Chapter 3

Solutions of Functional Equation

$f \circ S = S^k \circ f$

3.1 Some known results

Let f be a C^2 function on $\mathbb{C}$. *Newton's method, Halley's method* and *the Schwarzian derivative* of f are defined respectively as follows

$$\begin{aligned} N_f(z) &= z - \frac{f(z)}{f'(z)} \\ H_f(z) &= z - \frac{f(z)}{f'(z) - \frac{f(z)f''(z)}{2f'(z)}} \\ S(z) &= 2\frac{f'''(z)}{f'(z)} - 3\left(\frac{f''(z)}{f'(z)}\right)^2. \end{aligned}$$

Assume that f has a simple zero at ζ so that $f'(\zeta) \neq 0$. J. Palmore [69] and [70] studied the role of Schwarzian derivatives of N and H in controlling the order of convergence of N and H to ζ respectively. The main results in [69] and [70] are as follows:

Theorem 3.1.1 *[69] Let f be a differentiable function on $\mathbb{C}$. Let N and S be the Newton's function and the Schwarzian derivative of f. If f has a simple zero ζ such that $f'(\zeta) \neq 0$, then $N'''(\zeta) = S(\zeta)$. If $f''(\zeta) = 0$, then N has convergence to ζ of order 3 or greater. If $S(\zeta) = 0$, then N has convergence to ζ of order 4 or greater.*

Theorem 3.1.2 *[70] Let f be a differentiable function on $\mathbb{C}$. Let H and S be the Halley's function and the Schwarzian derivative of f. If f has a simple zero ζ such that $f'(\zeta) \neq 0$, then $H'''(\zeta) = -2\left(\frac{S(\zeta)}{2}\right)$. H has convergence to ζ of order 4 or greater if and only if $S(\zeta) = 0$*

P. Niamsup, J. Palmore and Y. Lenbury [67] studied the role of Schwarzian derivatives of composition function between H and N, namely $H \circ N$ and $N \circ H$, in controlling the order of the simple root ζ of f. They obtained the following results.

Theorem 3.1.3 *Let f be a differentiable function on $\mathbb{C}$. Let H, N and S be the Halley's function, Newton's function and Schwarzian derivative of f, respectively. If f has a simple zero ζ such that $f'(\zeta) \neq 0$, then the values of the first five derivatives of $H \circ N$ at ζ are zero, and $(H \circ N)^{(6)}(\zeta) = -\left(\frac{15}{2}\right)\left(\frac{f''(\zeta)}{f'(\zeta)}\right) \cdot S(\zeta)$. Therefore, $H \circ N$ has order of convergence to ζ equal to 6 or greater. The order of $H \circ N$ is controlled by the second derivative and by the Schwarzian derivative of f evaluated at ζ.*

Theorem 3.1.4 *Let f be a differentiable function on $\mathbb{C}$. Let H, N and S be the Halley's function, Newton's function and Schwarzian derivative of f, respectively. If f has simple zero ζ such that $f'(\zeta) \neq 0$, then the values of the first five derivatives of $N \circ H$ evaluated at ζ are zero, and $(N \circ H)^{(6)}(\zeta) = 10\left(\frac{f''(\zeta)}{f'(\zeta)}\right)$, $(H'''(\zeta))^2 = -\left(\frac{15}{2}\right)\left(\frac{f''(\zeta)}{f'(\zeta)}\right)^3 \cdot S(\zeta)$. Therefore, $N \circ H$ has order of convergence to ζ equal to 6 or greater and the order of convergence is controlled by the second derivative and by the Schwarzian derivative of f evaluated at ζ.*

A successive approximation $S(z)$ of $f(z)$ may be obtained by setting $f(z) = 0$ and then write this equation as $z = S(z)$. For example, if $f(z)$ is a quadratic polynomial with roots a and b such that $0 < |a| < |b| < 1$, that is $f(z) = (z-a)(z-b)$, then $S(z) = \frac{-ab}{z-(a+b)}$ is a successive approximation of $f(z)$ having $z = a$ as a global attractor. In general, Halley's method, Newton's method and successive approximation are iterative methods which can be used to locate roots of functions where the order of convergence of these methods are three, two and one, respectively. P. Niamsup and J. Palmore ([65] and [66]) studied the roles of Schwarzian derivative of Halley's method, Newton's method and the composite between two methods in controlling the order of convergence of these methods. The following relations between Halley's method, Newton's method and successive approximation for $f(z) = (z-a)(z-b), a, b \in \mathbb{C}$ such that $0 < |a| < |b|$ were given in [65] and [68], where $S(z) = \frac{-ab}{z-(a+b)}$. The following are some of these relations:

(1) $H \circ S = S^3 \circ H$,

(2) $N \circ S = S^2 \circ N$,

(3) $(H \circ N) \circ S = S^6 \circ (H \circ N)$,

(4) $H^i(S^j(0)) = S^{(j+1)3^i-1}(0), i, j \geq 0$,

(5) $N^i(S^j(0)) = S^{(j+1)2^i-1}(0), i, j \geq 0$,

(6) $(H \circ N)^j(S^j(0)) = S^{(j+1)6^i-1}(0), i, j \geq 0$.

In [71] and [72], J. Palmore investigated a rational function of the following form:

$$f_k(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}.$$

It was shown that when a and b are quadratic irrational numbers of the form

$$a = \frac{u + v^{\frac{1}{2}}}{w} \text{ and } b = \frac{u - v^{\frac{1}{2}}}{w}$$

where u, v and w are integers such that $v > 0$, v is not the square of an integer and $w \neq 0$, then $f_k(z)$ is a rational function of integers u, v and w. This is important when we study a computable orbit converging to a under the iteration of f_k. It was also shown that

$$f_k^{(i)}(0) = S^{k^i-1}(0)$$

where $k \geq 2$ and $i \geq 1$. That is, the order of convergence of f_k to a is equal to k. Note that f_2 is the usual Newton's method for f and f_3 is the Halley's method for f.

