Chapter 5

Dynamics of Composite Functions Outside a Small Set

5.1 Some known results

D. Sullivan [78] proved that the Fatou set of a rational function has no wandering domain, thus solving an open problem in the papers of Fatou and Julia. On the other hand this is not true for transcendental entire functions. I. N. Baker [7] constructed an entire function f such that $F(f)$ has wandering domains. Since then several entire functions which have wandering domains with various different properties had been constructed, see for instance [10], [38]. Also at the same time there has been a move to classify those entire functions which do not have wandering domains, see for instance [9], [40], [43]. In particular this is the case for functions which have only a finite number of asymptotic or critical values. Such functions are denoted as having *finite type*.

I. N. Baker and A. P. Singh [15] studied the dynamics of composite entire functions $g(z) = a+be^{(2\pi i/c)}$ where a, b, and c are nonzero constants. They obtained the following results.

Theorem 5.1.1 *[15] Let $p(z)$ be a non-constant entire function and let $g(z) = a + be^{(2\pi i/c)}$ where a, b, c are nonzero constants. If $h = g(p)$ has no wandering domains then neither does $p(g)$.*

In particular for a polynomial $p(z)$ it is known in [9] that $e^{p(z)}$ has no wandering domains and consequently it follows immediately that $p(e^z)$ has no wandering domains (also proved in [9]). As another application of the above theorem, they showed that $e^{e^z} - e^z$ has no wandering domains. Also $e^{e^z} - e^z$ is not of finite type and so provided an example of entire function which is not of finite type and having no wandering domains.

Theorem 5.1.2 *[15] Let g be a transcendental entire function having at least one fixed point. Then there exists an entire function f such that $g(f)$ has a wandering domain.*

The proof of this theorem is based on the proof of theorems in [13], [11] and so on the method of construction of wandering domain first introduced by A. E. Eremenko and M. Lü Lyubich [38].

W. Bergweiler and Y. Wang [26] studied the dynamics of composite entire functions without assuming any special forms of functions. The following are results obtained in [26]:

Theorem 5.1.3 *[26] Let f and g be nonlinear entire functions and $z \in \mathbb{C}$. Then $z \in J(f \circ g)$ if and only if $g(z) \in J(g \circ f)$.*

It follows that if U_0 is a component of $F(f \circ g)$, then $g(U_0)$ is contained in a component V_0 of $F(g \circ f)$. The result of M. Heins [46] already mentioned implied that $V_0 \backslash g(U_0)$ contains at most one point.

Theorem 5.1.4 *[26] Let f and g be nonlinear entire functions. Let U_0 be a component of $F(f \circ g)$ and let V_0 be a component of $F(g \circ f)$ that contains $g(U_0)$. Then*
(i) *U_0 is wandering if and only if V_0 is wandering.*
(ii) *If U_0 is periodic, then so in V_0. Moreover, V_0 is of the same type according to the classification of periodic components as U_0.*
In particular, $f \circ g$ has a wandering domain if and only if $g \circ f$ has a wandering domain.

Several examples of entire functions which have no wandering domains were then constructed by using Theorem 5.1.4.

W. Bergweiler and A. Hinkkanen [25] generalized the results in [15] by considering dynamical connection of transcendental entire functions f and h satisfying

$$g \circ f = h \circ g, \tag{5.33}$$

where g is a continuous and open function of the complex plane into itself. Then we say that f and h are *semiconjugated* (by g) and call g as a *semiconjugacy.* They obtained the following result.

Theorem 5.1.5 *[25] If f and h are transcendental entire functions, if g is a non-constant continuous function and if (5.33) holds, then*

$$g(J(f)) \subset J(h). \tag{5.34}$$

If, in addition, $g(\mathbb{C})$ is an open set and, in particular, if g is an open mapping, then $\mathbb{C}\backslash g(\mathbb{C})$contains at most one point.

