Chapter 3

Lim's Theorems for Multivalued Mappings in CAT(0) Spaces

Let X be a complete CAT(0) space. We prove that, if C is a nonempty bounded closed convex subset of X and $T : C \to K(X)$ a nonexpansive mapping satisfying the weakly inward condition, i.e., there exists $p \in C$ such that $\alpha p \oplus (1-\alpha)Tx \subset \overline{I_C(x)}$ $\forall x \in C$, $\forall \alpha \in [0,1]$, then T has a fixed point. In Banach spaces, this is a result of Lim [39]. The related result for unbounded $\mathbb{R}$−trees is given.

3.1 Introduction

In 1980 and 2001, Lim [39] and respectively Xu [54] had proved differently the same result concerning the existence of a fixed point for a nonself nonexpansive compact valued mapping defining on a bounded closed convex subset of a uniformly convex space and satisfying the weak inward condition. While Lim used the method of asymptotic radius, Xu used his characterization of uniform convexity. Recently in 2003, Bae [3] considered a closed valued mapping defined on a closed subset of a complete metric space. It was shown that if the mapping is weakly contractive and is metrically inward, then it has a fixed point.

Having all these results, we are interested in extending the Lim-Xu's result to a special kind of metric spaces, namely, CAT(0) spaces. Our proofs follow the ideas of the proofs in Lim [39], Bae [3], and Xu [54].

In Section 3.2, we give some basic notions and in Section 3.3 and Section 3.4 we prove our results.

3.2 Preliminaries

Given a metric space X, one way to describe a metric space ultrapower $\widetilde{X}$ of X is to first embed X as a closed subset of a Banach space Y (see, e.g., [46]), then let $\widetilde{Y}$ denote a Banach space ultrapower of Y relative to some nontrivial ultrafilter $\mathcal{U}$. Then take

$$\widetilde{X} := \left\{ \tilde{x} = [\{x_n\}] \in \widetilde{Y} : x_n \in X \text{ for each } n \right\}.$$

One can then let $\tilde{d}$ denote the metric on $\widetilde{X}$ inherited from the ultrapower norm $\|\cdot\|_{\mathcal{U}}$ in $\widetilde{Y}$. If X is complete, then so is $\widetilde{X}$ since $\widetilde{X}$ is a closed subset of the Banach space $\widetilde{Y}$. In particular, the metric $\tilde{d}$ on $\widetilde{X}$ is given by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{\mathcal{U}} \|x_n - y_n\| = \lim_{\mathcal{U}} d(x_n, y_n),$$

with $\{u_n\} \in [\{x_n\}]$ if and only if $\lim_{\mathcal{U}} \|x_n - u_n\| = 0$.

We present now a brief discussion on CAT(0) spaces (see Bridson and Haefliger [6] and Burago et al. [9] for more details). Although CAT(κ) spaces are defined for all real numbers κ, we restrict ourselves to the case that $\kappa = 0$.

A metric space is a CAT(0) space (the term is due to M. Gromov – see, e.g., [6], p. 159) if it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. For a precise definition and a detailed discussion of the properties of such spaces will be stated below. It is well-known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include the classical hyperbolic spaces, Euclidean buildings (see [8]), the complex Hilbert ball with a hyperbolic metric (see [20]; also inequality (4.3) of [49] and subsequent comments).

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. Obviously, c is an isometry and $d(x, y) = l$. The image of c is called a geodesic segment joining x and y and, when unique, denoted $[x, y]$. A metric space is said to be a geodesic space if any two of its points are joined by a geodesic segment. If there is exactly one geodesic segment joining x to y for all $x, y \in X$, we say that (X, d) is uniquely geodesic.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for a geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

(X, d) is said to be a CAT(0) space if all geodesic triangles satisfy the CAT(0) comparison axiom:

For every geodesic triangle $\triangle$ in X and its comparison triangle $\overline{\triangle}$ in $\mathbb{R}^2$, if $x, y \in \triangle$ and $\bar{x}, \bar{y}$ are their comparison points in $\overline{\triangle}$ respectively, then

$$d(x,y) \leq d(\bar{x},\bar{y}).$$

Let X be a CAT(0) space, and let C be a nonempty closed convex subset of X. The following facts will be needed:

(i) (X, d) is uniquely geodesic.

(ii) $(\widetilde{X}, \tilde{d})$ is a CAT(0) space.

