Chapter 4

Boundedness of Fatou Components of Composite Entire and Meromorphic Functions

4.1 Some known results

If f is a polynomial of degree at least two, then the Fatou set $F(f)$ contains the component $F_\infty = \{z : f^n(z) \to \infty\}$, which is unbounded and completely invariant. If f is a transcendental entire function then, from Picard's theorem and the invariance of $J(f)$ is unbounded, so that $F(f)$ no longer contains a neighborhood of ∞.

I. N. Baker [8] raised the question of whether every component of $F(f)$ must be bounded if f is of sufficiently small growth. The appropriate growth condition would be of order less than $\frac{1}{2}$, since I. N. Baker [8] showed that for any sufficiently large positive a , the function $f(z) = z^{\frac{-1}{2}}\sin z^{\frac{1}{2}} + z + a$ is of order $\frac{1}{2}$ and has an unbounded component U of $F(f)$ containing a segment $[x_0, \infty)$ of the positive real axis. In the positive direction to this problem, a few results have been obtained.

I. N. Baker [8] proved the following result.

Theorem 4.1.1 *[8] Let $f \in \mathcal{E}$. Suppose that there exist sequences R_n, $\rho_n \to \infty$ and $c(n) > 1$ such that*

(i) $M(R_n, f) = R_{n+1}$,

(ii) $R_n \le \rho_n \le R_n^{c(n)}$,

(iii) $m^*(\rho_n, f) > R_{n+1}^{c(n+1)}$ for all sufficiently large n.

Then all the components of the Fatou set $F(f)$ are bounded.

Note that $m^*(r,f) = \min\{|f(z)| : \ |z| \le r\}$.

He also proved that every component of $F(f)$ is bounded if

$$\log M(r,f) = O\{(\log r)^t\} \tag{4.18}$$

as $r \to \infty$, where $1 < t < 3$.

G. M. Stallard [76] improved the sufficient condition (4.18) to

$$\log\ \log M(r,f) = O\left[\frac{(\log\ r)^{\frac{1}{2}}}{(\log\ \log r)^{\epsilon}}\right] \tag{4.19}$$

for some $\epsilon > 0$. Furthermore, every component of $F(f)$ is bounded provided that f is of order less than $\frac{1}{2}$ and

$$\frac{\log M(2r,f)}{\log M(r,f)} \to c \text{ as } r \to \infty \tag{4.20}$$

where $c \ge 1$ is a finite constant that depends on f . By a method which is somewhat different from those of I. N. Baker or G. M. Stallard, J. M. Anderson and A. Hinkkanen [2] obtained the same result under another regularity condition on the growth of f instead of (4.20), namely, for some positive constant c,

$$\frac{\phi'(x)}{\phi(x)} \ge \frac{1+c}{x} \tag{4.21}$$

for all sufficiently large x , where $\phi(x) = \log M(e^x, f)$.

X. H. Hua and C. C. Yang [49] proved the following results.

Theorem 4.1.2 *[49] Suppose that f is a transcendental entire function of lower order $\mu = \mu(f) < 1/2$. Assume that for any $m > 1$,*

$$\log M(r^m, f) \ge m^2 \log M(r,f)$$

holds for all sufficiently large r. Then any component of $F(f)$ is bounded.

Now we will show that the condition $\rho < 1/2$ in the above theorem is sharp.

Example 4.1.3 *[8] Take a positive constant a such that*

$$\frac{1}{2}a(x+1+a/2)^{-1/2} > e|z|^{-1/2}$$

hold for all $x > a^2, y^2 < 4(x+1)$ *and* $z = x + iy$. *Let*

$$f(z) = \sin(z^{1/2})/z^{1/2} + z + a$$

and

$$D = \{z = x + iy : x > a^2, y^2 < 4(x+1)\}.$$

Then the lower order $\mu(f)$ *of* $f(z)$ *is* $1/2$ *and*

$$\log M(2r, f)/\log M(r, f) \to 2^{1/2}$$

as $r \to \infty$.
On the other hand, if $z \in D$, *then* $|f(z) - (z+a)| < e|z|^{-1/2}$. *Let* $z_0 \in \partial D$. *It is easy to verify that*

$$|z + a - z_0| > \frac{1}{2}a(x + 1 + a/2)^{-1/2} > e|z|^{-1/2}.$$

Thus $f(D) \subset D$, *and so,* $D \subset F(f)$. *Therefore* $F(f)$ *contains an unbounded component.*

Theorem 4.1.4 *[48] Suppose that* f *is a transcendental entire function with lower order* $\mu = \mu(f) < 1/2$. *Then every pre-periodic component is bounded.*

For example, take $f(z) = \cos\sqrt{z + (3\pi/2)^2}$. Then $F(f)$ has an unbounded component U which contains the origin. Note that $f(0) = 0, |f'(0)| < 1$. Thus U is an immediate attractive domain. Obviously, $\mu(f) = 1/2$. Thus the restriction on the lower order is sharp.

