

$$f(z) = \lambda z + a_2 z^2 + \cdots,$$

where $\lambda(\neq 0)$ is the multiplier. We show that $f(z)$ can be reduced to a simple normal form. The following result is proved by G. Koenigs [54] in 1884.

Theorem 2.5.2 (Koenigs linearization theorem) *[54] If λ satisfies $|\lambda| \neq 0, 1$, then there exists a local holomorphic change of coordinate $w = \phi(z)$ with $\phi(0) = 0$ such that $\phi \circ f \circ \phi^{-1}$ is the linear map $w \mapsto \lambda w$ for all w in some neighborhood of the origin. Furthermore, ϕ is unique up to multiplication by a non-zero constant. We call ϕ a* Koenigs map.

If z_0 is an attractive fixed point, then there exists a disk $D = \{|z - z_0| < r\}$ in which the sequence $f^n(z)(n = 1, 2, \ldots)$ converges uniformly to z_0.

Next we deal with super-attractive fixed points. Let f be holomorphic in some neighborhood of the origin. If $z = 0$ is a super-attractive fixed point of f, then

$$f(z) = az^p + a_{p+1}z^{p+1} + \cdots \quad a \neq 0,\ p \geq 2.$$

In 1904, Böttcher [31] proved the existence of the conjugation.

Theorem 2.5.3 (Böttcher) *[31] If $z = 0$ is a super-attractive fixed point of f, then there is a conformal map $w = \phi(z)$ of a neighborhood of 0 onto a neighborhood of 0 that conjugates $f(z)$ to w^p. The conjugating function is unique up to multiplication by a $(p-1)^{th}$ root of unity.*

2.5.4 Rationally indifferent fixed points

Now we consider the case that $z = 0$ is a rationally indifferent fixed point of f, that is, the multiplier is a root of unity. Choose a neighborhood N of the origin that is small enough so that f maps N conformally onto some neighborhood N_0 of the origin.

A connected open set U, with compact closure $\overline{U} \subset N \cap N_0$, will be called an *attracting petal* for f at the origin if

$$f(\overline{U}) \subset U \cup \{0\}$$

and

$$\bigcap_{k \geq 0} f^k(\overline{U}) = \{0\}.$$

Similarly, $V \subset N \cap N_0$ is a *repelling petal* for f if V is an attracting petal for f^{-1}.

Theorem 2.5.4 *[62] Let*

$$f(z) = z + az^{n+1} + \textit{higher terms } (a \neq 0, n \geq 1)$$

be holomorphic in some neighborhood N of the origin. Then there exist n disjoint attracting petals U_i and n disjoint repelling petals V_i so that the union of these $2n$ petals, together with the origin itself, forms a neighborhood N_0 of the origin. These petals alternate with each other, so that each U_i intersects only V_i and V_{i-1} (where V_0 is defined to be V_n).

Now we show the existence of a local conjugation near rationally indifferent fixed point $z = 0$.

Theorem 2.5.5 *[62] Let*

$$f(z) = \lambda z + az^{n+1} + \textit{higher terms } (a \neq 0, n \geq 1)$$

be holomorphic in some neighborhood N of the origin, where λ is a primitive q^{th} root of unity. Then there exists a local holomorphic change of coordinate $w = \phi(z)$ such that $\phi \circ f^q \circ \phi^{-1}$ is the linear map $z \mapsto w + 1$ for all w in some neighborhood of the origin.

2.5.5 Irrationally indifferent fixed points

Once more we consider holomorphic functions of the form

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots,$$

defined in some neighborhood of the origin, where the multiplier λ is of the form

$$\lambda = e^{2\pi i\theta}, \ \theta \in \mathbb{R} \backslash \mathbb{Q}.$$

Next we will study whether or not this function conjugates to the linear map $w \mapsto \lambda w$.

In 1938, Cremer proved that if $|\lambda| = 1$ and $\liminf_{n\to\infty} |\lambda^n - 1|^{\frac{1}{n}} = 0$, then there is a holomorphic function $f(z) = \lambda z + \cdots$ such that no linearization is possible. Later in 1942, Siegel gave an example of unimodular λ for which linearization is possible. In order to state this result clearly, some facts in number theory are needed.

A real number θ is *Diophantine* if it is badly approximable by rational numbers, in the sense that there exist $c > 0$ and $\mu < \infty$ so that

$$\left|\theta - \frac{p}{q}\right| \geq \frac{c}{q^{\mu}}$$

for all integers p and $q, q \neq 0$. This occurs if and only if $\lambda = e^{2\pi i\theta}$ satisfies

$$|\lambda^q - 1| \geq c' q^{1-\mu}, \; q \geq 1,$$

for some constant c'. In fact, as $q\theta - p \to 0$,

$$|\lambda^q - 1| = |e^{2\pi i(q\theta - p)} - 1| \sim 2\pi q \left|\theta - \frac{p}{q}\right|.$$

For fixed $\mu > 2$, if E is the set of $\theta \in [0,1]$ such that $|\theta - \frac{p}{q}| < q^{-\mu}$ infinitely often, then the measure of E satisfies

$$|E| \leq \sum_{q=n}^{\infty} 2 \cdot q^{-\mu} \cdot q = O(n^{2-\mu}) \to 0.$$

Thus almost all real numbers are Diophantine.

We call an irrationally indifferent fixed point *a Siegel point* or *Cremer point* depending whether a local linearization is possible or not. Similarly we can define the Siegel cycle and the Cremer cycle.

Theorem 2.5.6 (Siegel)*[33] If θ is Diophantine, and if f has fixed point at 0 with multiplier $e^{2\pi i\theta}$, then there exists a local change of coodinate $z = h(w)$, which conjugates f to the irrational rotation $w \mapsto \lambda w$.*

As an application of this theorem, we obtain the following result.

Corollary 2.5.7 *[48] Let z_0 be a Siegel point. Then there exists a disk $D : \{|z - z_0| < r\}$ and a sequence of positive integers $n_k (k = 1, 2, \ldots)$ tending to ∞ such that in D, the sequence $f^{n_k} (k = 1, 2, \ldots)$ converges uniformly to z_0.*

2.6 The Fatou and Julia sets

In this section, we will deduce some elementary properties of the Fatou and Julia sets of meromorphic functions. There are some difference properties between rational functions and transcendental functions, due to the existence of Picard exceptional values.

2.6.1 Definition of the Fatou and Julia sets

Let f be a meromorphic function. We define

$$F(f) = \{z \in \overline{\mathbb{C}} : \text{the sequence } \{f^n\} \text{ is well-defined and normal at } z\}$$

and $J(f) = \overline{\mathbb{C}} \setminus F(f)$,. They are called the *Fatou set* and the *Julia set* of f respectively. According to the definition, it is easily verified that $F(f)$ is open (possibly empty) and $J(f)$ is closed.