(iii) (X, d) satisfies the (CN) inequality:

$$d(x,y_0)^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2$$

for all $x, y_1, y_2 \in X$ and y_0 the midpoint of the segment $[y_1, y_2]$.
Note that the converse is also true. In fact, a geodesic metric space is a CAT(0) space if and only if it satisfies (CN)·inequality (cf. [30]).

(iv) Let p, x, y be points in X, let $\alpha \in (0, 1)$, and m_1 and m_2 denote, respectively, the points of $[p, x]$ and $[p, y]$ satisfying

$$d(p, m_1) = \alpha d(p, x) \text{ and } d(p, m_2) = \alpha d(p, y).$$

Then

$$d(m_1, m_2) \leq \alpha d(x, y).$$

(v) For every $x \in X$, there exists a unique point $p(x) \in C$ such that

$$d(x, p(x)) = \text{dist}(x, C).$$

With the same C and $p(x)$, if $x \notin C$, $y \in C$, and $y \neq p(x)$, then $\angle_{p(x)}(x, y) \geq \frac{\pi}{2}$, where $\angle_z(x, y)$ is the Alexandrov angle between the geodesic segments $[z, x]$ and $[z, y]$ for all $x, y, z \in X$ (see [6, p. 176]).

Let (X, d) be a metric space and C a nonempty subset of X. A closed valued mapping $T : C \to 2^X \backslash \emptyset$ is said to be metrically inward ([3]) if for each $x \in C$,

$$Tx \subset MI_C(x)$$

where $MI_C(x)$ is the metrically inward set of C at x defined by

$$MI_C(x) = \{z \in X : z = x \text{ or there exists } y \in C \text{ such that } y \neq x$$

$$\text{and } d(x,z) = d(x,y) + d(y,z)\}.$$

In case X is a Banach space, the inward set of C at x is defined by

$$I_C(x) = \{x + \lambda(y - x) : y \in C, \lambda \geq 1\}.$$

In general, $I_C(x) \subset MI_C(x)$ for each $x \in C$, and the equality may not be true.

We use the notation $(1-\alpha)u \oplus \alpha v, \alpha \in [0,1]$, to denote the points of the segment $[u,v]$ with distance $\alpha d(u,v)$ from u. For $C \subset X$ and a fixed element $p \in C$, $(1-\alpha)p \oplus \alpha C := \{(1-\alpha)p \oplus \alpha v : v \in C\}$. C is said to be convex if for each pair of points $x, y \in C$, we have $[x,y] \subset C$. We also adopt all the notion and definitions of Section 2.3, but with the norm $\|\cdot\|$ replaced by the distance d.

For a nonempty subset C of a CAT(0) space X, it is easy to see that the (metrically) inward set $MI_C(x)$ becomes

$$MI_C(x) = \left(\bigcup \{z : (x,z] \cap C \neq \emptyset\}\right) \bigcup \{x\} := I_C(x).$$

Definition 3.2.1. A multivalued mapping $T : C \to F(X)$ is said to be inward on C if for some $p \in C$,

$$\alpha p \oplus (1-\alpha)Tx \subset I_C(x) \;\; \forall x \in C, \forall \alpha \in [0,1],$$

and weakly inward on E if

$$\alpha p \oplus (1-\alpha)Tx \subset \overline{I_C(x)} \;\; \forall x \in C, \forall \alpha \in [0,1], \tag{3.3}$$

where $\overline{A}$ denotes the closure of a subset A of X.

When C is convex, it is easy to see that

$$I_C(x) = \left(\bigcup \{[x,y] : (x,y] \cap C \neq \emptyset\}\right) \bigcup \{x\}.$$

Note that in a normed space setting, the inward (resp. weakly inward) condition is equivalent to saying that $Tx \subset I_C(x)$ (resp. $Tx \subset \overline{I_C(x)}$) since in this case, $I_C(x)$ is convex. This is also true in $\mathbb{R}$−trees. For a precise definition of an $\mathbb{R}$−tree we refer the readers to Espinola and Kirk [17].

3.3 Lim's theorems

The following simple result is needed.