Y. Wang [80] gave a positive answer to Baker's problem for all functions of positive lower order. He showed that for an entire functions f, if $\rho < \frac{1}{2}$ and $\mu > 0$, then every component of $F(f)$ is bounded. Therefore only the case $\mu = 0$ remains open.

J. H. Zheng [83] investigated this subject for the case of meromorphic function and proved the following result.

Theorem 4.1.5 *[83] Let* $f(z)$ *be a transcendental meromorphic function. If we have*

$$\limsup_{r \to +\infty} \frac{m(r, f)}{r} = +\infty,$$

where $m(r, f) = \min\{|f(z)| : |z| = r\}$, *then the Fatou set,* $F(f)$, *of* f *has no unbounded preperiodic or periodic components*
In particular, f *has no Baker domains.*

J. H. Zheng and S. Wang [86] gave, among other things, a sufficient condition for non-existence of the unbounded Fatou components, which says that a transcendental meromorphic function $f(z)$ has no unbounded Fatou components provided that for a $d > 1$ and for all sufficiently large $r > 0$ there exists an $\tilde{r} \in [r, r^d]$ such that

$$\log m(\tilde{r}, f) \geq d \log M(r, f), \tag{4.22}$$

then every component of $F(f)$ is bounded. from this result, they considered the case of the composition of finitely many entire functions and proved the non-existence of unboubded Fatou components of $f_N \circ f_{N-1} \circ \ldots \circ f_1(z)$ with order $0 < \mu(f_j) \leq \rho(f_j) < \frac{1}{2}(j = 1, 2, 3, \ldots, N)$. Their results as follows:

Theorem 4.1.6 *[86] Let* $f_j(z)(j = 1, 2, \ldots, m; m \geq 1)$ *be transcendental entire functions and such that for some number* $h > 1$ *and all the sufficiently large* r *, there exists an* $r_j \in (r, r^h)$ *satisfying*

$$m(r_j, f_j) > M(r, f_j)^h, j = 1, 2, \ldots, m.$$

Set $g(z) = f_m \circ f_{m-1} \circ \cdots \circ f_1(z)$.
Then $F(g)$ *has no unbounded components.*

The condition of Theorem 4.1.6, can be achieved and from this they obtained the following consequence.

Corollary 4.1.7 *[86] Let* $f_j(z)(j = 1, 2, \ldots, m; m \geq 1)$ *be transcendental entire functions with finite order and such that for some* $\alpha \in (0, 1)$ *and* $d > 1$ *and all the sufficiently large* r*, there exists an* $r_j \in (r, r^d)$ *satisfying*

$$m(r_j, f_j) > M(r, f_j)^{\alpha}, j = 1, 2, \cdots, m, \tag{4.23}$$

and for some small $\epsilon > 0$*, there exists an* $\tilde{r}_j \in (r, r^d)$ *such that*

$$\log\ M(\tilde{r}_j, f_j) > r^{\epsilon}, j = 1, 2, \cdots, m.$$

Set

$$g(z) = f_m \circ f_{m-1} \circ \cdots \circ f_1(z).$$

Then $F(g)$ has no unbounded components.

Corollary 4.1.8 *[86] Let $f_j(z)(j = 1, 2, \ldots, m; m \geq 1)$ be transcendental entire functions of growth order and lower order lying in $(0, \frac{1}{2})$.*
Set $g(z) = f_m \circ f_{m-1} \circ \cdots \circ f_1(z)$.
Then $F(g)$ has no unbounded components.

In [32], C. Cao and Y. Wang extend the result in [86] by assuming instead one of functions $f_j(z)(j = 1, 2, 3, \ldots, N)$ considered has a positive lower order in terms of the argument due to I. N. Baker [8]. They show that if $f_1, f_2, \ldots, f_N$ are non-constant holomorphic functions in the plane, each having order less than $\frac{1}{2}$ and if the lower order of $f_j > 0$, for some $j \in \{1, 2, \ldots, N\}$, then the Fatou set of the function $h = f_N \circ f_{N-1} \circ \cdots \circ f_1$ has no unbounded components.