A set S is called *forward invariant* (or invariant) under f if $z \in S$ implies that $f(z) \in S$ or $f(z)$ is undefined; A set S is called *backward invariant* under f if $z \in S$ implies that $w \in S$ for all w satisfying $f(w) = z$. A set S is called *completely invariant* if it is both forward and backward invariant.

For $f \in \mathcal{R}$, the Julia set $J(f)$ and the Fatou set $F(f)$ are completely invariant, that is,

$$f(J(f)) = J(f) = f^{-1}(J(f))$$

and

$$f(F(f)) = F(f) = f^{-1}(F(f)).$$

However, for transcendental function, we will see the difference. The reason is that f possibly has finite Picard exceptional values. For $f \in \mathcal{M} \cup \mathcal{E}$

$$F(f) = f^{-1}(F(f)) = f(F(f)) \cup \{PV(f) \cap F(f)\}.$$

In particular, if f has no Picard exceptional values, then $f^{-1}(F(f)) = F = f(F(f))$.

2.6.2 Some properties of the Fatou and Julia sets

A point $a \in \mathbb{C}$ is said to be *a Fatou exceptional value* of the meromorphic function f if the set $O^-(a)$ is finite.

We denote by $FV(f)$ all the Fatou exceptional values of f. Note that according to Nevanlinna's second fundamental theorem, we have $FV(f)$ contains at most two points, and $PV(f) \subset FV(f)$.

For any $b \in \overline{\mathbb{C}} \backslash FV(f)$, we have

$$J(f) \subset \overline{\left(\bigcup_{n=0}^{\infty} f^{-n}(b)\right)}.$$

Furthermore, if $b \in J(f) \backslash FV(f)$, then

$$J(f) = \overline{\left(\bigcup_{n=0}^{\infty} f^{-n}(b)\right)}.$$

Note that the above results follow from Nevanlinna's second fundamental theorem.

Now we define the set

$$P_0 = P_0(f) = O^-(\infty) = \{z \in \overline{\mathbb{C}} : f^n(z) = \infty \text{ for some } n \in \mathbb{N}\}.$$

Let f be a function which has either at least two poles or exactly one pole, which is not a Picard exceptional value. Then $J(f) = P_0'$. From the definition of the Julia set, we see that $P_0 \subset J(f)$. For $f \in \mathcal{E}$ which has no unbounded component, the Fatou exceptional value always belongs to $J(f)$.

Next we give some basic properties.

Theorem 2.6.1 *[48] Let f be a meromorphic function. Then*
1. if $f \in \mathcal{E} \cup \mathcal{R} \cup \mathcal{P}$, then $F(f^p) = F(f)$ and $J(f^p) = J(f)$ for any positive integer p. (Here we not include $f \in \mathcal{M}$ because f^n may not be meromorphic in $\mathbb{C}$ so that $F(f^n)$ and $J(f^n)$ are not completely defined.);
2. $J(f)$ contains an infinite number of points;
3. $J(f)$ is perfect, that is, $J(f) = \{J(f)\}'$;
4. $J(f)$ is the closure of repelling periodic points of f;
5. (Expansivity of the Julia set) for any $z_0 \in J(f)$ and any neighborhood D of z_0, if A is a bounded and closed subset of $\mathbb{C}$ and $A \cap FV(f) = \emptyset$, then there exists an

integer $N > 0$ such that for any $n \geq N$, $A \subset f^n(D)$;
6. if $J(f)$ has an interior point, then $J(f) = \overline{\mathbb{C}}$ for $f \in \mathcal{R}$ and $J(f) = \mathbb{C}$ for $f \in \mathcal{M}$.

Theorem 2.6.2 *[48] If $z_0 \in J(f)$, then for each finite value a, there exist a sequence of points $\zeta_k \to z_0$ and a sequence of positive integers $n_k \to \infty$ such that*

$$f^{n_k}(\zeta_k) = a \ \ (k = 1, 2, \ldots),$$

except at most for two finite values. If f is entire, then there is at most one such exceptional value.

For any meromorphic function f, the Fatou set $F(f)$ contains all attracting points, super-attracting points and all Siegel points of f; the Julia set $J(f)$ contains all repelling points, all rationally indifferent points and all Cremer points.

Since for $f \in \mathcal{E} \cup \mathcal{R} \cup \mathcal{P}$, $F(f^p) = F(f)$ and $J(f^p) = J(f)$ then we obtain that the Fatou set $F(f)$ contains all attracting cycles, super-attracting cycles and all Siegel cycles of f; the Julia set $J(f)$ contains all repelling cycles, all rationally indifferent cycles and all Cremer cycles.

2.7 The components of the Fatou set

For a meromorphic function f, we focus on the behavior of the components of the Fatou set $F(f)$ and the behavior of f on its Fatou set. We will see that there are some essential differences among rational functions, transcendental entire functions and transcendental meromorphic functions.

2.7.1 Types of the components

Let U be a maximum domain of normality of the iterates of f namely, a component of $F(f)$. This domain is also called *a stable domain or a Fatou component.*

Consider a fixed component U of $F(f)$. There are several possibilities for the orbit of U under f.

(1) If $f^n(U) \subset U$ for some integer $n \geq 1$, then we call U *a periodic component* of

$F(f)$. The minimum n is the period of the component, In particular, if $n = 1$, then such a component U is said to be *an invariant component or a fixed component.*

(2) If $f^m(U)$ is periodic for some integer $m \geq 1$, then we call U *a pre-periodic component* of $F(f)$. In particular, if U is pre-periodic but not periodic, then we call U *a really pre-periodic component.*

(3) Otherwise, all $f^n(U)$ are disjointed, and we call U *a wandering domain.*

We have the classification theorem that was stated by I. N. Baker, J. Kotus and Y. L*ü* Lyubich [18] as follows.

Theorem 2.7.1 *[18] Let f be a transcendental meromorphic function and let U be a periodic component of period m. Then we have the five possibilities:*

(1) *U contains an attracting periodic point z_0 of period m, then $f^{nm}(z) \to z_0$ for $z \in U$ as $n \to \infty$, and U is called* the immediate attractive basin *of z_0. Furthermore, U is called a* super-attracting domain *or an* attracting domain *provided that z_0 is super-attracting or not.*

(2) *∂U contains a periodic point z_0 of period m and $f^{nm}(z) \to z_0$ for $z \in U$ as $n \to \infty$. Then $(f^m)'(z_0) = 1$. In this case, U is called* a parabolic domain (or Leau domain).