The special case when g is entire is important and Theorem 5.1.5 is easy to prove in this case. Theorem 5.1.5 is also easy to prove if we assume that g is open or discrete. Even in the case when g is entire, however, it is not clear whether we also have

$$g^{-1}(J(f)) \subset J(f) \tag{5.35}$$

and thus

$$g^{-1}(J(f)) = J(f). \tag{5.36}$$

If g is homeomorphism of $\mathbb{C}$ onto itself satisfying (5.33), then g is called a *conjugacy.* In this case (5.36) clearly holds. It is also known that (5.36) holds if $g(z) = e^z$ (see [23]). In [26] it was shown that (5.36) holds if g is entire and if there exists an entire function k such that $f = k \circ g$ and $h = g \circ k$. Note that if f and h have this special form, then (5.33) is always satisfied.

W. Bergweiler and A. Hinkkanen [25] also gave the concept of the set $A(f)$ where the iterates of function f tend to ∞ about as fast as possible:
$A(f)$=$\{z \in \mathbb{C}$: there exists $L \in \mathbb{N}$ such that $|f^n(z)| > M(R, f^{n-L})$ for $n > L\}$ is not empty for function f such that

$$\lim_{n\to\infty} \frac{\log\, \log M(R, f^n)}{n} = \infty$$

for all large $R > 0$. They obtained the following result.

Theorem 5.1.6 *[25] Let f and h be transcendental entire functions and let $g : \mathbb{C} \to \mathbb{C}$ be open and continuous such that $g \circ f = h \circ g$. If $A(f) \subset J(f)$ then $g^{-1}(J(h)) = J(f)$. In particular, this is the case if f has no wandering domains.*

Consider the special case when $f = h$ and g is entire. Then $f \circ g = g \circ f$ and we say that f and g *commute*. Theorem 5.1.6 implied that $g^{-1}(J(f)) = J(f)$ which means that $J(f)$ is completely invariant under g. Now $J(g)$ is known to be the smallest closed completely invariant set with at least three points (see, for example [20], p. 67 for the special case of rational functions). Therefore $J(g) \subset J(f)$. They have the following corollary.

Corollary 5.1.7 *[25] Let f and g be commutative transcendental entire functions. If f has no wandering domains or, more generally, if $A(f) \subset J(f)$ then $J(g) \subset J(f)$.*

The conclusion that $J(g) \subset J(f)$ if f has no wandering domains was obtained by J. K. Langley [57] under an additional growth restriction on g. The Corollary 5.1.7 implies that if neither f nor g has wandering domains, then $J(f) = J(g)$. It is conjectured that this remains valid in general, i.e. even if f and g are allowed to have wandering domains. It is known to be true for rational functions (see ([9] section 4), ([41] pp. 364–365) or ([50] p. 143)).

There are other results for commuting entire functions which have a generalization to the situation of a semiconjugacy. One such result is:

Theorem 5.1.8 *[25] Let f and h be entire functions such that f is either transcendental or polynomial of degree at least 2 and h is not the identity mapping. Then they are only countably many entire functions g such that (5.33) holds.*

Of course, if h is the identity function then (5.33) becomes $g \circ f = g$, which is satisfied by all constant functions g but not any non-constant function g unless $f(z) = wz + c$ for some root of unity w and some $c \in \mathbb{C}$ (see, for example [44]). If f has this form, then there are uncountably many non-constant solutions g of the equation $g \circ f = g$.

Theorem 5.1.8 generalizes the result of I. N. Baker in ([4], Theorem 1, p. 244) where it was proved that if f is a given entire function, either transcendental or a polynomial of degree at least two, then there are only countably many entire functions g commuting with f.

Theorem 5.1.9 *[25] Let f and h be transcendental entire functions and let $g : \mathbb{C} \to \mathbb{C}$ be open and continuous such that (5.33) holds. If f has no wandering domains, then h has no wandering domains.*

Theorem 5.1.9 generalizes the results in [26], where the conclusion was obtained if g is entire and if $f = k \circ g$ and $h = g \circ k$ for some entire function k. This was used in [26] to exhibit certain new classes of entire functions with no wandering domains. If $f = k \circ g$ and $h = g \circ k$ as in [26], then , by symmetry, f has wandering domains if and only if h has wandering domains. It is possible that f has wandering domains while h does not. An example is $f(z) = z + e^z + 1 = 2\pi i, g(z) = e^z$ and $h(z) = ze^{z+1}$. In [25], W. Bergweiler and A. Hinkkanen also gave an example which shows that the non-constant continuous function g need not be open or discrete in order to satisfy (5.33), even if $f = h$ so that f there can commute. This example showed that for a given transcendental entire function f there can sometimes be uncountably many non-constant, continuous, and non-constant entire functions g commuting with f (then (5.33) also holds with $f = h$).

