Chapter 1
Introduction

Let X be a set and $T : X \to X$ a mapping. The solutions we seek are represented by points invariant under T. These are the points satisfying

$$x = Tx. \tag{1.1}$$

Such points are said to be fixed under T or fixed points of T. The set of all solutions of (1.1) is called the fixed point set of T and denoted by $\mathrm{Fix}(T)$. If the mapping T does not have a fixed point we often say that T is fixed point free.

The presence or absence of a fixed point is an intrinsic property of T. One of the first and most celebrated results on this matter is the one proved by Brouwer [7] in 1912.

Theorem 1.0.1. (Brouwer) *If B stands for the closed unit ball of $\mathbb{R}^n$, then each continuous mapping $T : B \to B$ has a fixed point.*

An important generalization of Brouwer's theorem was discovered in 1930 by Schauder [50].

Theorem 1.0.2. (Schauder) *Let X be a Banach space. If K is a nonempty compact convex subset of X, then each continuous mapping $T : K \to K$ has a fixed point.*

The fixed point theorem, generally known as the Banach Contraction Principle, appeared in explicit form in Banach's thesis in 1922 where it was used to establish the existence of a solution for an integral equation. Since then, because of its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis.

Theorem 1.0.3. (Banach [4]) *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction. Then T has a unique fixed point x_0. Moreover, for each $x \in X$, we have that*

$$\lim_{n\to\infty} T^n(x) = x_0.$$

Fixed point theorems for single-valued mappings are useful in the existence theory of differential equations, integral equations, partial differential equations, random differential equations, and in other related areas. It has very fruitful applications in eigenvalue problems as well as in boundary value problems, including approximation theory, variational inequality, and complementarity problems.

Fixed point theory for a multivalued (set-valued) mapping was originally initiated by von Neumann [45] in the study of game theory. In 1941, Kakutani [23] proved a generalization of Brouwer's theorem to multivalued mappings.

Theorem 1.0.4. *If C is a nonempty bounded closed convex subset of $\mathbb{R}^n$ and $T : C \rightarrow FC(C)$ is an upper semicontinuous multivalued mapping, then T has a fixed point.*

The multivalued analogue of Scauder's fixed point theorem was given by Bohnenblust and Karlin [5].

Theorem 1.0.5. *If K is a nonempty compact convex subset of a Banach space and $T : K \rightarrow FC(K)$ is an upper semicontinuous multivalued mapping, then T has a fixed point.*

Nadler [43] gave the following as a multivalued analogue of Banach Contraction Principle.

Theorem 1.0.6. *Let (X, d) be a complete metric space and $T : X \rightarrow FB(X)$ a multivalued contraction mapping. Then T has a fixed point in X.*

The fixed point theory of multivalued nonexpansive mappings is however much more complicated and difficult than the corresponding theory of single-valued nonexpansive mappings. Let X be a Banach space and C a bounded closed convex subset of X and $T : C \rightarrow K(C)$ a multivalued nonexpansive mapping. A very general problem is the following : Does T have a fixed point under the suitable condition on X which assure the existence of fixed point for single-valued mappings? The answer to this question is unknown, but some papers have appeared showing geometrical properties on X which state fixed point results for multivalued mappings. One breakthrough was achieved by T. C. Lim in 1974 by using Edelstein's method of asymptotic centers [16].

Theorem 1.0.7. (Lim [37]) *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and $T : C \rightarrow K(C)$ a nonexpansive mapping. Then T has a fixed point.*

Lim's original proof was later simplified independently by Lim himself [38] and Goebel [18]. It was extended to a nonself mapping which satisfies the inwardness condition independently by Downing and Kirk [15] and by Reich [48].

Theorem 1.0.8. ([15],[48]) *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and $T : C \to K(X)$ a nonexpansive mapping satisfying the inwardness condition:*

$$Tx \subset I_C(x), \ \ x \in C.$$

Then T has a fixed point.

The slightly more general formulation below is due to T. C. Lim [39].

Theorem 1.0.9. (Lim [39]) *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and $T : C \to K(X)$ a nonexpansive mapping satisfying the weak inwardness condition:*

$$Tx \subset \overline{I_C(x)}, \ \ x \in C.$$

Then T has a fixed point.

Another important result for multivalued nonexpansive mappings was obtained by W. A. Kirk and S. Massa in 1990.

Theorem 1.0.10. (Kirk and Massa [31]) *Let C be a nonempty bounded closed convex subset of a Banach space X and $T : C \to KC(C)$ a nonexpansive mapping. Suppose that the asymptotic center in C of each bounded sequence of X is nonempty and compact. Then T has a fixed point.*

Theorem 1.0.10 applies to all $k-$uniformly rotund ($k-$UR) Banach spaces [53] but it does not apply to a nearly uniformly convex (NUC) Banach space [21] as in such a space the asymptotic center of a bounded sequence is not necessarily compact (cf. [35]). However, Dominguez and Lorenzo [14] recently obtained a fixed point theorem for multivalued nonexpansive mappings in such spaces.

In 2001, Xu [54] gave a different proof of Theorem 1.0.9. He also extended the Kirk-Massa theorem to nonself mappings satisfying the inwardness condition.

Theorem 1.0.11. (Xu [54]) *Let C be a nonempty bounded closed convex subset of a Banach space X and $T : C \to KC(X)$ a nonexpansive nonself-mapping which satisfies the inwardness condition. Suppose that the asymptotic center in C of each bounded sequence of X is nonempty and compact. Then T has a fixed point.*

The following theorem is due to Lami Dozo [36]. He obtained a fixed point theorem for multivalued mappings in a Banach space which satisfies Opial's condition [47].

Theorem 1.0.12. (Lami Dozo [36]) *Let X be a Banach space which satisfies Opial's condition. If C is a nonempty convex weakly compact subset of X and $T : C \to K(C)$ is a nonexpansive mapping, then T has a fixed point.*

The purpose of this thesis is to study the existence of fixed points for multivalued nonexpansive mappings in CAT(0) and modular function spaces. In Chapter 2 we collect some basic concepts and results in metric and Banach spaces. In Chapter 3 we give an analog of Lim's theorem in CAT(0) spaces and obtain a common fixed point theorem for a pair of single-valued and multivalued nonexpansive mappings defined on a nonempty bounded closed convex subset of a CAT(0) space. In Chapter 4 we introduce an ultrapower approach to proving fixed point theorems for multivalued nonexpansive mappings in CAT(0) and Banach spaces. In Chapter 5 we obtain a fixed point theorem for multivalued nonexpasive mappings in modular function spaces and apply the result to obtain fixed point theorems for multivalued nonexpansive mappings in the Banach spaces L_1 and l_1.