Proposition 3.3.1. *Let C be a nonempty closed convex subset of a complete CAT(0) space X, $x \in X$, and $p(x)$ the unique nearest point of x in C. Then*

$$d(x,p(x)) < d(x,y) \ \forall y \in \overline{I_C(p(x))} \setminus \{p(x)\}.$$

Proof. Let $y \in \overline{I_C(p(x))} \setminus \{p(x)\}$, there is a sequence $\{y_n\}$ in $I_C(p(x))$ and $y_n \to y$. For all large n we can find $z_n \in (p(x), y_n] \cap C$. Since $z_n \in C$ and $z_n \neq p(x)$, $\angle_{p(x)}(x, z_n) \geq \frac{\pi}{2}$ (see [6, p. 176]). Thus in the comparison triangle $\overline{\triangle}(p(x), x, y_n)$, the angle at $\overline{p(x)}$ is also greater than or equal to $\frac{\pi}{2}$ (see [6, p. 161]). By the law of cosines

$$d(x,p(x))^2 + d(y_n, p(x))^2 \leq d(x, y_n)^2.$$

Taking $n \to \infty$ we obtain

$$d(x,p(x)) < d(x,y).$$

□

One of powerful tools for fixed point theory is the following result.

Theorem 3.3.2. (Caristi [10]) *Assume (M,d) is a complete metric space and $g : M \to M$ is a mapping. If there exists a lower semicontinuous function $\psi : M \to [0,\infty)$ such that*

$$d(x,g(x)) \leq \psi(x) - \psi(g(x)) \ \textit{for any } x \in M,$$

then g has a fixed point.

We can now state our main theorem.

Theorem 3.3.3. *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X and $T : C \to K(X)$ a nonexpansive mapping. Assume T is weakly inward on C. Then T has a fixed point.*

By combining the idea of the proofs in [39], [3], and [54], we thus first establish the following lemma which is an analogue of a result of Lim in [40]. However, in applying the lemma, we choose to use the ultrapower technique which seems to be alternative.

Lemma 3.3.4. *Let C be a nonempty closed subset of a complete CAT(0) space X and $T : C \to F(X)$ $k-$contractive for some $k \in [0,1)$. Assume T satisfies, for all $x \in C$,*

$$Tx \subset \overline{I_C(x)}. \tag{3.4}$$

Then T has a fixed point.

Proof. Let $M = \{(x,z) : z \in Tx,\ x \in C\}$ be the graph of T. Give a metric ρ on M by $\rho((x,z),(u,v)) = \max\{d(x,u), d(z,v)\}$. It is easily seen that (M,ρ) is a complete metric space. Choose $\varepsilon > 0$ so that $\varepsilon + (k+2\varepsilon)(1+\varepsilon) < 1$.

Now define $\psi : M \to [0,\infty)$ by $\psi(x,z) = \frac{d(x,z)}{\varepsilon}$. Then ψ is continuous on M. Suppose that T has no fixed points, i.e., $\operatorname{dist}(x,Tx) > 0$ for all $x \in C$. Let $(x,z) \in M$. By (3.4), we can find $z' \in I_C(x)$ satisfying $d(z,z') < \varepsilon \operatorname{dist}(x,Tx)$. Now choose $u \in (x,z'] \cap C$ and write $u = (1-\delta)x \oplus \delta z'$ for some $0 < \delta \leq 1$. Note that the number δ varies as a function of x. However, for any such δ, we always have

$$\delta\varepsilon + (1-\delta) + (k+2\varepsilon)\delta(1+\varepsilon) < 1. \tag{3.5}$$

Since T is $k-$contractive and $d(x,u) > 0$, we can find $v \in Tu$ satisfying

$$\begin{aligned} d(z,v) &\leq D(Tx,Tu) + \varepsilon d(x,u) \\ &\leq (k+\varepsilon)d(x,u). \end{aligned}$$

Now we define a mapping $g : M \longrightarrow M$ by $g(x,z) = (u,v) \quad \forall (x,z) \in M$. We claim that g satisfies

$$\rho((x,z), g(x,z)) < \psi(x,z) - \psi(g(x,z)) \quad \forall (x,z) \in M. \tag{3.6}$$

Caristi's theorem then implies that g has a fixed point, which contradicts to the strict inequality (3.6) and the proof is complete.

So it remains to prove (3.6).
In fact, it is enough to show that

$$\rho((x,z),(u,v)) < \frac{1}{\varepsilon}\left(d(x,z) - d(u,v)\right).$$

But $d(z,v) \leq d(x,u)$, it only needs to prove that $d(x,u) < \frac{1}{\varepsilon}\left(d(x,z) - d(u,v)\right)$.

