# Chapter 2
# Basic Concepts

In this chapter we collect information that will be needed for an understanding of the research work. Although details are included in some cases, many of the fundamental principles of functional analysis are merely stated without proof.

## 2.1 Metric spaces

We begin with some basic definitions.

**Definition 2.1.1.** Let $X$ be a set and $d$ a function from $X \times X \to [0, \infty)$ such that for all $x, y, z \in X$ we have
(M1) $d(x, y) = 0$ if and only if $x = y$;
(M2) $d(x, y) = d(y, x)$; and
(M3) $d(x, y) \leq d(x, z) + d(z, y)$.

A function $d$ satisfying the above condition is said to be a distance function or a metric and the pair $(X, d)$ a metric space. We sometimes write $X$ for a metric space $(X, d)$.

**Example 2.1.2.** The real line $\mathbb{R}$ with $d(x, y) = |x - y|$ is a metric space. The metric $d$ is called the usual metric for $\mathbb{R}$.

**Example 2.1.3.** (The Hausdorff metric) Let $(X, d)$ be a metric space and let $FB(X)$ denote the family of all nonempty bounded closed subsets of $X$. For $A \in FB(X)$ and $\rho > 0$ define the $\rho-$neighborhood of $A$ to be the set

$$N_\rho(A) = \{x \in X : \text{dist}(x, A) < \rho\}.$$

where $\text{dist}(x, A) = \inf_{a \in A} d(x, a)$. Now for $A, B \in FB(X)$ set

$$D(A, B) = \inf\{\rho > 0 : A \subset N_\rho(B) \text{ and } B \subset N_\rho(A)\}.$$

Then $(FB(X), D)$ is a metric space, and $D$ is call the Hausdorff metric on $FB(X)$.

**Definition 2.1.4.** A sequence $\{x_n\}$ in a metric space $(X, d)$ is said to be a Cauchy sequence if for each $\varepsilon > 0$, there exists a positive integer $N$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n \geq N$.

**Definition 2.1.5.** A sequence $\{x_n\}$ in $(X, d)$ is said to converge to a point $x \in X$ if for each $\varepsilon > 0$, there exists a natural number $N$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq N$. In this case we write either $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

**Definition 2.1.6.** A metric space $X$ is said to be complete if every Cauchy sequence in $X$ converges to a point in $X$.

**Definition 2.1.7.** Suppose $(X, d)$ and $(Y, \rho)$ are metric spaces. A mapping $T : X \to Y$ is said to be an isometry if $\rho(Tx, Ty) = d(x, y)$ for each $x, y \in X$. If $T$ is surjective, then we say that $X$ and $Y$ are isometric.

The following is an important characterization of closed sets in a metric space.

**Theorem 2.1.8.** *A subset $C$ of a metric space $X$ is closed if and only if*

$$\{x_n\} \subset C \text{ and } \lim_{n\to\infty} x_n = x \Rightarrow x \in C.$$

The following is a characterization of compactness that is quite useful.

**Theorem 2.1.9.** *A subset $K$ of a metric space $X$ is compact if and only if any sequence $\{x_n\}$ in $K$ has a subsequence $\{x_{n_k}\}$ which converges to a point in $K$.*

Finally, we include with another important fact about compactness that will be used repeatedly in what follows.

**Theorem 2.1.10.** *Let $X$ be a compact metric space, and let $f : X \to \mathbb{R}$ be a continuous mapping. Then there is a point $x_0 \in X$ such that*

$$f(x_0) = \inf\{f(x) : x \in X\}.$$

## 2.2 Banach spaces

Let $X$ denote any nonempty set that contains with each of its elements $x$ and each real number $\alpha$ a unique element $\alpha \cdot x$, written as $\alpha x$, called a scalar multiple of $x$. (One could also include complex numbers $\alpha$ as well, but we restrict ourselves here to the real case.) Also assume that for each two elements $x, y \in X$ there exists a unique element $x + y \in X$ called the sum of $x$ and $y$. The system $(X, \cdot, +)$ is called a linear space (over $\mathbb{R}$) if the following conditions are satisfied. Here $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$.