J. H. Zheng [85] studied the connections between the Fatou components and the singularities of the inverse function of functions in class **M** satisfying the equation $h \circ f = g \circ h$ where h is meromorphic in $\mathbb{C}$. Several examples of Baker domains and wandering domains of transcendental entire functions which have special properties were also given in [85].

In this thesis, we extend Theorem 5.1.3 and Theorem 5.1.4 to functions meromorphic outside a small set which have certain properties such as those in subclasses of class **M** defined in chapter 2. By using these results, we will give an example of transcendental meromorphic function and function in class **M** which do not have wandering domains or Baker domains.

5.2 Lemmas

In this section, we give several lemmas which will be used in the proof of our main results. Throughout this chapter, we denote $f \circ g$ by fg and $E(f)$ by E_f.

Lemma 5.2.1 *Let f, $g \in \mathbf{M}$. If z_0 is a periodic point of fg, then $g(z_0)$ is a periodic point of gf.*

Proof Let z_0 be a periodic point of period n of fg, namely $(fg)^n(z_0) = z_0$. Then $z_0 \notin E((fg)^n) = \left(\bigcup_{j=0}^{n-1} \left((fg)^j\right)^{-1}(E_g)\right) \bigcup \left(\bigcup_{j=0}^{n-1} \left((gf)^j g\right)^{-1}(E_f)\right)$. Thus, $g(fg)^n(z_0)$ is defined and equal to $g(z_0)$. Since $g(fg)^n(z_0) = (gf)^n(g(z_0))$, it follows that $g(z_0)$ is a periodic point of gf. This completes the proof.

Recall that the singularities of the inverse function of function f in class $\mathbf{M}$, denoted by sing(f^{-1}), is the union of the set of critical values of f, denoted by $CV(f)$, and the set of asymptotic values of f, denoted by $AV(f)$ together with all limit points of $CV(f) \cup AV(f)$. We denote the set of limit points of a set E by E'.

Lemma 5.2.2 *Let f, $g \in \mathbf{M}$. Assume the following conditions hold*

(i) $\infty \in E_f \cap E_g$.

(ii) *If for some $z_0 \in E_{fg}$ and for some path $\gamma(t)$, $0 \le t < 1$, we have $\gamma \cap E_{fg} = \emptyset$ and $\gamma \to z_0$ as $t \to 1$, then $g(\gamma)' \cap (E_f \setminus \{\infty\}) = \emptyset$.*

Then we have

$$CV(fg) \subset CV(f) \cup f(CV(g)),$$
$$AV(fg) \subset AV(f) \cup f(AV(g)),$$

and

$$sing(fg)^{-1} \subset sing(f^{-1}) \cup f(sing(g^{-1})).$$

Proof Let α be a critical value of fg. Then there exists z_0 such that $(fg)'(z_0) = f'(g(z_0))g'(z_0) = 0$ and $(fg)(z_0) = \alpha$. Thus, $z_0 \notin E_g \cup g^{-1}(E_f)$. If $f'(g(z_0)) = 0$, then $g(z_0)$ is a critical point for f and we have $(fg)(z_0) \in CV(f)$. If $g'(z_0) = 0$, then z_0 is a critical point of g and so $g(z_0) \in CV(g)$. Thus, $(fg)(z_0) \in f(CV(g))$.

Therefore, $CV(fg) \subset CV(f) \cup f(CV(g))$. Now let α be an asymptotic value of fg. Then there exists $z_0 \in E_{fg}$ and a path $\gamma(t), 0 \leq t < 1$ such that $\gamma \cap E_{fg} = \emptyset$ and $\gamma \to z_0$ as $t \to 1$ and $(fg)(z) \to \alpha$ along γ.

Case 1: z_0 is finite.