P. Niamsup and J. Palmore [66] studied the functional equation

$$f \circ S = S^k \circ f \tag{3.8}$$

where $k \geq 2$ where $S(z) = \frac{-ab}{z-(a+b)}$ is a successive approximation of quadratic polynomial $P(z)$ with roots a and b in $\mathbb{C}$ such that $0 < |a| < |b|$, that is, $P(z) = (z-a)(z-b)$, and f is a rational function of degree k of the form

$$f_k(z) = \frac{a_k z^k + a_{k-1} z^{k-1} + \ldots + a_1 z + a_0}{b_k z^k + b_{k-1} z^{k-1} + \ldots + b_1 z + b_0}, \tag{3.9}$$

where $a_i, b_j \in \mathbb{C}$ $(i, j = 0, 1, 2, \ldots, k)$, $(a_0, b_0) \neq (0, 0)$.

They began by showing that (3.8) has a rational solution.

Theorem 3.1.5 *The functional equation (3.8) has a rational solution, namely*

$$f_k(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}.$$

From Theorem 3.1.5, they obtained the following result which is more general than the result in [68].

Corollary 3.1.6 *For $k \geq 2$, we have $f_k^{(i)}(S^j(-(ab/z_k-(a+b)))) = S^{(j+1)k^i-1}(-(ab/z_k-(a+b)))$ for $i, j \geq 0$, where z_k is a fixed point of f_k. In particular, for $b_k = 0$, f_k has a fixed point at ∞ and hence $b_k^{(i)}(S^j(0)) = S^{(j+1)k^i-1}(0)$ for $i, j \geq 0$.*

For all rational solutions of (3.8) when $k = 2$, they obtained the following result.

Theorem 3.1.7 *Let f_2 be a rational solution of (3.8), then f_2 is of the following form*

(a) *If $a_2 \neq 0$, then*

$$f_2(z) = \frac{z^2 + (-2abb_2)z + (-ab + ab(a+b)b_2)}{b_2z^2 + (2 - 2(a+b)b_2)z + (-abb_2 - (a+b) + (a+b)^2b_2)}$$

where b_2 is any complex number. Moreover, if $b_2 = 0$, then f_2 is the Newton's method for P and if b_2 is a nonzero complex number, then we obtain $f_2(z) = T_2(N(z))$ where $T_2(z) = (z - abb_2/b_2z + (1 + (a+b)b_2))$.

(b) *If $a_2 = 0$ and $a_1 \neq 0$, then*

$$f_2(z) = \frac{z - (a + b/2)}{(-(1/2ab))z^2 + ((a + b/ab))z - ((a^2 + ab + b^2/2ab))}.$$

Note that $f_2(z) = (S^{-1} \circ N \circ S)(z)$.

(c) *If $a_2 = a_1 = 0$ and $a_0 \neq 0$, then there are no rational solutions for (3.8) of this form.*

Conversely, if T is any mapping such that $T \circ S = S \circ T$, then $N \circ T$ and $T \circ N$ are solutions of (3.8).

For general positive integer k, they obtained the following main result.

Theorem 3.1.8 *Let f_k be a rational solution of (3.8) of the form (3.9).*
(a) *If $a_k = 1$, then*

$$f_k = T_k \circ f_{0,k}$$

where $f_{0,k}(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}$, $T_k(z) = \frac{z - abb_k}{b_k z + (1-(a+b)b_k)}$ *and* $b_k \in \mathbb{C}$.
(b) *If $a_k = 0$, and $a_{k-1} \neq 0$ then there is only one rational solution in this form for (3.8) and we can explicitly find such a solution.*
(c) *If $a_k = a_{k-1} = 0$, then there are no nonzero rational solutions for (3.8) of this form.*

Conversely, if T is any mapping such that $T \circ S = S \circ T$ then $f_0 \circ T$ and $T \circ f_0$ are solutions of (3.8).

Remark 3.1.9 (1) *When $k = 3$, $f_{0,3}(z)$ is the Halley's method for P.*
(2) *If $P(z)$ is a polynomial of degree 3 or more with distinct roots, then any successive approximation of $P(z)$ would have degree 2 or more. From which it follows that (3.8) does not hold (since degree of S is not equal to degree of S^k).*
(3) *From [71], [72] and Theorem 3.1.8, f_k is a rational function with integer coefficients if and only if $z_k \in \mathbb{Z}$, $a = (s(u - \sqrt{v})/w)$ and $b = (s(u + \sqrt{v})/w)$ where $u, v, w, s \in \mathbb{Z}\backslash\{0\}, u > 0, v > 0, v^2 \notin \mathbb{Z}^+$.*

In this thesis, we propose to study meromorphic solutions of the functional equation (3.8) and to study the Julia set of rational solutions of (3.8). Under some certain conditions, we propose to give the explicit form of f .

Moreover, we study meromorphic solutions f of the following functional equation

$$f \circ R = R^k \circ f,$$

where $k \geq 2$ and R is a Möbius transformation which has only one fixed point, say $a \in \mathbb{C}$ (so a is a global attractor of R).

3.2 Main results

Let S be a Möbius transformation which has two fixed points, say a and b in $\mathbb{C}$. Without loss of generality we may assume that a is an attracting fixed point and b

is a repelling fixed point of S. We are interested in finding meromorphic solutions f on $\mathbb{C}$ of the following functional equation

$$f \circ S = S^k \circ f \tag{3.10}$$

where $k \geq 2$.

In [66], the rational solutions f of (3.10) are solved directly from a linear system of equations. In this thesis, we study the functional equation (3.10) more analytically. We will show that for a given complex number α distinct from a and b, there exists a unique solution of (3.10) which fixes α, a, and b. We also show that the Julia sets of rational solutions of (3.10) are circles on the sphere.