(1) $x + y = y + x$;

(2) $x + (y + z) = (x + y) + z$;

(3) $\alpha(x + y) = \alpha x + \alpha y$;

(4) $x + y = x + z$ implies $y = z$;

(5) $(\alpha + \beta)x = \alpha x + \beta x$;

(6) $(\alpha\beta)x = \alpha(\beta x)$;

(7) $1x = x$.

A finite subset $\{x_1, \ldots, x_n\}$ of a linear space $X$ is said to be linearly independent if for any $\alpha_1 \ldots, \alpha_n \in \mathbb{R}$ with $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ implies $\alpha_1 = \cdots = \alpha_n = 0$. If, in addition, every $x \in X$ is a linear combination of $x_1, \ldots, x_n$, that is $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$ for some $\alpha_1 \ldots, \alpha_n \in \mathbb{R}$, then we say that $X$ has the dimension $n$.

A function $\|\cdot\|$ from a (real) linear space $X$ into $\mathbb{R}$ is called a norm if it satisfies the following properties for all $x, y \in X$ and $\alpha \in \mathbb{R}$:

(1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;

(2) $\|\alpha x\| = |\alpha| \|x\|$;

(3) $\|x + y\| \leq \|x\| + \|y\|$.

From this norm we can define a metric, induced by the norm $\|\cdot\|$, by

$$d(x, y) = \|x - y\|, \quad x, y \in X.$$

A linear space $X$ equipped with the norm $\|\cdot\|$ is called a normed linear space. A normed linear space $(X, \|\cdot\|)$ which is complete is called a Banach space.

A subset $C$ of a Banach space $X$ is said to be convex if $\alpha x + (1 - \alpha)y \in C$ for each $x, y \in C$ and $\alpha \in [0, 1]$.

If $A \subset X$, the set

$$\overline{\text{conv}}(A) = \cap\{C \subset X : C \text{ is closed, convex and } C \supset A\}$$

is called the closed convex hull of $A$. It is not difficult to see that $\overline{\text{conv}}(A)$ is closed and convex.

The modulus of convexity of a Banach space $X$ is the function $\delta : [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\right\}.$$

The characteristic (or coefficient) of convexity of a Banach space $X$ is the number

$$\varepsilon_0 = \varepsilon_0(X) = \sup\{\varepsilon \ge 0 : \delta(\varepsilon) = 0\}.$$

**Definition 2.2.1.** A Banach space $X$ with modulus of convexity $\delta$ is said to be uniformly convex if $\delta(\varepsilon) > 0$ for each $\varepsilon \in (0,2]$, or equivalently, if $\varepsilon_0(X) = 0$.

**Definition 2.2.2.** A Banach space $X$ is said to be strictly convex if

$$\|\frac{1}{2}(x+y)\| < 1,$$

whenever $x$ and $y$ are different points of the unit sphere of $X$.

**Proposition 2.2.3.** *If $C$ is a nonempty closed convex subset of a strictly convex space $X$ and if $T : C \to C$ is nonexpansive, then the set $Fix(T)$ is closed and convex.*

A function $f : X \to \mathbb{R}$ is said to be linear if $f(\alpha x + y) = \alpha f(x) + f(y)$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$. In addition, if there is $M > 0$ such that $|f(x)| \le M\|x\|$ for all $x \in X$, we say that $f$ is a bounded linear functional. It is not difficult to see that the class of all bounded linear functionals of $X$, denoted by $X^*$, is a Banach space equipped with the norm defined by

$$\|f\| = \sup\{|f(x)| : x \in B_X\} = \sup\{|f(x)| : x \in S_X\}$$

where $B_X = \{x \in X : \|x\| \le 1\}$ is the unit ball of $X$ and $S_X = \{x \in X : \|x\| = 1\}$ is the unit sphere of $X$.