Subcase 1.1: $g(z) \to z_0$ along γ.

In this subcase, α is an asymptotic value of f.

Subcase 1.2: $g(z) \not\to z_0$ along γ and $g(z)$ is eventually bounded along γ (namely, there exists $\delta > 0$ such that $|g(z)|$ is bounded on $\{z \in \gamma : |z - z_0| < \delta\}$).

In this subcase, there exist a sequence $\{z_n\}$ on γ and a finite point w_0 such that $\lim_{n\to+\infty} z_n = z_0$ and $\lim_{n\to+\infty} g(z_n) = w_0$. By (ii), $w_0 \notin E_f$ and it follows that $f(w_0) = \lim_{n\to+\infty} f(g(z_n)) = \alpha$. By (ii) and the fact that poles of f cannot accumulate at a finite point outside E_f, we can find a neighborhood U_{w_0} of w_0 such that $U_{w_0} \cap (E_f \cup P_f) = \emptyset$, where P_f is the set of poles of f (if there exists a sequence w_n of points in E_f such that $\lim_{n\to+\infty} w_n = w_0$, then $w_0 \in \overline{E_f} = E_f$. This is impossible by (ii)). Thus, f is analytic in U_{w_0}. Let $\rho > 0$ be a fixed sufficiently small positive real number. Then for some $\varepsilon > 0$, we have $|f(w) - \alpha| > \varepsilon$ for $w \in \{w : |w - w_0| = \rho\}$. Next, as α is an asymptotic value of $f \circ g$, $|f(g(z)) - \alpha| < \varepsilon$ for all $z \in \{z : |z - z_0| < \delta\}$ on γ, for some $\delta > 0$. In particular, if $|z_n - z_0|$ are sufficiently small, then $|f(g(z)) - \alpha| < \varepsilon$ for all z such that $|z - z_0| < |z_n - z_0|$ and $|g(z_n) - w_0| < \rho$. Thus, $|g(z) - w_0| < \rho$ for all z which is arbitrarily closed to z_0 and hence w_0 is an asymptotic value of g. This gives $\alpha \in f(AV(g))$.

Subcase 1.3: $g(z)$ is not eventually bounded along γ.

In this subcase, there exists a sequence $\{z_n\}$ on γ such that $\lim_{n\to+\infty} z_n = z_0$ and $\lim_{n\to+\infty} g(z_n) = \infty$. If there are infinitely many points z_{n_k} of the sequence z_n such that $g(z_{n_k}) = \infty$, then we modify the path γ slightly so as to avoid the poles of g while preserving all other conditions. Thus, eventually along $\{z_n\}$, g is defined and unbounded; that is, there exists a sequence $\{\alpha_n\}$ on γ such that $\lim_{n\to+\infty} \alpha_n = z_0$, $g(\alpha_n) \neq \infty$ and $\lim_{n\to+\infty} g(\alpha_n) = \infty$. If $g(z) \to \infty$, along γ, then $\alpha \in AV(f)$ since $\infty \in E_f$. Otherwise, there is a sequence β_n on γ such that $\lim_{n\to+\infty} g(\beta_n) = w_0$ for some finite w_0. By (ii), $w_0 \notin E_f$ and it follows that

$f(w_0) = \lim_{n\to+\infty} f(g(\beta_n)) = \alpha$. By the same argument as in Subcase 1.2, we can find a neighborhood U_{w_0} of w_0 such that f is analytic in U_{w_0}. Let $\rho > 0$ be a fixed sufficiently small positive real number. Then for some $\varepsilon > 0$, we have $|f(w) - \alpha| > \varepsilon$ for $w \in \{w : |w - w_0| = \rho\}$. Next, as α is an asymptotic value of $f \circ g$, $|f(g(z)) - \alpha| < \varepsilon$ for all $z \in \{z : |z - z_0| < \delta\}$ on γ, for some constant δ. In particular, if β_n are sufficiently close to z_0 on γ, then $|f(g(z)) - \alpha| < \varepsilon$ for all z beyond β_n on γ and $|g(\beta_n) - w_0| < \rho$. Thus, $|g(z) - w_0| < \rho$ for all z sufficiently close to z_0 on γ. Thus g must be bounded on γ which contradicts to the assumption that $g(z)$ is not eventually bounded along γ. Therefore, this subcase cannot occur at all.