Let S and f be as above. We have

Theorem 3.2.1 *For any* $i, j \in \mathbb{N}$,

$$f^i \circ S^j = S^{jk^i} \circ f^i. \tag{3.11}$$

Proof. Fix $i = 1$ and let $P(j)$ be $f \circ S^j = S^{jk} \circ f$. Then for $j = 2$,

$$\begin{aligned}(f \circ S) \circ S &= (S^k \circ f) \circ S \\ &= S^k \circ (f \circ S) \\ &= S^k \circ (S^k \circ f) \\ &= S^{2k} \circ f.\end{aligned}$$

This implies $P(2)$ holds. Assume that $P(n)$ holds. Then

$$\begin{aligned}f \circ S^{n+1} &= (f \circ S^n) \circ S \\ &= (S^{nk} \circ f) \circ S \\ &= S^{nk} \circ (f \circ S) \\ &= S^{nk} \circ (S^k \circ f) \\ &= S^{(n+1)k} \circ f\end{aligned}$$

which implies that $P(n+1)$ holds. Therefore $f \circ S^j = S^{jk} \circ f$ holds for all $j \in \mathbb{N}$. Similarly for a fixed $j \in \mathbb{N}$, let $Q(i)$ be $f^i \circ S^j = S^{jk^i} \circ f^i$.

Then for $i = 2$,

$$\begin{aligned} f^2 \circ S^j &= f \circ (f \circ S^j) \\ &= f \circ (S^{jk} \circ f) \\ &= (f \circ S^{jk}) \circ f \\ &= (S^{(jk)k} \circ f) \circ f \\ &= S^{jk^2} \circ f^2. \end{aligned}$$

This implies that $Q(2)$ holds. Assume that $Q(n)$ holds. Then

$$\begin{aligned} f^{(n+1)} \circ S^j &= f \circ (f^n \circ S^j) \\ &= f \circ (S^{jk^n} \circ f^n) \\ &= (f \circ S^{jk^n}) \circ f^n \\ &= (S^{jk^{(n+1)}} \circ f) \circ f^n \\ &= S^{jk^{(n+1)}} \circ f^{(n+1)} \end{aligned}$$

which implies that $Q(n+1)$ holds. Therefore $f^i \circ S^j = S^{jk^i} \circ f^i$ holds for all $i \in \mathbb{N}$. We conclude that $f^i \circ S^j = S^{jk^i} \circ f^i$ for all $i, j \in \mathbb{N}$. This completes the proof.

Theorem 3.2.2 *Let f be a solution of (3.10). If $f(b) \neq a$, then a and b are fixed points of f.*

Proof. Firstly, we show that a, b are not poles of f. For if a was a pole of f, then $f(a) = \infty$. From (3.11) and for $i = 1$ we have

$$\infty = f(a) = f \circ S^j(a) = S^{jk}(f(a)) = S^{jk}(\infty).$$

This implies that $a = \infty$ or $b = \infty$ which is a contradiction. Thus a and b are not poles of f. From (3.11) if we take $i = 1$, then for $z \notin f^{-1}(b) \cup \{a, b\}$ we have, by continuity of f,

$$f\left(S^j(z)\right) = S^{jk}\left(f(z)\right).$$

Thus

$$\begin{aligned} f(a) &= f\left(\lim_{j\to+\infty} S^j(z)\right) = \lim_{j\to+\infty} f\left(S^j(z)\right) \\ &= \lim_{j\to+\infty} S^{jk}\left(f(z)\right) = a \end{aligned}$$

which implies that a is a fixed point of f. From (3.11) if we take $z = b$, then

$$f(b) = S^{jk}(f(b)).$$

As we assume that $f(b) \neq a$ we conclude that $f(b) = b$. This completes the proof.

Remark 3.2.3 *Let f be a solution of (3.10) such that $f(b) \neq a$. Then a, b are super-attracting fixed points of f.*

Proof. Consider

$$f \circ S(z) = S^k \circ f(z),$$

by differentiating both sides we obtain

$$f'(S(z))S'(z) = S'(S^{k-1} \circ f(z)) \cdot S'(S^{k-2} \circ f(z)) \cdot \ldots \cdot S'(f(z)) \cdot f'(z).$$

For $z = a$,

$$f'(a) \cdot S'(a) = [S'(a)]^k \cdot f'(a)$$

and since $S'(a) \neq 0$, we conclude that $f'(a) = 0$. That is, a is a super-attracting fixed point of f.

Similarly, for $z = b$,

$$f'(b) \cdot S'(b) = [S'(b)]^k \cdot f'(b)$$

and since $S'(b) \neq 0$, we conclude that $f'(b) = 0$. That is, b is a super-attracting fixed point of f. This completes the proof.

Theorem 3.2.4 *For a given complex number α distinct from a and b, there exists a unique solution of (3.10) which fixes α, a and b.*

Proof. Let f and g be solutions of (3.10) which fix a, b and α. From (3.11), take $i = 1$ we have

$$f \circ S^j(\alpha) = S^{jk} \circ f(\alpha) = S^{jk}(\alpha)$$

and

$$g \circ S^j(\alpha) = S^{jk} \circ g(\alpha) = S^{jk}(\alpha).$$

Since $\alpha \neq a, b$ and a is a global attractor of S, $S^j(\alpha) \to a$ as $j \to \infty$. This implies that a is a limit point of $\{S^j(\alpha) : j \in \mathbb{N}\}$. As

$$\{S^j(\alpha) : j \in \mathbb{N}\} \subseteq \{z \in \overline{\mathbb{C}} : f(z) = g(z)\},$$

we have, by the Identity Theorem (see [35]), $f \equiv g$ on $\overline{\mathbb{C}}$. Therefore, there is a unique solution of (3.10) which fixes α, a and b where α is a complex number distinct from a and b. This completes the proof.