The most well-known theorem in Banach space theory is the Hahn-Banach theorem: for each $x \in X$ there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

The topology induced by a norm is too strong in the sense that it has many open sets. Indeed, in order that each bounded sequence in $X$ has a norm convergent subsequence, it is necessary and sufficient that $X$ be finite dimensional. This fact leads us to consider other weaker topologies on normed spaces which are related to the linear structure of the spaces to search for subsequential extraction principles. So it is worthwhile to define the weaker topology for a Banach space $X$. We say that a net $(x_\alpha)$ in $X$ converges weakly to $x$, denoted by

$$w - \lim_\alpha x_\alpha = x,$$

if $\lim_\alpha f(x_\alpha) = f(x)$ for all $f \in X^*$. A subset $K$ of $X$ is weakly closed if it is closed in the weak topology, that is, if it contains the weak limit of each of its weakly convergent nets. The weakly open sets are now taken as those sets whose complements are weakly closed. The resulting topology on $X$ is called the weak topology on $X$. Sets which are compact in this topology are said to be weakly compact.

It is important to know that the weak topology on a Banach space is a Hausdorff topology, and that weak limits are unique. This is because the functionals in $X^*$ separate points in $X$, that is, given any two points $x \neq y \in X$ there exists an $f \in X^*$ such that $f(x) \neq f(y)$. This is an another consequence of the Hahn-Banach Theorem.

For $x \in X$ and $f \in X^*$ define $i(x)(f) = f(x)$. It is easily seen that $i(x) \in X^{**}$ and that, in fact, the mapping $i : X \to X^{**}$ is an isometric isomorphism, called the canonical embedding of $X$ into $X^{**}$. If $i(X) = X^{**}$, then $X$ is said to be reflexive.

Analogously we can also consider the weak convergence in $X^*$. Moreover, there is an other important mode of convergence: if $(x_n^*)$ is a sequence in the dual space $X^*$ and $x^* \in X^*$, then we say that $x_n^*$ converges $*$-weakly to $x^*$, denoted by $x_n^* \xrightarrow{w^*} x^*$, if $x_n^*(x) \to x^*(x)$ for all $x \in X$. By the definition, we see that norm convergence implies weak convergence and, in turn, implies weak* convergence in the dual space.

We now collect for later using some well-known properties of the weak and weak* topology.



**Proposition 2.2.4.** *A convex subset $C$ of a Banach space is weakly closed if and only if it is closed.*

The above leads to the following.

**Proposition 2.2.5.** *If a subset $C$ of a Banach space is weakly compact, then $\overline{conv}(C)$ is also weakly compact.*

Neither of the above facts holds for the weak* topology. However, the following holds for both topologies.

**Proposition 2.2.6.** *If $C \subset X$ is weakly compact (or weak* compact if $X$ is a dual space), then $C$ is bounded.*

The following fact holds only for the weak* topology (except, of course, in reflexive spaces).

**Proposition 2.2.7.** (Alaoglu's Theorem) *The unit ball $B_{X^*}$ (hence any ball) in a dual space $X^*$ is always compact in the weak* topology.*

The following says that in the weak topology compactness is equivalent to sequential compactness. This fact holds also for the weak* topology on $X^*$ if $X$ is separable, because in this case the weak* topology is metrizable, but it does not holds in general for the weak* topology.

**Proposition 2.2.8.** (Eberlein-Smulian Theorem) *For any weakly closed subset $A$ of a Banach space the following are equivalent.*

(1) *Each sequence $(x_n)$ in $A$ has a subsequence which converges weakly to a point of $A$.*

(2) *Each net $(x_\alpha)$ in $A$ has a subnet which converges weakly to a point of $A$.*

(3) *$A$ is weakly compact.*

The following lists are the several properties which characterize reflexivity.