(3) *There exists an analytic homeomorphism $\phi : U \to \triangle$ such that $\phi \circ f^m \circ \phi^{-1}(z) = e^{2\pi\alpha i}z$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, where $\triangle = \{z : |z| < 1\}$. In this case U is called* a Siegel disc. *We have the following commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{f^m} & U \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ \triangle & \xrightarrow{e^{2\pi\alpha i}z} & \triangle \end{array}$$

(4) *U is doubly connected and f^m is conjugate to either a rotation of an annulus or to a rotation followed by an inversion. This U is called* a Herman ring (an Arnold-Herman ring). *The Siegel disc and Herman ring are referred to as rotation domains.*

(5) *$f^{nm}(z) \to z_0 \in \partial U$ for $z \in U$ as $n \to \infty$ but f^m is not holomorphic at z_0, and U is called* a Baker domain (or infinite Fatou component, or essentially parabolic domain, or domain at ∞).

Theorem 2.7.2 *[48] Any $f \in \mathcal{E}$ does not have Herman rings and the only case for the Baker domain in the classification theorem is $z_0 = \infty$.*

The existence of the Baker domain can be seen from the following examples.

Example 2.7.3 *Let $f(z) = z + 1 + e^{-z}$. Then the right half-plane is $f-$invariant and $Re(f^n(z)) \to +\infty$ for z in the right half-plane.*

Example 2.7.4 *Let $f(z) = \frac{1}{z} - e^z$. We have*

$$f^2(z) = \frac{z}{1 - z\exp(-z)} - \exp\left(\frac{1}{z} - e^z\right) \sim z + z^2 + \cdots$$

as $z \to 0$ in $W(\varepsilon) = \{z : \frac{3}{4}\pi < \arg z < \frac{5}{4}\pi, |z| < \varepsilon\}$. There is a component U of $F(f)$, which contains $(-\varepsilon, 0)$ for small positive ε. Further, $f^2(U) \subset U, f^{2n} \to 0$ in U. Obviously f^2 is undefined at 0.

Theorem 2.7.5 *[20] If $f \in \mathcal{R}$, then every Fatou component of f is eventually periodic. This implies that f has no wandering domains.*

For any component U of a Fatou set $F(f)$, it is easy to see that $f(U)$ is contained in some component V of $F(f)$. If f is rational, we have $f(U) = V$ (see A. F. Beardon [20]). However, if f is transcendental, it is possible that $f(U) \neq V$. Bergweiler-Rohde and Herring independently proved that for any entire function f, $V \setminus f(U)$ contains at most one point which is an asymptotic value of f. Concerning this question, Hua-Yang proved that for $f \in \mathcal{P}$, if U and V are two Fatou components such that $f(U) \subset V$, then $V = f(U)$.

2.7.2 Singular points

Singular points play an important role in the study of dynamics of meromorphic functions. Let f be a meromorphic function. A point $a \in \mathbb{C}$ is said to be *a non-singular point*(of the inverse function f^{-1}) if it has a neighborhood V such that $f : f^{-1}(V) \to V$ is an unbranched cover. The set of singular points is denoted by *sing*(f^{-1}). *Singular points*(or singularities) are of the following types (see R. Nevanlinna [64]):

1. a is *a critical value* (or an algebraic singularity), i.e., there exists $z_0 \in \mathbb{C}$ such that $f(z_0) = a, f'(z_0) = 0$. Such a point z_0 is called *a critical point* of f. We denote by $CV(f)$ all these values;

2. a is *an asymptotic value*(or transcendental singularity), i.e., there exists a curve Γ going to ∞ such that $f(z) \to a$ as $z \to \infty$ along Γ. All these values are denoted by $AV(f)$. In particular, if a has a simply connected neighborhood V such that for some component U of the set $f^{-1}(V)$ the mapping $f : U \to V\backslash\{a\}$ is an universal covering, then a is called *a logarithmic branch point*, and U is called *an exponential tract.* In addition, if there exists a neighborhood V_0 and a component U_0 of $f^{-1}(V_0)$ such that $f(z) \neq a$ for $z \in U_0$, then a is called *direct*, otherwise it is *indirect.* Obviously, a logarithmic branch point is direct;

3. limit points of types 1 and 2.

For any transcendental meromorphic function f, we have

$$sing(f^{-1}) \neq \emptyset.$$

In fact, for any transcendental meromorphic function f, we let $\Im$ be the Riemann surface defined by the inverse $z = f^{-1}(w)$. $z = f^{-1}(w)$ is single-valued function and maps $\Im$ conformally onto $\mathbb{C}$. By Iversen's Theorem, the asymptotic values of $f(z)$ correspond to the boundary of $\Im$ and vice versa. A corollary of Iversen's Theorem is that any Picard value of a meromorphic function is an asymptotic value of the function. Based on singularities, for the family of meromorphic functions, we introduce two subclasses $\mathcal{S}$ and $\mathcal{B}$.

$$\mathcal{S} = \{f : f \text{ has only finitely many critical and asymptotic values.}\}$$

According to A. E. Eremenko and Y. L*ü* Lyubich, the letter $\mathcal{S}$ was choosen in honor of Speiser, who introduced this class.

$$\mathcal{B} = \{f : sing(f^{-1}) \text{ is bounded}\}.$$

We remark that $\mathcal{B}\backslash\mathcal{S} \neq \emptyset$. For example, let

$$g_a(z) = \pi^2 - a \sin\sqrt{z}/\sqrt{z},$$

where $\pi^2 < a < 2\pi^2$. Then all critical points of g_a are real and positive and are denoted by z_j, $0 < z_1 < z_2 < \cdots$. The critical values are denoted by $c_j = g_a(z_j)$. Obviously, $c_j \to \pi^2$ as $j \to \infty$. We also note that π^2 is the only asymptotic value of g_a. Hence $g_a \in \mathcal{B}\backslash\mathcal{S}$.

A meromorphic function f is said to be of *finite type* or *bounded type* if $f \in \mathcal{S}$ or $f \in \mathcal{B}$ respectively. Note that if $f \in S$, then the Fatou set $F(f)$ has no Baker and Wandering domains.

If f is of finite order, then we have the following best estimate which is so called Denjoy-Carleman-Ahlfors Theorem.