Remark 3.2.5 *Let f be a solution of (3.10) which fixes a, b and α where α is a complex number distinct from a and b. For all Möbius transformation $T(z)$ which fixes a, b, and $T(f(\alpha)) = \alpha$ then*

$$T(z) = \frac{(-b\alpha + \alpha f(\alpha) + ab - a\alpha)z + (ab\alpha - abf(\alpha))}{(f(\alpha) - \alpha)z + (\alpha f(\alpha) + ab - bf(\alpha) - af(\alpha))}.$$

We can show that $T \circ S = S \circ T$.

Theorem 3.2.6 *Let f be a solution of (3.10). Then $f \circ T$ and $T \circ f$ are solutions of (3.10) where T is any transformation which satisfies $S \circ T = T \circ S$.*

Proof. Put $g = f \circ T$ and $h = T \circ f$. Then

$$\begin{aligned} g \circ S &= (f \circ T) \circ S \\ &= f \circ (T \circ S) \\ &= f \circ (S \circ T) \\ &= (f \circ S) \circ T \\ &= (S^k \circ f) \circ T \\ &= S^k \circ (f \circ T) \\ &= S^k \circ g. \end{aligned}$$

That is, g is a solution of (3.10). And

$$\begin{aligned} h \circ S &= (T \circ f) \circ S \\ &= T \circ (f \circ S) \\ &= T \circ (S^k \circ f) \\ &= (T \circ S) \circ (S^{k-1} \circ f) \\ &= (S \circ T) \circ (S^{k-1} \circ f) \\ &= S \circ (T \circ S) \circ (S^{k-2} \circ f) \\ &= S \circ (S \circ T) \circ (S^{k-2} \circ f) \\ &= S^2 \circ T \circ (S^{k-2} \circ f) \\ &\vdots \\ &= S^k \circ (T \circ f) \\ &= S^k \circ h. \end{aligned}$$

That is, h is a solution of (3.10). This completes the proof.

Theorem 3.2.7 *Let f and g be solutions of (3.10) such that f fixes a, b and α ($\alpha \neq a, b$) and g fixes a, b and β ($\beta \neq a, b$). Then g can be expressed in the form*

$$g = T \circ f$$

where T is a Möbius transformation which fixes a, b and $T(f(\beta)) = \beta$.

Proof. From Remark 3.2.5, $S \circ T = T \circ S$. By Theorem 3.2.6, $T \circ f$ is a solution of (3.10). Since

$$T \circ f(a) = T(a) = a$$

$$T \circ f(b) = T(b) = b$$

$$T \circ f(\beta) = T(f(\beta)) = \beta,$$

this implies $T \circ f$ is a solution of (3.10) which fixes a, b and β. By Theorem 3.2.4, we obtain $g = T \circ f$.

Theorem 3.2.8 *Let f be a solution of (3.10) which fixes a, b. Then f is a rational function.*

Proof. First, we consider $S(z) = \lambda z, |\lambda| \neq 0, 1$. Let g be a solution of (3.10) which fixes $0, \infty$ and S is defined as above. So

$$g(\lambda z) = \lambda^k g(z). \tag{3.12}$$

Set

$$g(z) = \sum_{n=1}^{\infty} a_n z^n$$

where $a_n \in \mathbb{C}, \forall n$. We have

$$g(\lambda z) = \sum_{n=1}^{\infty} a_n \lambda^n z^n$$

and

$$\lambda^k g(z) = \sum_{n=1}^{\infty} a_n \lambda^k z^n.$$

From (3.12), we obtain

$$\sum_{n=1}^{\infty} a_n \lambda^n z^n = \sum_{n=1}^{\infty} a_n \lambda^k z^n$$

$$\sum_{n=1}^{\infty} a_n (\lambda^n - \lambda^k) z^n = 0$$

$$a_n (\lambda^n - \lambda^k) = 0, \forall n$$

For $n \neq k, a_n = 0$, so that $g(z) = a_k z^k$. This implies that g is a rational function.

Now, we consider S which fixes a, b. Then S is conjugate to a map $z \mapsto \lambda z, |\lambda| \neq 0, 1$ by the Möbius transformation that send $z = a$ to 0 and $z = b$ to ∞, namely

$$M(z) = \frac{-z + a}{-z + b}.$$

Let f be a solution of (3.10) which fixes a, b. Then f is conjugate to g with the same Möbius transformation. Therefore f is a rational function. This completes the proof.

Proposition 3.2.9 *Let f be a solution of (3.10) which fixes a and b. For any complex number α distinct from a and b, if $f(\alpha) = \alpha$ then f is conjugate to a map $\left(\frac{-z+b}{-z+a}\right)^{k-1} z^k$.*

Proof. Assume that f is a solution of (3.10) which fixes a and b. Given $\alpha \neq a, b$. From Theorem 3.2.8, f is conjugate to a map g where $g(z) = Kz^k, \exists K \neq 0$ by the Möbius transformation that send $z = a$ to 0 and $z = b$ to ∞, namely

$$M(z) = \frac{-z+a}{-z+b}.$$

Since $f(\alpha) = \alpha$, we have

$$M^{-1}gM(\alpha) = \alpha.$$
$$g(M(\alpha)) = M(\alpha).$$

That is, $M(\alpha)$ is a fixed point of g. But g has fixed points at $0, \infty$ and $(k-1)^{th}$ roots of $\frac{1}{K}$. Then

$$\begin{aligned} M(\alpha) &= \left(\frac{1}{K}\right)^{\frac{1}{k-1}} \\ \frac{-\alpha+a}{-\alpha+b} &= \left(\frac{1}{K}\right)^{\frac{1}{k-1}} \\ \left(\frac{-\alpha+a}{-\alpha+b}\right)^{k-1} &= \frac{1}{K} \\ K &= \left(\frac{-\alpha+b}{-\alpha+a}\right)^{k-1}. \end{aligned}$$

This implies that f is conjugate to a map $\left(\frac{-z+b}{-z+a}\right)^{k-1} z^k$. This completes the proof.