**Proposition 2.2.9.** *For a Banach space $X$ the following are equivalent.*

(1) *$X$ is reflexive.*

(2) *$X^*$ is reflexive.*

(3) *$B_X$ is weakly compact in $X$.*

(4) *Any bounded sequence in $X$ has a weakly convergent subsequence.*

(5) *For any $f \in X^*$ there exists $x \in B_X$ such that $f(x) = \|f\|$.*

(6) *For any bounded closed convex subset $C$ of $X$ and any $f \in X^*$ there exists $x \in C$ such that $f(x) = \sup\{f(y) : y \in C\}$.*

(7) *If $(C_n)$ is any descending sequence of nonempty bounded closed convex subsets of $X$, then $\cap_{n=1}^{\infty} C_n \neq \emptyset$.*

We conclude this section by noting that Property (7) above offers a quick way, which we will not prove here, to confirm the following fact.

**Theorem 2.2.10.** *If $X$ is a uniformly convex Banach space, then $X$ is reflexive.*

## 2.3 Multivalued mappings

Let $X$ be a Banach space and $C$ a nonempty subset of $X$. We shall denote by $F(C)$ the family of nonempty closed subsets of $C$, by $FB(C)$ the family of nonempty bounded closed subsets of $C$, by $FC(C)$ the family of nonempty closed convex subsets of $C$, by $K(C)$ the family of nonempty compact subsets of $C$, and by $KC(C)$ the family of nonempty compact convex subsets of $C$. Let $D(\cdot,\cdot)$ be the Hausdorff distance on $F(X)$, i.e.,

$$\begin{aligned} D(A,B) &= \inf\{\rho > 0 : A \subseteq N_\rho(B) \text{ and } B \subseteq N_\rho(A)\} \\ &= \max\left\{\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\right\} \end{aligned}$$

where $\operatorname{dist}(a,B) = \inf\{\|a-b\| : b \in B\}$ is the distance from the point $a$ to the subset $B$.

**Definition 2.3.1.** A multivalued mapping $T : C \to F(X)$ is said to be a contraction if there exists a constant $k \in [0,1)$ such that

$$D(Tx,Ty) \le k\|x-y\|, \quad x, y \in C. \tag{2.2}$$

In this case $T$ is said to be $k-$contractive. If (2.2) is valid when $k = 1$, then $T$ is called nonexpansive. A point $x$ is a fixed point for a multivalued mapping $T$ if $x \in Tx$.

The following method and results deal with the concept of asymptotic centers. For a bounded sequence $\{x_n\}$ in a Banach space $X$ and $C$ a bounded subset of $X$ we associate the number

$$r(C,\{x_n\}) = \inf\{\limsup_{n\to\infty} \|x_n - x\| : x \in C\}$$

and the set

$$A(C,\{x_n\}) = \{x \in C : \limsup_{n\to\infty} \|x_n - x\| = r(C,\{x_n\})\}.$$

$r(C,\{x_n\})$ and $A(C,\{x_n\})$ are called, respectively, the asymptotic radius and the asymptotic center of $\{x_n\}$ relative to $C$.

If $X$ is reflexive and $C$ is closed and convex, then $A(C,\{x_n\})$ is always a nonempty closed convex subset of $C$ for any bounded sequence $\{x_n\}$ in $X$. To see this observe that for each $\varepsilon > 0$ the set

$$C_\varepsilon = \{x \in C : \limsup_{n\to\infty} \|x_n - x\| \le r(C,\{x_n\}) + \varepsilon\}$$

is nonempty by definition of $r(C, \{x_n\})$ and straightforward argument shows that each of the set $C_\varepsilon$ is closed and convex. Hence

$$A(C, \{x_n\}) = \cap_{\varepsilon>0} C_\varepsilon,$$

and the letter set is nonempty by weak compactness of $C$.

Clearly, $A(C, \{x_n\})$ is a nonempty weakly compact convex set as $C$ is (cf. [19]).