Lemma 2.7.6 (Denjoy-Carleman-Ahlfors Theorem) [64] *If the inverse function of a meromorphic function f has n direct singularities, $n \geq 2$, then*

$$\liminf_{r\to+\infty} \frac{T(r,f)}{r^{\frac{n}{2}}} > 0.$$

Consequently, the inverse function to a meromorphic function of finite order ρ has at most $\max\{2\rho, 1\}$ *direct singularities. Moreover, an entire function of finite order ρ has at most 2ρ finite asymptotic values.*

Example 2.7.7 *The entire function*

$$w(z) = \int_0^z \frac{\sin t^q}{t^q} dt \quad (q > 0 \text{ is an integer})$$

is of order q and has $2q$ finite asymptotic values

$$e^{\nu\pi i/q} \int_0^\infty \frac{\sin t^q}{t^q} dt, \quad \nu = 1, \ldots, q.$$

Thus the above theorem is the best possible.

Concerning the indirect singularity, W. Bergweiler and A. E. Eremenko [24] proved the following result.

Lemma 2.7.8 *[24] For a meromorphic function f of finite order, every indirect singularity of f^{-1} is a limit of critical values.*

In 1959, W. K. Hayman conjectured that for any transcendental meromorphic function f and any $n \geq 1, f^n f'$ takes every nonzero and finite value infinitely often. The final solving of this conjecture is mainly based on the above lemma. Now we consider relations between $f^n(sing(f^{-1}))$ and $sing(f^{-n})$. Put

$$E(f) = \bigcup_{n \geq 0} f^n(sing(f^{-1})).$$

A point $a \in E(f)$ if and only if $a \in sing(f^{-n})$ for some positive integer n.

The post-singular set of f is defined to be the closure of the forward orbit of singular points:

$$\overline{E} := \overline{E(f)} = E(f) \cup E'(f),$$

where $E'(f)$ is the derived set of $E(f)$.

Theorem 2.7.9 *[48] A cycle of a super-attracting domain contains at least one critical value (and critical point). A cycle of an attracting domain or a parabolic domain contains infinitely many singular values (and singular points). The boundary of the cycle of a Siegel disc and a Herman ring is contained in $\overline{E}$.*

For $f \in \mathcal{E}$, if z_0 is a Cremer point of f, then $z_0 \in E'(f)$.

2.7.3 Connected components

The following results is standard.

Proposition 2.7.10 *[20]* (i) *The closure of a connected set is connected.*
(ii) *A compact set K in $\overline{\mathbb{C}}$ is disconnected if and only if there exists a Jordan curve γ which separates K.*
(iii) *A domain D is simply connected $\Leftrightarrow \overline{\mathbb{C}} \setminus D$ is connected $\Leftrightarrow$ the boundary ∂D is connected $\Leftrightarrow$ each component of D is simply connected.*

Theorem 2.7.11 *[48] Let $f \in \mathcal{E}$ satisfy one of the following statements*
(i) *f is bounded on some curve Γ going to ∞;*
(ii) *$F(f)$ has an unbounded component,*
then all Fatou components are simply connected.

Theorem 2.7.12 *[48] Let $f \in \mathcal{E}$ and all Fatou components of f is bounded. Then $J(f)$ is connected if and only if $J(f) \cup \{\infty\}$ is connected in $\overline{\mathbb{C}}$.*

Theorem 2.7.13 *[20] Let $P(z)$ be a nonlinear polynomial. Then*
(i) *$\infty \in F(P)$ and F_∞ is completely invariant, where F_∞ is a component of $F(P)$ which contains ∞;*
(ii) *unbounded component of $F(P)$ is either simply connected or infinitely connected; and*
(iii) *each bounded component of $F(P)$ is simply connected.*

2.7.4 Completely invariant components

In this subsection, we will study the completely invariant components of the Fatou set. Recall that a set U is completely invariant with respect to f if $z \in U$ if and only if $f(z) \in U$.

Theorem 2.7.14 *[20] Let $f \in \mathcal{M}$. If U is a completely invariant component of $F(f)$, then*
(i) *U is unbounded;*
(ii) *all components of $F(f)$ are simply connected when $f \in \mathcal{E}$;*
(iii) *$\partial U = J(f)$.*

For any meromorphic function f, if the Fatou set $F(f)$ has two or more completely invariant components $U_0, U_1, \ldots$, then each component is simply connected. For rational functions, there are at most two completely invariant components. But for transcendental entire functions, the case is different. I. N. Baker showed that if $f \in \mathcal{E}$, then $f(z)$ has at most one completely invariant component and if $f \in \mathcal{M} \cap \mathcal{S}$, $F(f)$ has at most two completely invariant components.

Now we will discuss the number of the Fatou components. For $f \in \mathcal{E}$, the number of the Fatou components is either 0, 1 or ∞, and the number of multiply connected Fatou components is either 0 or ∞ and for $f \in \mathcal{R}$, the number of the Fatou component is either 0, 1, 2 or ∞. A natural problem is when the Fatou set has only one component and when it has infinitely many components. Wang-Hua proved the following result.

Theorem 2.7.15 *[48] Let $f \in \mathcal{E}$ and $z \in J(f)$. If V is an open neighborhood of z, then*

1. *$F(f)$ has only one component if and only if V meets one component of $F(f)$;*
2. *$F(f)$ has infinitely many components if and only if V meets infinitely many components of $F(f)$.*

For $f \in \mathcal{R}$, we have the following result.

Theorem 2.7.16 *[20] Let $f \in \mathcal{R}$ and F_0 be a completely invariant component of $F(f)$. Then*

(i) *$\partial F_0 = J(f)$;*

(ii) *F_0 is either simply connected or infinitely connected;*

(iii) *all other components of $F(f)$ are simply connected; and*

(iv) *F_0 is simply connected if and only if $J(f)$ is connected.*

2.7.5 Limit functions

A function $\phi(z)$ is a *limit function* of $\{f^n\}$ on a Fatou component U if there is some subsequence of $\{f^n\}$ that converge locally uniformly to ϕ on U. We denote by $\mathcal{L}(U)$ all such functions.

For any Fatou component U, $\mathcal{L}(U)$ does not contain any repelling fixed point of f. If ζ is a constant limit function then either ζ is a fixed point of f or $\zeta = \infty$. For any nonconstant entire function, all constant limit functions of f^n in (pre-)periodic components are in $E'(f) \cup \{\infty\}$, except possibly in (pre-images of) super-attracting components.

Theorem 2.7.17 *[48] Suppose that U is a forward invariant Fatou component and every limit function in $\mathcal{L}(U)$ is constant. Then $\mathcal{L}(U)$ contains exactly one element, with value b, say, and $f^n(z)$ converges to b locally and uniformly in U. This implies that U is of the type* (1), (2) *and* (3) *in theorem 2.7.1.*

I. N. Baker [5] further proved the following result on the location of constant limit functions.

Theorem 2.7.18 *[5] For $f \in \mathcal{E} \cup \mathcal{P}$, any constant limit function of a sequence $f^{n_k}(z)$ in a component of $F(f)$ belongs to $L = \overline{E}(f) \cup A$, where $A = \{\infty\}$ or $\{0, \infty\}$ provided that f is entire or $f \in \mathcal{P}$ respectively.*

For some special Fatou component, we have better results.