For more detail of boundedness of components of $F(f)$ of transcendental entire function f, we refer to I. N. Baker [8], G. M. Stallard [76], Y. Wang [80], J. H. Zheng [81] and references cited therein.

J. H. Zheng [83] considered the case of meromorphic functions and gave a sufficient condition for non-existence of unbounded pre-periodic and periodic Fatou components. Therefore, it is an open problem for the case of wandering domains. In this thesis, we attempt to discuss the boundedness of Fatou components of composition of any finitely many transcendental meromorphic functions of order less than $\frac{1}{2}$ with finitely many poles and obtain a generalization of results of Zheng-Wang [86] and Cao-Wang [32].

4.2 Lemmas

In order to prove our main Theorem, we need two lemmas and the basic knowledge of the hyperbolic metric.

Lemma 4.2.1 *Let f be a meromorphic function of order less than $\frac{1}{2}$ with finitely many poles. There exist $d > 1$ and $R > 0$ such that for all $r > R$, there exists $\tilde{r} \in (r, r^d)$ satisfying*

$$|f(z)| \geq m(\tilde{r}, f) = M(r, f)$$

for all $z \in \{z : |z| = \tilde{r}\}$.

Lemma 4.2.1 follows directly from [3], satz 1. Actually, $f(z)$ in Lemma 4.2.1 can be written into the form

$$f(z) = g(z) + R(z)$$

where $g(z)$ is an entire function with order $\rho(g) = \rho(f) < 1/2$ and $R(z)$ is a rational function such that $R(z) \to 0$ as $z \to \infty$. It is well-known that Lemma 4.2.1 is true for g, and hence it is easy to see that Lemma 4.2.1 holds for f.

Lemma 4.2.2 *Let f be a transcendental meromorphic function with only finitely many poles, finite order ρ and positive lower order μ. Then for any $d > 1$ such that $d\mu > \rho$, we have*

$$\lim_{r\to\infty} \frac{\log M(r^d, f)}{\log M(r, f)} = \infty.$$

Lemma 4.2.2 follows immediately from the proof of Corollary 2 of J. H. Zheng [86].

In what follows, let us recall some properties on the hyperbolic metric, see [1], [19], etc. An open set W in $\mathbb{C}$ is called *hyperbolic* if $\mathbb{C}\backslash W$ contains at least two points. Let U be a hyperbolic domain in $\mathbb{C}$. $\lambda_U(z)$ is the density of the hyperbolic on U and $\rho_U(z_1, z_2)$ stands for the hyperbolic distance between z_1 and z_2 in U, i.e.

$$\rho_U(z_1, z_2) = \inf_{\gamma \in U} \int_\gamma \lambda_U(z)|dz|,$$

where γ is a Jordan curve connecting z_1 and z_2 in U. If U is simply-connected and $d(z, \partial U)$ is a Euclidean distance between $z \in U$ and ∂U, then for any $z \in U$,

$$\frac{1}{2d(z, \partial U)} \leq \lambda_U(z) \leq \frac{2}{d(z, \partial U)}. \tag{4.24}$$

Let $f : U \to V$ be an analytic function, where U and V are hyperbolic domains. By the principle of hyperbolic metric, we have

$$\rho_V(f(z_1), f(z_2)) \le \rho_U(z_1, z_2), \tag{4.25}$$

for $z_1, z_2 \in U$.

4.3 Main theorem

In this section, we mainly prove the following result.

Theorem 4.3.1 *Let $f_j(z)(j = 1, 2, 3, \ldots, m)$ be transcendental meromorphic functions of order less than $\frac{1}{2}$ with at most finitely many poles and at least one of them has positive lower order. Let $g(z) = f_m \circ f_{m-1} \circ \ldots \circ f_1(z)$. Then either $g(z)$ has no unbounded Fatou components or at least one unbounded Fatou component is multiply connected.*

Proof Suppose that $F(g)$ has an unbounded component U and every unbounded component of $F(g)$ is simply-connected. Then in view of Theorem 4.1.5 under our assumption, U is wandering and every U_n is unbounded. Therefore U has an unbounded component Γ of its boundary and so $U_n \subset \mathbb{C}\backslash\Gamma$ and $\mathbb{C}\backslash\Gamma$ is certainly simply connected. Take a point $z_0 \in U$. Then there exists a sufficiently large $R_0 > |z_0|$ so that each $f_j(z)$ has no poles in $\{z : |z| > R_0\}$.