Theorem 3.2.10 *Let $S(z) = \lambda z$, $|\lambda| \neq 0$ and let f be a meromorphic solution of (3.10). Then f is of the following form*

(a) *If $|\lambda| = 1$,*

(a.1) *and if λ is not an m^{th}-root of 1,*

(a.1.1) *and if $f(0) = 0$, then $f(z) = a_k z^k, a_k \neq 0$.*

(a.1.2) *and if $f(0) = \infty$, then there are no solutions for (3.10).*

(a.2) *and if λ is an m^{th}-root of 1,*

(a.2.1) *and if $f(0) = 0$,*

(a.2.1.1) *and if $m < k$ then $f(z) = a_t z^t, a_t \neq 0$ or $f(z) = \sum_{p \geq 0} a_{t+pm} z^{t+pm}$, where t is the remainder of k divided by m.*

(a.2.1.2) *and if $m = k$ then $f(z) = a_k z^k, a_k \neq 0$ or $f(z) = \sum_{p \geq 1} a_{pm} z^{pm}$.*

(a.2.1.3) *and if $m > k$ then $f(z) = a_k z^k, a_k \neq 0$ or $f(z) = \sum_{p \geq 0} a_{k+pm} z^{k+pm}$.*

(a.2.2) *and if* $f(0) = \infty$ *(*$f(z) = \sum_{n \geq -i} a_n z^n, i \in \mathbb{Z}^+$*),*

(a.2.2.1) *and if* $m < k$ *then* $f(z) = \sum_{p \geq j} a_{t+pm} z^{t+pm}$*, where* j *is the smallest integer* p *so that* $t + pm \geq -i$ *and* t *is the remainder of* k *divided by* m*.*

(a.2.2.2) *and if* $m = k$ *then* $f(z) = \sum_{p \geq j} a_{pm} z^{pm}$*, where* j *is the smallest integer* p *so that* $pm \geq -i$*.*

(a.2.2.3) *and if* $m > k$ *then* $f(z) = \sum_{p \geq j} a_{k+pm} z^{k+pm}$*, where* j *is the smallest integer* p *so that* $k + pm \geq -i$*.*

(b) *If* $|\lambda| \neq 1$*,*

(b.1) *and if* $f(0) = 0$*, then* $f(z) = a_k z^k, a_k \neq 0$*.*

(b.2) *and if* $f(0) = \infty$*, then there are no solutions for (3.10).*

Proof. Since $S(z) = \lambda z$, $|\lambda| \neq 0$ and f is a meromorphic solution of (3.10), we have

$$f(\lambda z) = \lambda^k f(z). \tag{3.13}$$

We discuss two situations.

Case a $|\lambda| = 1$. We discuss the following two subcases.

Subcase a.1 λ is not an m^{th}-root of 1. We discuss the following two sub-subcases.

Sub-subcase a.1.1 $f(0) = 0$. We can write $f(z)$ in the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

where $a_n \in \mathbb{C}, \forall n \in \mathbb{N}$. By substitution $f(z)$ into (3.13), we obtain

$$\sum_{n=1}^{\infty} a_n \lambda^n z^n = \sum_{n=1}^{\infty} a_n \lambda^k z^n$$

$$\sum_{n=1}^{\infty} a_n(\lambda^n - \lambda^k) z^n = 0$$

$$a_n(\lambda^n - \lambda^k) = 0, \forall n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, $a_n = 0$ or $\lambda^n - \lambda^k = 0$. Since $\lambda^n \neq \lambda^k, \forall n \neq k$, we have $f(z) = a_k z^k, a_k \neq 0$.

Sub-subcase a.1.2 $f(0) = \infty$. We can write $f(z)$ in the form

$$f(z) = \sum_{n=-i}^{\infty} a_n z^n,$$

where $i \in \mathbb{Z}^+$ and $a_n \in \mathbb{C}, \forall n \geq -i$. By substitution $f(z)$ into (3.13), we obtain

$$\sum_{n=-i}^{\infty} a_n \lambda^n z^n = \sum_{n=-i}^{\infty} a_n \lambda^k z^n$$

$$\sum_{n=-i}^{\infty} a_n(\lambda^n - \lambda^k) z^n = 0$$

$$a_n(\lambda^n - \lambda^k) = 0, \forall n \geq -i.$$

Then for each $n \geq -i$, $a_n = 0$ or $\lambda^n - \lambda^k = 0$. Since $\lambda^n \neq \lambda^k, \forall n \neq k$. Therefore $f(z) = a_k z^k, a_k \neq 0$. But $f(0) = \infty$, this leads to a contradiction. Therefore, there are no solutions for (3.10).

Subcase a.2 λ is an m^{th}-root of 1. We discuss the following two sub-subcases.

Sub-subcase a.2.1 $f(0) = 0$. We can write $f(z)$ in the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

where $a_n \in \mathbb{C}, \forall n \in \mathbb{N}$.

If $m < k$, there exist $s,\ t \in \mathbb{Z},\ 0 \leq t < m$ such that $k = sm + t$. We can reduce (3.13) into the form

$$f(\lambda z) = \lambda^t f(z). \tag{3.14}$$

By substitution $f(z)$ into (3.14), we obtain

$$\sum_{n=1}^{\infty} a_n \lambda^n z^n = \sum_{n=1}^{\infty} a_n \lambda^t z^n$$

$$\sum_{n=1}^{\infty} a_n(\lambda^n - \lambda^t) z^n = 0$$

$$a_n(\lambda^n - \lambda^t) = 0, \forall n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, $a_n = 0$ or $\lambda^n - \lambda^t = 0$. If $a_n = 0, \forall n \neq t$, $f(z) = a_t z^t$, $a_t \neq 0$ and if $\lambda^n - \lambda^t = 0$, $\lambda^{n-t} = 1$ which implied that $n = t + pm,\ p \in \mathbb{Z}^+ \cup \{0\}$. Thus $f(z) = \sum_{p \geq 0} a_{t+pm} z^{t+pm}$.