Theorem 2.7.19 *[48] Let $f \in \mathcal{E} \cup \mathcal{P}$, and let U be a wandering domain of f. If $f \in \mathcal{E}$, then all limit functions of $\{f^n\}$ in U are constants and are contained in $(E'(f) \cap J(f)) \cup \{\infty\}$; If $f \in \mathcal{P}$ and U is a bounded annulus, then the only possible limit functions of $\{f^n\}$ in U are constants 0 and ∞.*

Theorem 2.7.20 *[48] Suppose that U is a forward invariant Fatou component and every limit function in $\mathcal{L}(U)$ is constant. Let b be the unique limit function in Theorem 2.7.17. Then exactly one of the following holds:*
(i) *$b = \infty$;*
(ii) *b is an attracting fixed point of f and $b \in U$;*
(iii) *b is a rationally indifferent fixed point of f and $b \in \partial U$.*

Next we give some results about nonconstant limit functions, see [48] for more details..

Theorem 2.7.21 *[48] Suppose that U is a forward invariant Fatou component and $\mathcal{L}(U)$ contains some non-constant limit functions. Then*
(i) *f is conformal in U;*
(ii) *the identity map I is in $\mathcal{L}(U)$;*
(iii) *any non-constant limit function is conformal in U;*
(iv) *$\mathcal{L}(U)$ does not contain any constant limit function.*
Moreover U is either a Siegel disc or Herman ring.

From the prove of the above theorem we immediately obtain the following result.

Theorem 2.7.22 *[48] Let $f \in \mathcal{M}$. If in a component D of $F(f)$, some subsequence of $\{f^n\}$ has a non-constant limit function, then there is a component D_1 of $F(f)$*

and a positive integer p such that $f^p(z)$ maps D_1 univalently onto D_1 and for some increasing sequence of integers n_k one has $f^{pn_k}(z) \to z$ in D_1. Moreover $f^N(D) \subset D_1$ for some N.

I. N. Baker showed that for any $f \in \mathcal{E} \cup \mathcal{P}$, if $\overline{E}(f)$ has an empty interior and a connected complement, then no sequence $\{f^{n_k}\}$ has a non-constant limit function in any component of $F(f)$.

2.8 The functions in class M

The iteration of meromorphic functions at once leads us out of a closed system: if we iterate meromorphic f in $\mathbb{C}$ we are also iterating f^2, which is not in general meromorphic, so there must be a more general version of the theory. This suggests trying to extend the classical theory of Julia and Fatou as far as possible while retaining its basic results. An early step in this direction was taken by H. Radström [74] who also pointed out that the Fatou - Julia approach is appropriate for analytic functions of the punctured plane $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ to itself. The case of the punctured plane has been developed by various authors, for example in ([11], [28], [52], [53], [55], [56], [59]).

It is of more than purely aesthetic interest that the class of functions to be studied in an extended theory should be closed under iteration or even arbitrary functional composition. One reason is that in the classical theory it is often very convenient in proofs to replace a function by its iterate of some higher order. For example points of period p of f can be discussed as fixed points of f^p . Also f and f^p have the same Fatou and Julia sets.

The smaller class which obviously includes the meromorphic functions with one essential singularity and which is closed under composition in the class **K**= $\{f$: there is a compact countable set $E(f) \subset \overline{\mathbb{C}}$ such that f is meromorphic in $\overline{\mathbb{C}}\backslash E(f)$ but in no larger set$\}$. This class has been studied by Bolsch ([29], [30]), independently, broader generalizations have been given in [47].

In [16], I. N. Baker, P. Domínguez and M. E. Herring gave a simplified presentation of some parts of [47]. They introduced the class **M**=$\{f$: there is a compact totally disconnected set $E = E(f)$ such that f is meromorphic in $E^c = \overline{\mathbb{C}} \backslash E$ and the cluster set of f at any point $z_0 \in E$ with respect to E^c, that is, the set $C(f, E^c, z_0)= \{w \in \overline{\mathbb{C}} : w = \lim_{n\to\infty} f(z_n)$ for some sequence $z_n \in E^c$ with $z_n \to z_0\}$ is equal to $\overline{\mathbb{C}}$. If $E = \emptyset$ we make the further assumption that f is neither constant nor univalent in $\overline{\mathbb{C}}\}$.

It was proved that the class **M** is closed under functional composition and for any $f, g \in \mathbf{M}, E(f \circ g) = E(g) \cup g^{-1}(E(f))$. The composition of a finite number of meromorphic functions is a member in the class **M** and has only at most countably many essential singularities.

For $f \in \mathbf{M}$, the set $Sing(f^{-1})$ of singular values of some branch of f^{-1} consists of *the critical values* $f(c)$, where $f'(c) = 0$, together with the set of all asymptotic values of f: w is *an asymptotic value* of f if there is some $z_0 \in E(f)$ and the path $\gamma(t), 0 \leq t < 1$, in E^c such that $\gamma(t) \to z_0$ and $f(\gamma(t)) \to w$ as $t \to 1$. Further, $E_j(f) = \cup_{k=0}^{j-1} f^{-k}(E(f))$ is the set of essential singularities of f^j. We use the following notations about singularities of the inverse function. For $f \in \mathbf{M}$, set

$$S_p(f) = \{a \in \mathbb{C} : a \text{ is a finite singularity of } f^{-p}\}. \tag{2.1}$$

and

$$P(f) = \cup_{p=1}^{+\infty} S_p(f). \tag{2.2}$$

That is, $P(f)$ is the set where some branch of f^{-n} has singularity for some $n \in \mathbb{N}$, or $P(f) = \cup_{j=0}^{\infty} f^j(Sing(f^{-1}) \backslash E_j(f))$, where $E_0(f) = \emptyset$. Thus $P(f)$ consists of the forward orbit of $Sing(f^{-1})$, so far as this is defined. For a set A, the derived set of A is denoted by A'.

2.8.1 The Fatou and Julia sets

For any $f \in \mathbf{M}$, we may define f^0 to be the identity function with $E_0 = \emptyset$, and, inductively, $f^1 = f, f^n = f \circ f^{n-1}$ for $n \geq 2$. We obtain $f^n \in \mathbf{M}$, for all $n \in \mathbb{N}$, with $E_n = E(f^n) = \cup_{j=0}^{n-1} f^{-j}(E)= \{$singularities of $f^n\}$. Clearly if we set

$$J_1(f) = \overline{\{\cup_{n=0}^{+\infty} E_n\}} \tag{2.3}$$

and

$$F_1(f) = \overline{\mathbb{C}} \backslash J_1(f), \tag{2.4}$$

then $F_1(f)$ is the largest open set in which all f^n are defined. Further, $f(F_1) \subset F_1$.