We first prove the following result: there exists $h > 1$ such that for all sufficiently large r and for an arbitrary curve γ which intersects $\{z : |z| < r\}$ and $\{z : |z| > r^h\}$, we have

$$g(\gamma) \cap \{z : |z| < R\} \neq \emptyset \text{ and } g(\gamma) \cap \{z : |z| > R^h\} \neq \emptyset \tag{4.26}$$

where $R = M_m(r, g)$, $M_1(r, g) = M(r, f_1), \ldots,\ M_m(r, g) = M(M_{m-1}(r, g), f_m)$.

Assume that $f_k(z)$ has positive lower order for $k \in \{1, 2, \ldots, m\}$. By Lemma 4.2.1, for each j, we have $t > 0$ such that for any $r > t$, there exists $\tilde{r_j} \in (r, r^d)$ such that

$$|f_j(z)| \ge M(r, f_j),\ \text{on } \Gamma_j := \{z : |z| = \tilde{r_j}\}, j = 1, 2, 3, \ldots, m \tag{4.27}$$

where each $f_j(z)$ has no poles in $\{z : |z| > t\}$ and $M(r, f_j)$ is increasing for $r > t$. Assume that γ is a curve under our consideration for $h = d^{2k}$, where d is as in Lemma 4.2.2 for f_k, namely,

$$\gamma \cap \{z : |z| < r\} \neq \emptyset \text{ and } \gamma \cap \{z : |z| > r^h\} \neq \emptyset.$$

From Lemma 4.2.1, there exists $\tilde{r}_1 \in (r^{d^{2k-1}}, r^{d^{2k}})$ such that

$$|f_1(z)| > M(r^{d^{2k-1}}, f_1) > M(r, f_1)^{d^{2k-2}}$$

on $\Gamma_1 := \{z : |z| = \tilde{r}_1\}$.
Let $R_1 = M(r, f_1)$. Then

$$f_1(\gamma) \cap \{z : |z| > R_1^{d^{2k-2}}\} \neq \emptyset$$

and from the maximum modulus principle, we have

$$f_1(\gamma) \cap \{z : |z| < R_1\} \neq \emptyset.$$

Then there exists $\tilde{R}_1 \in (R_1^{d^{2k-3}}, R_1^{d^{2k-2}})$ such that

$$|f_2(z)| \geq M(R_1^{d^{2k-3}}, f_2) > M(R_1, f_2)^{d^{2k-4}}$$

on $\Gamma_2 := \{z : |z| = \tilde{R}_1\}$.
Let $R_2 = M(R_1, f_2)$. Then

$$f_2 \circ f_1(\gamma) \cap \{z : |z| < R_2\} \neq \emptyset$$

and

$$f_2 \circ f_1(\gamma) \cap \{z : |z| > R_2^{d^{2k-4}}\} \neq \emptyset.$$

Then there exists $\tilde{R}_2 \in (R_2^{d^{2k-5}}, R_2^{d^{2k-4}})$ such that

$$|f_3(z)| \geq M(R_2^{d^{2k-5}}, f_3) > M(R_2, f_3)^{d^{2k-6}}$$

on $\Gamma_3 := \{z : |z| = \tilde{R}_2\}$.
Inductively, we set $R_{k-2} = M(R_{k-3}, f_{k-2})$. Then

$$f_{k-2} \circ f_{k-3} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| > R_{k-2}^{d^4}\} \neq \emptyset$$

and

$$f_{k-2} \circ f_{k-3} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| < R_{k-2}\} \neq \emptyset.$$

Then there exists $\tilde{R}_{k-2} \in (R_{k-2}^{d^3}, R_{k-2}^{d^4})$ such that

$$|f_{k-1}(z)| \geq M(R_{k-2}^{d^3}, f_{k-1}) > M(R_{k-2}, f_{k-1})^{d^2}$$

on $\Gamma_{k-1} := \{z : |z| = \tilde{R}_{k-2}\}$.
Set $R_{k-1} = M(R_{k-2}, f_{k-1})$. Then

$$f_{k-1} \circ f_{k-2} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| > R_{k-1}^{d^2}\} \neq \emptyset$$

and

$$f_{k-1} \circ f_{k-2} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| < R_{k-1}\} \neq \emptyset.$$

Then there exists $\tilde{R}_{k-1} \in (R_{k-1}^{d}, R_{k-1}^{d^2})$ such that

$$|f_k(z)| \geq M(R_{k-1}^{d}, f_k) > M(R_{k-1}, f_k)^{d^{2m}}$$

on $\Gamma_k := \{z : |z| = \tilde{R}_{k-1}\}$, where the last inequality follows from Lemma 4.2.2.
Set $R_k = M(R_{k-1}, f_k)$. Then