**Definition 2.3.2.** A bounded sequence $\{x_n\}$ is said to be regular relative to a bounded subset $C$ of a Banach space $X$ if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$; further, $\{x_n\}$ is said to be asymptotically uniform relative to $E$ if $A(C, \{x_n\}) = A(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

**Lemma 2.3.3.** ([18], [38]) *Let $\{x_n\}$ be a bounded sequence in a Banach space $X$ and $C$ a bounded subset of $X$. Then*
*(i) there always exists a subsequence of $\{x_n\}$ which is regular relative to $C$;*
*(ii) if $C$ is seperable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform relative to $C$.*

**Lemma 2.3.4.** *If $C$ is a nonempty bounded closed convex subset of a uniformly convex Banach space $X$, then for any bounded sequence $\{x_n\}$ in $X$, the asymptotic center $A(C, \{x_n\})$ consists of exactly one point.*

**Definition 2.3.5.** Let $C$ be a nonempty closed subset of a Banach space $X$. The inward set of $C$ at $x \in C$ is given by

$$I_C(x) = \{x + \lambda(y - x) : \lambda \geq 1, \, y \in C\}.$$

In case $C$ is a nonempty closed convex subset of a Banach space $X$, we have

$$I_C(x) = \{x + \lambda(y - x) : \lambda \geq 0, \, y \in C\}.$$

A multivalued mapping $T : C \to 2^X \backslash \emptyset$ is said to be inward (resp. weakly inward) on $C$ if

$$Tx \subset I_C(x) \;\; (\text{resp. } Tx \subset \overline{I_C(x)}) \;\; \text{for all } x \in C.$$

## 2.4 Ultrapower techniques

Ultrapowers of a Banach space are proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. We recall some basic facts about the ultrapowers. Let $\mathcal{F}$ be a filter on $\mathbb{N}$, that is $\mathcal{F} \subset 2^{\mathbb{N}}$, satisfying:

(1) If $A \in \mathcal{F}$ and $A \subset B \subset \mathbb{N}$, then $B \in \mathcal{F}$.

(2) If $A$, $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

A filter $\mathcal{U}$ on $\mathbb{N}$ is called an ultrafilter if it is maximal with respect to the ordering of filters on $\mathbb{N}$ given by set-inclusion. That is, if $\mathcal{U} \subset \mathcal{F}$ and $\mathcal{F}$ is a filter on $\mathbb{N}$, then $\mathcal{F} = \mathcal{U}$. An ultrafilter is called trivial if it is of the form $\{A : A \subset \mathbb{N}, n_0 \in A\}$ for some fixed $n_0 \in \mathbb{N}$, otherwise, it is called nontrivial.

Let $\{x_n\}$ be a sequence in a Hausdorff topological space $X$ and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$. The sequence $\{x_n\}$ is said to converge to $x$ with respect to $\mathcal{U}$, denoted by

$$\lim_{\mathcal{U}} x_n = x,$$

if for each neighborhood $U$ of $x$, $\{n \in \mathbb{N} : x_n \in U\} \in \mathcal{U}$.
Limits along $\mathcal{U}$ are unique and if $\{x_n\}$ is a bounded sequence in $\mathbb{R}$, then

$$\liminf_{n\to\infty} x_n \le \lim_{\mathcal{U}} x_n \le \limsup_{n\to\infty} x_n.$$

Moreover, if $E$ is a closed subset of $X$ and $\{x_n\} \subset E$, then $\lim_{\mathcal{U}} x_n$ belongs to $E$ whenever it exists.

We will use the fact that

(1) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset \mathbb{N}$, either $A \in \mathcal{U}$ or $\mathbb{N} \backslash A \in \mathcal{U}$.

(2) If $X$ is compact, then the $\lim_{\mathcal{U}} x_n$ of a sequence $\{x_n\}$ in $X$ always exists and is unique.