As in [16], for $f \in \mathbf{M}$, we define *the Fatou set* of f, denoted by $F(f)$, to be the largest open set in which (i) all iterates f^n are meromorphic and (ii) the family $\{f^n\}$ is a normal family; and *the Julia set* of f, denoted by $J(f)$, is defined to be the complement of $F(f)$, that is, $J(f) = \overline{\mathbb{C}} \backslash F(f)$. If the set $J_1(f)$ is either empty or contains one point or two points, then f is conjugate to a rational function or an entire function or a holomorphic function of the punctured plane $\mathbb{C}^*$, respectively. In these cases the condition (i) is trivial and the Fatou sets are determined by (ii). In all other cases, by Montel's theorem, we have $F(f) = F_1(f)$ and $J(f) = J_1(f)$. It is easy to see that for $f \in \mathbf{M}$, $F(f)$ is open and completely invariant. Let U be a connected component of $F(f)$, then $f^n(U)$ is contained in a component U_n of $F(f)$. If for some $n \in \mathbb{N}$, $U_n = U$, namely $f^n(U) \subset U$, then U is said to be *periodic*. If for some pair of $m \neq n$, $U_m = U_n$, but U is not periodic, then U is said to be *preperiodic*. If whenever $m \neq n$, $U_m \neq U_n$, then U is called *a wandering domain* of f. For a periodic component of $F(f)$ we have the following classification theorem [16]:

Theorem 2.8.1 *[16] Let U be a periodic component of the Fatou set of period p. Then precisely one of the following is true:*

(i) U is a (super)attracting domain of a (super)attracting periodic point a of f of period p such that $f^{np}|_U \to a$ as $n \to +\infty$ and $a \in U$.

(ii) U is a parabolic domain of a rational neutral periodic point b of f of period p such that $f^{np}|_U \to b$ as $n \to +\infty$ and $b \in \partial U$.

(iii) U is a Siegel disc of period p such that there exists an analytic homeomorphism $\phi : U \to \triangle$, where $\triangle = \{z : |z| < 1\}$, satisfying $\phi\left(f^p\left(\phi^{-1}(z)\right)\right) = e^{2\pi\alpha i} z$ for some irrational number α and $\phi^{-1}(0) \in U$ is an irrational neutral periodic point of f of period p.

(iv) U is a Herman ring of period p such that there exists an analytic homeomorphism $\phi : U \to A$, where $A = \{z : 1 < |z| < r\}$, satisfying $\phi\left(f^p\left(\phi^{-1}(z)\right)\right) = e^{2\pi\alpha i} z$

for some irrational number α.

(v) U is a Baker domain of period p such that $f^{np}|_U \to c \in J(f)$ as $n \to +\infty$ but f^p is not meromorphic at c. If $p = 1$, then $c \in E(f)$.

2.8.2 Subclasses of the class M

There are several subclasses of the class **M** which are introduced in [16] including those studied by Bolsch in [29] and [30]. To suit our purpose, we introduce some subclasses and their dynamical properties as follows.

Definition 2.8.2 *Let $f \in \mathbf{M}$. Then*

(i) *f is in class **K** if there is a compact countable set $E(f) \subset \overline{\mathbb{C}}$ such that f is meromorphic in $\overline{\mathbb{C}} \setminus E(f)$ but in no larger set.*

(ii) *f is in class $\mathbf{MP}_k$, where k is an integer not less than two, if $E(f) \neq \emptyset$ and for each $z_0 \in E(f)$ and open set U which contains z_0, f takes in $U \setminus E(f)$ every value in $\overline{\mathbb{C}}$ with at most k exceptions.*

(iii) *f is in class $\mathbf{MA}_k$, where $k \in \mathbb{N}$, if $E(f) \neq \emptyset$ and for each $z_0 \in E(f)$ the function f has the k islands property at z_0, namely for any neighborhood U of z_0 and k simply-connected domains $\triangle_i$ in $\overline{\mathbb{C}}$ which have disjoint closures and which are bounded by sectionally analytic Jordan curves, there is a simply-connected subdomain D in $U \setminus E(f)$ such that f maps D univalently onto one of the $\triangle_i$.*

(iv) *f is in class **MA** if f is in class $\mathbf{MA}_k$ for some $k \in \mathbb{N}$, that is, **MA** is a union of all $\mathbf{MA}_k$, $k \in \mathbb{N}$.*

(v) *f is in class **MS** if the set of singular values of f^{-1} is finite.*

(vi) *f is in class **MSR** if $f \in \mathbf{MS}$ and the complement of $E(f)$ is of class O_{AD} (If W is a domain in the plane and F is a function analytic in W, the Dirichlet integral of F is defined by $D_W(F) = \int\int_W |F'(z)|^2 dxdy$. An analytic function with finite Dirichlet integral is said to be of the class AD. The domain W is said to be of class O_{AD} if the only functions of class AD on W are constants).*

The followings results were established in [16]:

Theorem 2.8.3 *[16] Let $f \in \mathbf{M}$. Then the following statements are true.*

(i) $\mathbf{K} \subset \mathbf{MA}_k \subset \mathbf{MP}_{k-1}$, $\mathbf{K} \subset \mathbf{MP}_2 \cap \mathbf{MA}_5$, $\mathbf{K} \cap \mathbf{M} \subset \mathbf{MSR}$.

(ii) *The subclasses* $\mathbf{K}$*,* $\mathbf{MP}_k$*,* $\mathbf{MA}_k$*, and* $\mathbf{MS}$ *are closed under composition.*

(iii) *If* $f \in \mathbf{MS}$*, then* f *has no Baker domains.*

(iv) *If* $f \in \mathbf{MSR}$*, then* f *has no wandering domains.*

We do not assert that **MSR** is closed under composition. A subset which is closed under composition is $\mathbf{MS}_0 = \{\ f : f \in \mathbf{MS}$ and $E(f)$ has capacity zero$\}$, since $f, g \in \mathbf{MS}_0$ implied $\mathrm{Cap}\{E(g) \cup g^{-1}(E(f))\}$=0, see, for example ([75], p. 69). The most immediate application, for which we have many examples is for f in the class **K**. Clearly $\mathbf{MS} \cap \mathbf{K} \subset \mathbf{MS}_0 \subset \mathbf{MSR}$.