Case 2: $z_0 = \infty$.

Subcase 2.1: $g(z) \to \infty$ along γ.

In this subcase, α is an asymptotic value of f.

Subcase 2.2: $g(z) \not\to \infty$ along γ.

In this subcase, there exists a sequence $\{z_n\}$ on γ and a finite point w_0 such that $\lim_{n\to+\infty} z_n = \infty$ and $\lim_{n\to+\infty} g(z_n) = w_0$. By (ii), $w_0 \notin E_f$ and it follows that $f(w_0) = \lim_{n\to+\infty} f(g(z_n)) = \alpha$. The same argument as in Subcase 1.2 gives $\alpha \in f(AV(g))$.

From Case 1 and Case 2, we conclude that $AV(fg) \subset AV(f) \cup f(AV(g))$. This completes the proof.

Remark 5.2.3 *If f and g are transcendental entire functions, then all assumptions in Lemma 5.2.2 hold.*

5.3 Main results

We are now ready to state and prove our main results.

Theorem 5.3.1 *Let f, $g \in \mathbf{M}$. Assume that $\infty \in E_f \cap E_g$ and every point in $J(fg)$ and $J(gf)$ is a limit point of periodic points of fg and gf, respectively. Then the following statements hold*

(i) *If $z \in J(fg) \backslash E_g$, then $g(z) \in J(gf)$.*

(ii) *If* $g(z) \in J(gf)\backslash E_f$, *then* $z \in J(fg)$.

Proof Let $z \in J(fg)\backslash E_g$. By assumption, there exist periodic points z_k of fg, say $(fg)^{n_k}(z_k) = z_k$ where $z_k \neq z$ such that $z_k \to z$ as $k \to +\infty$. By Lemma 5.2.1, $g(z_k)$ are periodic points of gf and $g(z_k) \neq g(z)$ for all but finitely many k (otherwise, the set $\{w : g(w) - g(z) = 0\}$ has a limit point and hence g is a constant). As z, $z_k \notin E_g$ we have $g(z_k) \to g(z)$ as $k \to +\infty$ and hence $g(z)$ is a limit point of periodic points of gf. It follows that $g(z) \in J(gf)$. Similarly, by interchanging the role of f and g, if $w \in J(gf)\backslash E_f$, then $f(w) \in J(fg)$. Conversely, assume that $g(z) \in J(gf) \setminus E_f$, then $f(g(z)) \in J(fg)$ and by the complete invariance property of the Julia set we obtain $z \in J(fg)$. This completes the proof.

From Theorem 5.3.1, we have

Corollary 5.3.2 *If* U *is a component of* $F(fg)$, *then* $g(U)$ *is contained in a component* V *of* $F(gf)$.

Proof Let U be a component of $F(fg)$. Then $U \cap J(fg) = \emptyset$. We claim that $g(U) \cap J(gf) = \emptyset$. Suppose that $g(U) \cap (J(gf)\backslash E_f) \neq \emptyset$. Then there exists $z_0 \in U$ such that $g(z_0) \in (J(gf)\backslash E_f)$. By Theorem 5.3.1 (ii) we have $z_0 \in J(fg)$ which is impossible. Now if $g(U) \cap E_f \neq \emptyset$, then there exists $z_0 \in U$ such that $g(z_0) \in E_f$. Thus, $z_0 \in g^{-1}(E_f) \subset E_{fg} \subset J(fg)$ which is impossible. Therefore, $g(U) \cap J(gf) = \emptyset$ and hence $g(U)$ is contained in a component V of $F(gf)$. This completes the proof.