If $m = k$, by substitution $f(z)$ into (3.13), we obtain

$$\sum_{n=1}^{\infty} a_n \lambda^n z^n = \sum_{n=1}^{\infty} a_n \lambda^k z^n$$

$$\sum_{n=1}^{\infty} a_n(\lambda^n - \lambda^k) z^n = 0$$

$$a_n(\lambda^n - \lambda^k) = 0, \forall n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, $a_n = 0$ or $\lambda^n - \lambda^k = 0$. If $a_n = 0$, $\forall n \neq k$, $f(z) = a_k z^k$, $a_k \neq 0$ and if $\lambda^n - \lambda^k = 0$, $\lambda^{n-k} = 1$ which implies that $n = pm$, $p \in \mathbb{Z}^+$. Therefore $f(z) = \sum_{p\geq 1}^{\infty} a_{pm} z^{pm}$.

<u>If $m > k$</u>, by substitution $f(z)$ into (3.13), we obtain

$$\sum_{n=1}^{\infty} a_n \lambda^n z^n = \sum_{n=1}^{\infty} a_n \lambda^k z^n$$

$$\sum_{n=1}^{\infty} a_n(\lambda^n - \lambda^k) z^n = 0$$

$$a_n(\lambda^n - \lambda^k) = 0, \forall n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, $a_n = 0$ or $\lambda^n - \lambda^k = 0$. If $a_n = 0$, $\forall n \neq k$, $f(z) = a_k z^k$, $a_k \neq 0$ and if $\lambda^n - \lambda^k = 0$, $\lambda^{n-k} = 1$ which implies that $n = k + pm$, $p \in \mathbb{Z}^+ \cup \{0\}$. Therefore $f(z) = \sum_{p\geq 0}^{\infty} a_{k+pm} z^{k+pm}$.

<u>Sub-subcase a.2.2</u> $f(0) = \infty$. We can write $f(z)$ in the form

$$f(z) = \sum_{n=-i}^{\infty} a_n z^n$$

where $i \in \mathbb{Z}^+$ and $a_n \in \mathbb{C}$, $\forall n \geq -i$.

<u>If $m < k$</u>, there exist $s,\ t \in \mathbb{Z}$, $0 \leq t < m$ such that $k = sm + t$. We can reduce (3.13) into the form

$$f(\lambda z) = \lambda^t f(z). \tag{3.15}$$

By substitution $f(z)$ into (3.15), we obtain

$$\sum_{n=-i}^{\infty} a_n \lambda^n z^n = \sum_{n=-i}^{\infty} a_n \lambda^t z^n$$

$$\sum_{n=-i}^{\infty} a_n(\lambda^n - \lambda^t) z^n = 0$$

$$a_n(\lambda^n - \lambda^t) = 0, \forall n \geq -i.$$

Then for each $n \geq -i$, $a_n = 0$ or $\lambda^n - \lambda^t = 0$. If $a_n = 0, \forall n \neq t$, $f(z) = a_t z^t$, $a_t \neq 0$. But $f(0) = \infty$, this leads to a contradiction. Thus, this situation cannot occur. And if $\lambda^n - \lambda^t = 0$, $\lambda^{n-t} = 1$ which implies that $n = t + pm$, $p \in \mathbb{Z}$. Thus $f(z) = \sum_{p\in\mathbb{Z}} a_{t+pm} z^{t+pm}$. Since $a_{t+pm} = 0$, $\forall p < j$ where j is the smallest integer such that $t + pm \geq -i$, we have $f(z) = \sum_{p\geq j} a_{t+pm} z^{t+pm}$.

If $m = k$, by substitution $f(z)$ into (3.13), we obtain

$$\sum_{n=-i}^{\infty} a_n \lambda^n z^n = \sum_{n=-i}^{\infty} a_n \lambda^k z^n$$

$$\sum_{n=-i}^{\infty} a_n(\lambda^n - \lambda^k) z^n = 0$$

$$a_n(\lambda^n - \lambda^k) = 0, \forall n \geq -i.$$

Then for each $n \geq -i$, $a_n = 0$ or $\lambda^n - \lambda^k = 0$. If $a_n = 0$, $\forall n \neq k$, $f(z) = a_k z^k$, $a_k \neq 0$. But $f(0) = \infty$, this leads to a contradiction. Thus, this situation cannot occur. And if $\lambda^n - \lambda^k = 0$, $\lambda^{n-k} = 1$ which implies that $n = pm$, $p \in \mathbb{Z}$. Thus $f(z) = \sum_{p\in\mathbb{Z}} a_{pm} z^{pm}$. Since $a_{pm} = 0$, $\forall p < j$ where j is the smallest integer such that $pm \geq -i$, we have $f(z) = \sum_{p\geq j} a_{pm} z^{pm}$.

If $m > k$, by substitution $f(z)$ into (3.13), we obtain

$$\sum_{n=-i}^{\infty} a_n \lambda^n z^n = \sum_{n=-i}^{\infty} a_n \lambda^k z^n$$

$$\sum_{n=-i}^{\infty} a_n(\lambda^n - \lambda^k) z^n = 0$$

$$a_n(\lambda^n - \lambda^k) = 0, \forall n \geq -i.$$

Then for each $n \geq -i$, $a_n = 0$ or $\lambda^n - \lambda^k = 0$. If $a_n = 0$, $\forall n \neq k$, $f(z) = a_k z^k$, $a_k \neq 0$. But $f(0) = \infty$, this leads to a contradiction. Thus, this situation cannot occur. And if $\lambda^n - \lambda^k = 0$, $\lambda^{n-k} = 1$ which implies that $n = k + pm$, $p \in \mathbb{Z}$. Thus $f(z) = \sum_{p\in\mathbb{Z}} a_{k+pm} z^{k+pm}$. Since $a_{k+pm} = 0$, $\forall p < j$ where j is the smallest integer such that $k + pm \geq -i$, we have $f(z) = \sum_{p\geq j} a_{k+pm} z^{k+pm}$.