We noted a result on the connectivity of invariant and periodic components in the case when $E(f)$ is compact and countable, that is $f \in \mathbf{K}$. A. Bolsch [30] showed that if $f \in \mathbf{K}$ and U is a periodic component of $F(f)$, then the connectivity of U is 1, 2 or ∞.

Next we will show you some dynamics of some subclasses of the class **M**. I. N. Baker and A. P. Singh [15] proved that if $f \in \mathbf{MA}_k$ and $F(f)$ has a component H of connectivity at least k, then singleton components are dense in $J(f)$.

I. N. Baker , P. Domínguez and M. E. Herring [17] studied the completely invariant domains in the Fatou set for the subclass **MS** of the class **M**. They obtains two main results.

Theorem 2.8.4 *[17] Suppose that* $f \in \mathbf{MS}$ *and* $F(f)$ *has a simply-connected completely invariant domain* U_0*. If* w_0 *is an isolated point of* $E(f)$*, then* w_0 *is accessible in* U_0*.*

Theorem 2.8.5 *[17] Suppose that* $f \in \mathbf{MS}$*. If* $E(f)$ *has an isolated point, then* f *has at most two completely invariant domains.*

Note that a boundary point z of D is said to be *accessible* from the interior of D if one can find a simple Jordan line where all points but z are interiors in D and so that it links z to any point a inside D; the line may consist of finite number of segments of straight lines or of an infinitely of segments having z as limit point.

2.8.3 Standard properties of Julia sets

Next, we collect the main properties of $F(f)$ and $J(f)$ of the function f in class **M**. First we introduce *the backward orbit* $O^-(w)$ of a point $w \in \overline{\mathbb{C}}$, defined by $\{z\colon f^n(z) = w$ for some $n \in \mathbb{N}\}$ and *the α- limit set $\alpha(w)$ of w* which is the set of the limit points of $O^-(w)$. Recall that a generic property of points in a complete metric space is one which holds for all points outside a set of first category.

Theorem 2.8.6 *[16] Suppose that $f \in \mathbf{M}$ as defined above. Then*
(1) $F(f)$ is completely invariant in the sense that $z \in F(f)$ if and only if $f(z) \in F(f)$. Thus $z \in J(f)\backslash E$ if and only if $f(z) \in J(f)$;
(2) for every positive integer p, $F(f^p) = F(f)$ and $J(f^p) = J(f)$;
(3) if ψ is a Möbius transformation and $f_\psi = \psi \circ f \circ \psi^{-1}$, then $f_\psi \in \mathbf{M}, F(f_\psi) = \psi(F(f))$ and $J(f_\psi) = \psi(J(f))$;
(4) $J(f)$ is perfect;
(5) for a generic point $w \in \overline{\mathbb{C}}$, the set $\alpha(w)$ contains $J(f)$. If $f \in \mathbf{MP}_k$, then the exceptional set $X(f)$= $\{z : O^-(z)$ is a finite set $\}$ contains at most k elements and for w not in $X(f)$, $\alpha(w)$ contains $J(f)$, while if $w \in J(f)\backslash X(f)$ we have $\overline{O^-(w)} = J(f)$;
(6) if $J(f)$ has a non-empty interior, then $F(f) = \emptyset$;
(7) if $f \in \mathbf{MA}_k$ for some $k(\geq 5) \in \mathbb{N}$, then repelling periodic points are dense in $J(f)$;
(8) If $E(f)$ has the local Picard property, namely there exist no open set V with $E \cap V \neq \emptyset$ and no function f meromorphic in $V \backslash E(f)$ with an essential singularity at each point of $E \cap V$ such that f omits three values in $V \setminus E(f)$, then every point of $J(f)$ is a limit point of periodic points of f. (We do not assert that the periodic points are repelling.)

2.8.4 The Fatou components and singularities

I. N. Baker [14] studied the connections between the Fatou components and the singularities of the inverse functions. He showed that for $f \in \mathbf{MA}$, if U is a wandering domain of $F(f)$, then any limit function of a sequence of iterates in

U is a constant which lies in $(P(f))'$, and if U is a component of $F(f)$ such that $f^n \to a$ in U, where $a \in \overline{\mathbb{C}}$, the either (i) a is an attracting fixed point of f, (ii) a is a parabolic fixed point of f, or (iii) $a \in E(f) \cap Sing(f^{-1})$.

J. H. Zheng [85] also discuss this subject and obtained many results. First, we set $J_\infty(f) = \cup_{n=0}^{+\infty} E(f^n)$. If $J_\infty(f)$ has at least three points, then $J(f) = \overline{J_\infty(f)}$, so $F(f)$ is the largest open set in which all f^n, $n \in \mathbb{N}$ are meromorphic. If $J_\infty(f)$ consists of two points, then f is a holomorphic function of $\mathbb{C}^*$ onto itself up to a Mӧbius transformation. If $J_\infty(f)$ consists of one point, then f is a transcendental entire function. If $J_\infty(f)$ is empty, then f is a rational function. He showed that for $f \in \mathbf{M}$ and if U is a wandering domain of f , then every limit function of convergent subsequence of $\{f^n|_U\}$ lies in the derived set of $P(f)$. This result was proved in [27] for f being entire and in [82] and [84] deduced that for $f \in \mathbf{M}$ and a component U of $F(f)$, if $\{f^{np}|_U\} \to q($ as $n \to +\infty)$, then either q lies in the derived set of $S_p(f)$ or is a periodic point of f of period $k \le p$ and $f^p(q) = q$. Moreover for $e \in J_\infty(f)$, if $e \notin (S_p(f))'$, then there exist no components of $F(f)$ in which $f^{np}(z) \to e$ as $n \to +\infty$, which is a generalization of Theorem F in [16].

Now we give some sufficient conditions for $f \in \mathbf{M}$ has no wandering domains and Baker domains.

Theorem 2.8.7 *[85] Let $f \in \mathbf{M}$. If $(\mathrm{Sing}(f^{-1}))' \cap E(f) = \emptyset$, $J(f) \cap (P(f))'$ is finite and $(P(f))' \cap J_\infty(f) \backslash E(f) = \emptyset$, then f has no wandering domains.*

Theorem 2.8.8 *[85] Let $f \in \mathbf{M}$. Then f has no Baker domains of period $k \le p$, if one of the following statements holds:*
(1) $f(z)$ has no asymptotic values which lie in $J_\infty(f,p) = \cup_{n=0}^{p-1} f^{-n}(E(f))$;
(2) $(S_p(f))' \cap E(f) = \emptyset$.

As an application of Theorem 2.8.7 and Theorem 2.8.8, J. H. Zheng gave a sufficient condition to determine the Julia set of transcendental meromorphic function equal to Riemann sphere.