Theorem 5.3.3 *Let* f, $g \in \mathbf{M}$. *Assume that* $\infty \in E_f \cap E_g$ *and every point in* $J(fg)$ $J(gf)$ *is a limit point of periodic points of* fg *and* gf, *respectively. Let* U *be a component of* $F(fg)$ *and let* V *be the component of* $F(gf)$ *which contains* $g(U)$. *Then*

(i) U *is a wandering domain if and only if* V *is a wandering domain.*

(ii) *If* U *is periodic, then so is* V. *Moreover,* V *is of the same type according to the classification of periodic components as* U *unless* U *is a Siegel disc or Herman ring where in this case* V *is either a Siegel disc or Herman ring.*

Proof For each $n \in \mathbb{N}$, let U_n be the component of $F(fg)$ which contains $(fg)^n(U)$ and let V_n be the component of $F(gf)$ which contains $(gf)^n(V)$. As $U \cap E_g = \emptyset$ we see that $g((fg)^n(U)) = (gf)^n(g(U))$ which gives $g(U_n) \subset V_n$. By a similar argument used in the proof of Corollary 5.3.2, we may show that $f(V_n) \subset U_{n+1}$. As a result, if $U_m = U_n$, then $V_m = V_n$ and if $V_m = V_n$, then $U_{m+1} = U_{n+1}$. This gives the statement (i) of the theorem. Moreover, if $U_n = U$, then $V_n = V$, namely if U is periodic, then so is V. Assume that $U_n = U$ and for some sequence $\{n_j\}$ we have $(fg)^{n_j}|_U \to \phi$ as $j \to +\infty$ where $\phi \notin E_{fg}$. Let V^* be a domain in V such that a branch $g_V^{-1} : V^* \to U^* \subset U$ of the inverse function of g is defined. Then $(gf)^n|_{V^*} = g(fg)^n g_V^{-1}|_{V^*}$ and hence $(gf)^n(V^*) \to \psi = g\phi g_V^{-1}$. If U is a Siegel disc or Herman ring, then ϕ is a non-constant limit function of $\{(fg)^n\}$ on U, hence ψ is also a nonconstant limit function of $\{(fg)^n\}$ on V and hence V is either a Siegel disc or Herman ring. If U is an attracting domain, then ϕ is a constant limit function lying in $F(fg)$, hence ψ is also a constant limit function lying in $F(gf)$ and V must be an attracting domain. Similarly, if U is a parabolic domain, then so is V. By the same arguments, if V is an attracting or parabolic domain, then so is U_1; and if V is a Siegel disc or Herman ring, then U_1 is either a Siegel disc or Herman ring. It follows that if U is a Baker domain, then so is V. This completes the proof.

5.4 Example

We now give an example of transcendental meromorphic function and function in class **M** which do not have wandering domains or Baker domains.

Example 5.4.1 *Let* $f(z) = e^{iz} + z$ *and* $g(z) = \tan z$. *Then* g *has finite order and has no critical values; hence, by Lemma 2.7.6 and Lemma 2.7.8,* g *has only finitely many asymptotic values. In fact,* $AV(g) = \{-i,\ i\}$. *For* f *we may easily show that* $CV(f) = \left\{i + \left(\frac{\pi}{2} + 2k\pi\right) : k \in \mathbb{Z}\right\}$ *and* f *has no finite asymptotic values. We may show that* $g(CV(f)) = \{-\cot i\}$, *hence, by Lemma 5.2.2,* $AV(gf) \subset \{-i,\ i\}$ *and* $CV(gf) \subset \{-\cot i\}$. *Since* $E_{gf} = E_f \cup f^{-1}(E_g) = \{\infty\}$, gf *is a transcendental meromorphic function on* $\mathbb{C}$ *and* $gf \in \mathbf{K} \cap \mathbf{MS} \subset \mathbf{MSR}$. *By Theorem 2.8.1,* gf

has no wandering domains or Baker domains. We conclude from Theorem 5.3.3 that $fg = e^{i \tan z} + \tan z$ *has no wandering domains or Baker domains. Note that* $CV(fg) = \left\{ i + \frac{\pi}{2} + 2k\pi : k \in \mathbb{Z} \right\}$, *hence* $fg \notin \mathbf{MS}$ *or not even of bounded type.*

Remark 5.4.2 *Theorem 5.3.3 generalizes Theorem 5.1.4 obtained in [26] and in fact we may find other examples of transcendental entire or meromorphic functions which have no wandering domains or Baker domains.*