Case b $|\lambda| \neq 1$. We discuss the following two subcases.

Subcase b.1 $f(0) = 0$. We can write $f(z)$ in the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

where $a_n \in \mathbb{C}$, $\forall n \in \mathbb{N}$. By substitution $f(z)$ into (3.13), we obtain

$$\sum_{n=1}^{\infty} a_n \lambda^n z^n = \sum_{n=1}^{\infty} a_n \lambda^k z^n$$

$$\sum_{n=1}^{\infty} a_n(\lambda^n - \lambda^k) z^n = 0$$

$$a_n(\lambda^n - \lambda^k) = 0, \forall n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, $a_n = 0$ or $\lambda^n - \lambda^k = 0$. Since $|\lambda| \neq 1$, we have that for all $n \neq k$, $\lambda^n \neq \lambda^k$. This implies $a_n = 0,\ \forall n \neq k$. Therefore $f(z) = a_k z^k, a_k \neq 0$.

Subcase b.2 $f(0) = \infty$. We can write $f(z)$ in the form

$$f(z) = \sum_{n=-i}^{\infty} a_n z^n,$$

where $i \in \mathbb{Z}^+$ and $a_n \in \mathbb{C}, \forall n \geq -i$. By substitution $f(z)$ into (3.13), we obtain

$$\sum_{n=-i}^{\infty} a_n \lambda^n z^n = \sum_{n=-i}^{\infty} a_n \lambda^k z^n$$

$$\sum_{n=-i}^{\infty} a_n(\lambda^n - \lambda^k) z^n = 0$$

$$a_n(\lambda^n - \lambda^k) = 0, \forall n \geq -i.$$

Then for each $n \geq -i$, $a_n = 0$ or $\lambda^n - \lambda^k = 0$. Since $|\lambda| \neq 1$, we have that for all $n \neq k$, $\lambda^n \neq \lambda^k$. This implies $a_n = 0,\ \forall n \neq k$. Therefore $f(z) = a_k z^k, a_k \neq 0$. But $f(0) = \infty$, this leads to a contradiction. Therefore, there are no solutions for (3.10). This completes the proof.

Remark 3.2.11 *Let $S(z) = \lambda z$ and let f be a meromorphic solution of (3.10).Then*
(a) *if $|\lambda| = 1$ but λ is not an m^{th}- root of 1 and $|\lambda| \neq 1$, then 0 and ∞ are super-attracting fixed points of f and $f(z) = a_k z^k, a_k \neq 0$, that is, there are no meromorphic solutions f with deg $f \neq k$;*
(b) *if $|\lambda| = 1$ and λ is an m^{th}- root of 1, then f need not fix ∞. For example, let $S(z) = -z$ and $k = 3$, then $f(z) = \frac{z}{z^2+1}$ is the meromorphic solution of (3.10) which $f(\infty) = 0$.*

Remark 3.2.12 *Consider (3.9), we obtain the following results:*
(a) *If $a_k = 1$, so $f(a) = a, f(b) = b$, and $f(\infty) = \frac{1}{b_k}$, $b_k \in \mathbb{C}$. If M_k is a Möbius transformation that $M_k(a) = a, M_k(b) = b$ and $M_k(\frac{1}{b_k}) = \infty$, then by Theorem (3.2.4) $M_k \circ f_k$ is a unique solution of (3.10) that fixes a, b and ∞. This implies* (a) *of Theorem (3.1.8).*
(b) *If $a_k = 0, a_{k-1} \neq 1$, so $f(a) = a, f(b) = b$, and $f(\infty) = 0$. If M_k is a Möbius transformation that $M_k(a) = a, M_k(b) = b$ and $M_k(0) = \infty$, then by Theorem (3.2.4) $M_k \circ f_k$ is a unique solution of (3.10) such that fixes a, b and ∞. This*

implies (b) *of Theorem (3.1.8).*

(c) *If* $a_k = a_{k-1} = 0$*, from Theorem (3.2.8)* f *is conjugate to a map* $z \mapsto Kz^k, K \neq 0$ *by the Möbius transformation* $M(z) = C\left(\frac{z-a}{z-b}\right), C \neq 0$*. Then* f *is of the form*

$$f(z) = \frac{bKC^k(z-a)^k - aC(z-b)^k}{KC^k(z-a)^k - C(z-b)^k}.$$

Therefore (3.10) has no solutions. This implied (c) *of Theorem (3.1.8).*

Example 3.2.13 *For* $k = 2$*, Newton's method* N *is the rational solution of (3.10) which fixes* a, b *and* ∞*. Let* f *be a solution of (3.10) which fixes* a, b *and* α ($\alpha \neq a, b$)*.*

Then $f = T \circ N$ *where* T *is a Möbius transformation which fixes* a, b *and* $T(f(\alpha)) = \alpha$*.*

Theorem 3.2.14 *The Julia set of the rational solutions of (3.10) are circles on the sphere.*

Proof. In [66], we know that

$$f_k(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}$$

is the rational solution of (3.10). Let f be a rational solution of (3.10) which fixes a, b and α ($\alpha \neq a, b$). Theorem 3.2.7 shows that $f = T_k \circ f_k$ where T_k is a Möbius transformation which fixes a, b and $T_k(f_k(\alpha)) = \alpha$. For $k \geq 2$, the function f_k is conjugate to a map $w \mapsto w^k$ and T_k is conjugate to a map $w \mapsto Kw^k$ where $|K| < 1$ by the Möbius transformation that send $w = 0$ to a and $w = \infty$ to b, namely

$$M(w) = \frac{bw - a}{w - 1}.$$

The inverse M^{-1}, of M is given by

$$M^{-1}(z) = \frac{-z + a}{-z + b}.$$

This implies that f is conjugate to the map Kz^{k^2} where $|K| < 1$. So we obtain that $J(f)$ is a circle on the sphere. This completes the proof.