Theorem 2.8.9 *[85] Let $f \in \mathbf{M}$ with $(\mathrm{Sing}(f^{-1}))' \cap E(f) = \emptyset$. Assume that $J(f) \cap (P(f))'$ is finite and $(P(f))' \cap J_\infty(f) \backslash E(f) = \emptyset$ and for every $b \in \mathrm{Sing}(f^{-1})$, b is pre-periodic or $b \in J_\infty(f)$ or $f^n(b) \to E(f)$ as $n \to \infty$. Then $J(f) = \overline{\mathbb{C}}$.*

From Theorem 2.8.9, he also showed that there exist μ and λ such that the Julia set of a transcendental meromorphic function $f(z) = \mu + z + e^z + \frac{\lambda}{e^z - 1}$ is the Riemann sphere. Note that such function is not of bounded type.

The method of A. E. Eremenko and M. Lü Lyubich ([40], pp. 993–994) gives the following results.

Theorem 2.8.10 *[40] If $f \in \mathbf{M}$ and if $e \in E(f)$ is not a limit point of singularities of f^{-1} then there is no invariant component of $F(f)$ in which $f^n \to e$.*

Corollary 2.8.11 *[40] If $f \in \mathbf{MSR}$, then every component of $F(f)$ is eventually periodic and the only periodic components are associated with attracting or parabolic periodic points or are part of a cycle of Siegel discs or Herman rings.*

I. N. Baker, P. Domínguez and M. E. Herring [16] studied the relation between singular orbits and Fatou domains. The results which hold for rational functions generalize with little change in the proof.

Lemma 2.8.12 *[16] If $f \in \mathbf{M}$ and if $g_{n(k)}$ are branches of the inverses of $f^{n(k)}$ which are meromorphic in the domain U, $k \in \mathbb{N}$, then $\{g_{n(k)}\}$ is normal in U.*

Lemma 2.8.13 *[16] (i) If $f \in \mathbf{M}$ and $G_1, \ldots, G_p$ is periodic cycle of Fatou components in which the iterates converge to either an attracting or parabolic periodic cycle of points then $G_1 \cup \ldots \cup G_p$contains the forward orbit of some singular point of f^{-1}.*

(ii) If $G_1, \ldots, G_p$ is a cycle of Seigel discs or Herman rings (i.e. each G_i is a Seigel disc or Herman ring of f^p), then each point of $\cup_i(\partial G_i)$ is a limit point in the orbit of singularities of f^{-1}.

They also studied dynamics of the subclasses **MSR** of the class **M**.

Theorem 2.8.14 *[16] Suppose that $f \in \mathbf{MSR}$ and suppose that there is an attracting fixed point whose Fatou component G contains all the singular points of f^{-1}. Then $J(f)$ is totally disconnected.*

2.8.5 Semiconjugation of functions in class M

J. H. Zheng [85] also discussed the connection between dynamics of two functions $f(z)$ and $g(z)$ in the class $\mathbf{M}$ satisfying the functional equation

$$h(f(z)) = g(h(z)) \tag{2.5}$$

where $h(z)$ is meromorphic in $\mathbb{C}$. He proved that for f, g and h which satisfy (2.5), if $J(f) = \overline{J_\infty(f)}$ and either $\infty \in E(f)$ or $f(\infty) \neq \infty$, then $h(J(f)) = J(g)$ and $h(F(f)) = F(g)$. Combining this result with Bergweiler's result [23], he obtain that for $f(z)$ and $g(z)$ in $\mathbf{M}$ with either $\infty \in E(f)$ or $f(\infty) \neq \infty$, if $\exp f(z) = g(e^z)$, then $\exp J(f) = J(g)$ and $\exp F(f) = F(g)$.

Two function $f(z)$ and $g(z)$ in class $\mathbf{M}$ are called *permutable* if $f \circ g = g \circ f$ in $\overline{\mathbb{C}} \backslash (E(f) \cup E(g) \cup g^{-1}(E(f)) \cup f^{-1}(E(g)))$.

As an immediate application of the relation between $J(f)$ and $F(f)$ which satisfy (2.5), for any two permutable transcendental meomorphic functions $f(z)$ and $g(z)$ in $\mathbb{C}$, we have $J(f) = J(g)$.

The following result is about a dynamic connection between Fatou components of f and g which satisfy (2.5).

Theorem 2.8.15 *[85] Let f, g and h be functions in class* $\mathbf{M}$ *such that (2.5) holds. If h maps any component of $F(f)$ onto a hyperbolic domain, then the following statement hold.*

(i) *If f has no wandering domains, then g has no wandering domains;*

(ii) *If U is a periodic component of $F(f)$, then $h(U)$ is contained in a periodic component of $F(g)$, and they are of the same type, unless U is a Baker domain or Herman ring. If U is a Herman ring, then the component V of $F(g)$ containing $h(U)$ must be a Siegel disc or Herman ring.*

The result (i) in Theorem 2.8.15 was proved in [25] for f and g being entire. From Theorem 2.8.15 and Theorem E in [16], if it of finite type, then the composition of two transcendental meromorphic functions has no wandering domains. If f and g are both meromorphic functions in $\mathbb{C}$ and f is of finite type such that $f \circ g$ is of finite type, then $g \circ f$ has no wandering domains, for g must map any component of $F(f \circ g)$ onto a hyperbolic domain.

To complete this subsection, we will introduce the concepts of orbits which approach a singular point.

For transcendental entire functions f, A. E. Eremenko [37] studied the set

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}. \tag{2.6}$$

A. E. Eremenko also proved that for such functions (a) $I(f) \neq \emptyset$ and indeed (b) $J(f) = \partial I(f)$ and (c) $J(f) \cap I(f) \neq \emptyset$. It has been observed that if f is a function of $\mathbb{C}^*$ to itself with essential singularities at 0 and ∞ then the same results hold, and by symmetry we could replace ∞ by 0 in the definition of $I(f)$.

P. Dom*í*nguez [36] showed that Eremenko's results can applied to transcendental meromorphic functions if one replaces the definition (2.6) by

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty \text{ and } f^n(z) \neq \infty\}. \tag{2.7}$$

Consider now for $f \in \mathbf{M}$ and for $e \in E(f)$, we define
$I(f, e)$=$\{z \in \mathbb{C} :\ f^n(z)$ is defined for all n and $f^n(z) \to \infty$ as $n \to \infty\}$.

Theorem 2.8.16 *[16] If for some $k \in \mathbb{N}$ we have $f \in \mathbf{MP}_k$, then for $e \in E(f)$ we have:*

(1) $I(f, e) \neq \emptyset$;

(2) $J(f) = \partial I(f, e)$;

(3) $J(f) \cap I(f, e) \neq \emptyset$.