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| > R_k^{d^{2m}}\} \neq \emptyset$$

and

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| < R_k\} \neq \emptyset.$$

Then there exists $\tilde{R}_k \in (R_k^{d^{2m-1}}, R_k^{d^{2m}})$ such that

$$|f_{k+1}(z)| \geq M(R_k^{d^{2m-1}}, f_{k+1}) > M(R_k, f_{k+1})^{d^{2m-2}}$$

on $\Gamma_{k+1} := \{z : |z| = \tilde{R}_k\}$.
Inductively, we set $R_{m-1} = M(R_{m-2}, f_{m-1})$. Then we have

$$f_{m-1} \circ f_{m-2} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| > R_{m-1}^{d^{2k+2}}\} \neq \emptyset$$

and

$$f_{m-1} \circ f_{m-2} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| < R_{m-1}\} \neq \emptyset.$$

Then there exists $\tilde{R}_{m-1} \in (R_{m-1}^{d^{2k+1}}, R_{m-1}^{d^{2k+2}})$ such that

$$|f_m(z)| \geq M(R_{m-1}^{d^{2k+1}}, f_m) > M(R_{m-1}, f_m)^{d^{2k}} = M_m(r,g)^h,$$

on $\Gamma_m := \{z : |z| = \tilde{R}_{m-1}\}$.
Moreover, there exists a point $z_{m1} \in \gamma$ such that

$$|f_{m-1} \circ f_{m-2} \circ \cdots \circ f_1(z_{m1})| = \tilde{R}_{m-1}.$$

Thus, $|g(z_{m1})| > M_m(r,g)^h > M(R_0,g)^h > |g(z_0)|^h$.
By setting $R_{m1} = M_m(r,g)$, we obtain

$$g(\gamma) \cap \{z : |z| = R_{m1}^h\} \neq \emptyset$$

and

$$g(\gamma) \cap \{z : |z| = R_{m1}\} \neq \emptyset.$$

Repeating the previous process above inductively, there is a point $z_{mn} \in \gamma$ such that

$$|g^n(z_{mn})| \geq M(R_{mn}, g)^h \geq M(R_0, g)^h > |g^n(z_0)|^h, \tag{4.28}$$

where $R_{mn} = M_m(R_{n-1}, g)$.
Since $g^n(U) \subseteq U_n \subset \mathbb{C}\backslash\Gamma$ and U is unbounded, we have U_n is an unbounded component of $F(g)$. For an arbitrary point $a \in J(g)$, we obtain, by (4.24), that

$$\lambda_{U_n}(z) \geq \lambda_{\mathbb{C}\backslash\Gamma}(z) \geq \frac{1}{2d(z,\Gamma)} \geq \frac{1}{2|z-a|} \geq \frac{1}{2(|z|+|a|)}. \tag{4.29}$$

It follows that

$$\begin{aligned} \rho_{U_n}(g^n(z_0), g^n(z_{mn})) &\geq \int_{|g^n(z_0)|}^{|g^n(z_{mn})|} \frac{dr}{2(r+|a|)} \\ &= \frac{1}{2}\log\frac{|g^n(z_{mn})|+|a|}{|g^n(z_0)|+|a|}. \end{aligned} \tag{4.30}$$

Set $A = \max\{\lambda_U(z_0, z) : z \in \gamma\}$. Clearly $A \in (0, +\infty)$. From (4.25), noting $z_{mn} \in \gamma \subset U$, we have

$$\rho_{U_n}(g^n(z_0), g^n(z_{mn})) \leq \rho_U(z_0, z_{mn}) \leq A. \tag{4.31}$$

Therefore, by combining (4.28), (4.30) and (4.31) we obtain

$$|g^n(z_0)|^h < M(R_0, g^n)^h < |g^n(z_{mn})| + |a| \leq (|g^n(z_0)| + |a|)e^{2A}. \tag{4.32}$$

This is impossible, since a and e^{2A} are constants, $h > 1$ and $|g^n(z_0)| \to +\infty$ as $n \to +\infty$. Therefore, $F(g)$ has no unbounded Fatou components. This completes the proof.

Corollary 4.3.2 *Let $f_j(z)(j = 1, 2, 3, ..., m)$ be transcendental entire functions with order less than $\frac{1}{2}$ and at least one of them has positive lower order.*
Let $g(z) = f_m \circ f_{m-1} \circ \ldots \circ f_1(z)$.
Then $g(z)$ has no unbounded Fatou components.

Remark 4.3.3 *A different proof of Corollary 4.3.2 may be found in [32].*