We now consider the case when the Möbius transformation has exactly one fixed point in the complex plane. Let R be a Möbius transformation which has only one fixed point, say $a \in \mathbb{C}$ (so a is the global attractor of R). We are interested in finding meromorphic solutions f on $\mathbb{C}$ of the following functional equation

$$f \circ R = R^k \circ f \tag{3.16}$$

where $k \geq 2$ and R is defined as above.

Remark 3.2.15 *We can show that*

$$f^i \circ R^j = R^{jk^i} \circ f^i \text{ for } i,\ j \in \mathbb{N}. \tag{3.17}$$

Theorem 3.2.16 *Let f be a solution of (3.16). Then a is a fixed point of f.*

Proof. First, we show that a is not a pole of f. For if a is a pole of f, then $f(a) = \infty$. From (3.17) and for $i = 1$, we have

$$\infty = f(a) = f \circ R^i(a) = R^{jk}(f(a)) = R^{jk}(\infty).$$

Since a is an attracting fixed point of R, $R^{jk}(\infty) \to a$ as $j \to \infty$. This implies that $a = \infty$ which is a contradiction. Thus a is not a pole of f.
Take $i = 1$ and $z = a$ in (3.17) we obtain

$$f(a) = R^{jk}(f(a)).$$

This implies that $f(a)$ is a fixed point of $R^{jk}(z)$.
Taking $j \to \infty$,

$$f(a) = \lim_{j\to\infty} R^{jk}(f(a)) = a,$$

since a is a global attracting fixed point of R. Therefore a is a fixed point of f.

Theorem 3.2.17 *For a given complex number α distinct from a. There exists a unique solution of (3.16) which fixes α and a.*

Proof. Let f and g be solutions of (3.16) which fixes α and a. From (3.17), take $i = 1$ we obtain

$$f \circ R^j(\alpha) = R^{jk} \circ f(\alpha) = R^{jk}(\alpha)$$

and

$$g \circ R^j(\alpha) = R^{jk} \circ g(\alpha) = R^{jk}(\alpha).$$

Since $\alpha \neq a$ and a is a global attractor of R, $R^j(\alpha) \to a$ as $j \to \infty$. This implies that a is a limit point of $\{R^j(\alpha) : \ j \in \mathbb{N}\}$. As

$$\{R^j(\alpha) : \ j \in \mathbb{N}\} \subseteq \{z \in \overline{\mathbb{C}} : \ f(z) = g(z)\},$$

we have, by the identity Theorem (see [35]), $f \equiv g$ on $\overline{\mathbb{C}}$. Therefore, there is a unique solution of (3.16) which fixes a and α where α is a complex number distinct from a. This completes the proof.

Theorem 3.2.18 *Let f be a solution of (3.16). If T is any Möbius transformation such that $T \circ R = R \circ T$, then $f \circ T$ and $T \circ f$ are solutions of (3.16).*

Proof. Put $g = f \circ T$ and $h = T \circ f$. Then

$$\begin{aligned} g \circ R &= (f \circ T) \circ R \\ &= f \circ (T \circ R) \\ &= f \circ (R \circ T) \\ &= (f \circ R) \circ T \\ &= (R^k \circ f) \circ T \\ &= R^k \circ (f \circ T) \\ &= R^k \circ g. \end{aligned}$$

That is, g is a solution of (3.10). And

$$\begin{aligned} h \circ R &= (T \circ f) \circ R \\ &= T \circ (f \circ R) \\ &= T \circ (R^k \circ f) \\ &= (T \circ R) \circ (R^{k-1} \circ f) \\ &= (R \circ T) \circ (R^{k-1} \circ f) \\ &= R \circ (T \circ R) \circ (R^{k-2} \circ f) \end{aligned}$$

$$\begin{aligned} h \circ R &= R \circ (R \circ T) \circ (R^{k-2} \circ f) \\ &= R^2 \circ T \circ (R^{k-2} \circ f) \\ &\vdots \\ &= R^k \circ (T \circ f) \\ &= R^k \circ h. \end{aligned}$$

That is, h is a solution of (3.10). This completes the proof.

Theorem 3.2.19 *Let f be a solution of (3.16). Then f is conjugate to a map*

$$kz + P(e^{\frac{-2\pi i}{c} z}) + Q(e^{\frac{2\pi i}{c} z})$$

where P and Q are meromorphic functions.

Proof. Without loss of generality we may assume that $R(z) = z + c, c \neq 0$. Assume that g is a solution of (3.16). Then $g(z+c) = g(z) + kc$. Note that $g(z+c) = g(z) + kc$ if and only if $g(z) = kc + H(z)$ where $H(z) = g(z) - kc$. So $H(z) = H(z+c)$, that is, H is periodic. Since rational functions cannot have a period, this implies that $H(z) = P(e^{\frac{-2\pi i}{c} z}) + Q(e^{\frac{2\pi i}{c} z})$ where P and Q are meromorphic functions.

Now, we consider S which fixes a. Then S is conjugate to a map $z \mapsto z + c,\ c \neq 0$ by the Möbius transformation that send $z = a$ to ∞, namely

$$M(z) = \frac{1}{-z+a}.$$

Let f be a solution of (3.16). Then f is conjugate to g with the same Möbius transformation. This completes the proof.

Example 3.2.20 *$f(z) = 2z + e^{-z} + e^{z}$ is a solution of the functional equation $f \circ R = R^2 \circ f$ where $R(z) = z + 2\pi i$.*