(3) Suppose $\{x_n\}$ converges to $x$ in the topology of the space $X$. Then $\{x_n\}$ converges to $x$ with respect to any ultrafilter $\mathcal{U}$.

(4) Let $X$ be a linear topological vector space. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\lim_{\mathcal{U}} x_n$ and $\lim_{\mathcal{U}} y_n$ exist. Then

$$\lim_{\mathcal{U}}(x_n + y_n) = \lim_{\mathcal{U}} x_n + \lim_{\mathcal{U}} y_n$$

and

$$\lim_{\mathcal{U}} \alpha x_n = \alpha \lim_{\mathcal{U}} x_n,$$

for any scalar $\alpha \in \mathbb{R}$.

Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{U}$ a nontrivial ultrafilter on $\mathbb{N}$. Consider the Banach space

$$l_\infty(X) = \{x = \{x_n\} \subset X : \sup_n \|x_n\| < \infty\}.$$

The subset defined by

$$N_{\mathcal{U}} = \{x = \{x_n\} \in l_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\}$$

is a closed linear subspace of $l_\infty(X)$.

The Banach space ultrapower of $X$ (over $\mathcal{U}$) is defined to be the quotient space

$$X_{\mathcal{U}} = l_\infty(X)/N_{\mathcal{U}}$$

equipped with the quotient norm. When it is not necessary to mention the ultrafilter, we write $\widetilde{X}$ instead of $X_{\mathcal{U}}$. The equivalence class of an element $x = \{x_n\} \in l_\infty(X)$ is denoted by $\tilde{x} = [\{x_n\}]$. It follows from (2) above and the definition of the quotient norm that

$$\|\tilde{x}\| = \|[\{x_n\}]\| = \lim_{\mathcal{U}} \|x_n\|.$$

The mapping $\mathcal{J}\colon X \to \widetilde{X}$ defined by

$$\mathcal{J}(x) = [\{x, x, x, ..\}] = [\{x_n\}], \quad \text{where } x_n = x \text{ for all } n \in \mathbb{N}$$

is an isometric embedding of $X$ into $\widetilde{X}$. Using the map $\mathcal{J}$, one may identify $X$ with $\mathcal{J}(X)$ seen as a subspace of $\widetilde{X}$. When it is clear we will omit mention of the map $\mathcal{J}$ and simply regard $X$ as a subspace of $\widetilde{X}$.

If $C$ is a subset of a Banach space $X$, we associate to it subset $\widetilde{C}$ of $\widetilde{X}$ defined by

$$\widetilde{C} = \{[\{x_n\}] : x_n \in C \text{ for each } n \in \mathbb{N}\}.$$

For each $x \in X$, we let $\dot{x} = [\{x_n\}]$ where $x_n = x$ for each $n \in \mathbb{N}$, and let $\dot{X}$ and $\dot{C}$ denote the respective canonical isometric copy of $X$ and $C$ in $\widetilde{X}$.

The following properties hold:

(1) If $C$ is convex, then $\widetilde{C}$ and $\dot{C}$ are convex.

(2) If $C$ is closed, then $\widetilde{C}$ and $\dot{C}$ are closed.

(3) If $C$ is compact, then $\widetilde{C}$ and $\dot{C}$ are compact. Moreover, $\widetilde{C} = \dot{C}$.

(4) If $C$ is bounded, then $\widetilde{C}$ and $\dot{C}$ are bounded. Moreover, $\mathrm{diam}(\widetilde{C}) = \mathrm{diam}(\dot{C}) = \mathrm{diam}(C)$.

**Theorem 2.4.1.** *Let $X$ be a Banach space and $\mathcal{U}$ a nontrivial ultrafilter on $\mathbb{N}$. Then the following statements are equivalent :*

(1) *$\widetilde{X}$ is strictly convex.*

(2) *$\widetilde{X}$ is uniformly convex.*

(3) *$X$ is uniformly convex.*

References for more detailed treatment and proofs of all the result here stated are [1], [27] and [51].