Now,

$$\begin{aligned} d(x,u) &= \delta d(x,z') \\ &\leq \delta(d(x,z)+d(z,z')) \\ &\leq \delta(d(x,z)+\varepsilon \text{dist}(x,Tx)) \\ &\leq \delta(d(x,z)+\varepsilon d(x,z)) \\ &\leq \delta(1+\varepsilon)d(x,z). \end{aligned}$$

Therefore

$$d(x,u) \leq \delta(1+\varepsilon)d(x,z). \tag{3.7}$$

It follows that

$$\begin{aligned} d(z,v) &\leq (k+\varepsilon)d(x,u) \\ &\leq (k+\varepsilon)\delta(1+\varepsilon)d(x,z). \end{aligned}$$

Now we let $y=(1-\delta)x \oplus \delta z$, then

$$\begin{aligned} d(u,v) &\leq d(u,y)+d(y,z)+d(z,v) \\ &\leq \delta d(z,z')+(1-\delta)d(x,z)+(k+\varepsilon)\delta(1+\varepsilon)d(x,z) \\ &\leq \delta\varepsilon \text{dist}(x,Tx)+((1-\delta)+(k+\varepsilon)\delta(1+\varepsilon))d(x,z) \\ &\leq \delta\varepsilon d(x,z)+((1-\delta)+(k+\varepsilon)\delta(1+\varepsilon))d(x,z) \\ &\leq (\delta\varepsilon+(1-\delta)+(k+\varepsilon)\delta(1+\varepsilon))d(x,z). \end{aligned}$$

Thus

$$d(u,v) \leq (\delta\varepsilon+(1-\delta)+(k+\varepsilon)\delta(1+\varepsilon))d(x,z) \tag{3.8}$$

(3.7), (3.8), and (3.5) imply that

$$\begin{aligned} \varepsilon d(x,u)+d(u,v) &\leq \varepsilon\delta(1+\varepsilon)d(x,z)+(\delta\varepsilon+(1-\delta)+(k+\varepsilon)\delta(1+\varepsilon))d(x,z) \\ &= (\delta\varepsilon+(1-\delta)+(k+2\varepsilon)\delta(1+\varepsilon))d(x,z) \\ &< d(x,z). \end{aligned}$$

Therefore $d(x,u) < \frac{1}{\varepsilon}(d(x,z)-d(u,v))$ as desired. □

We are now ready to present the proof of Theorem 3.3.3.

Proof of Theorem 3.3.3. For each integer $n \geq 1$, the contraction $T_n : C \to K(X)$ is defined by

$$T_n(x) := \frac{1}{n}p \oplus (1-\frac{1}{n})Tx, \ x \in C,$$

where $p \in C$ is the existing point satisfying the weakly inward condition (3.3). Weak inwardness of T implies that such T_n satisfies the condition (3.4) in Lemma 3.3.4 and in turn it guarantees that T_n has a fixed point $x_n \in C$. Clearly,

$$\operatorname{dist}(x_n, Tx_n) \leq \frac{1}{n-1}\operatorname{diam}(C) \to 0.$$

Let $\widetilde{X}$ be a metric space ultrapower of X and

$$\dot{C} = \{\dot{x} = [\{x_n\}] : x_n \equiv x \in C\}.$$

Then $\dot{C}$ is a nonempty closed convex subset of $\widetilde{X}$. Since T is compact-valued, we can take $y_n \in Tx_n$ such that

$$d(x_n, y_n) = \operatorname{dist}(x_n, Tx_n), n \geq 1.$$

Let $\tilde{x} = [\{x_n\}]$ and $\tilde{y} = [\{y_n\}]$, then $\tilde{x} = \tilde{y}$. Since $\dot{C}$ is a closed convex subset of a complete CAT(0) space $\widetilde{X}$, $\tilde{x}$ has a unique nearest point $\dot{v} \in \dot{C}$, i.e., $\tilde{d}(\tilde{x}, \dot{v}) = \operatorname{dist}(\tilde{x}, \dot{C})$. As Tv is compact, we can find $v_n \in Tv$ satisfying

$$d(y_n, v_n) = \operatorname{dist}(y_n, Tv) \leq D(Tx_n, Tv).$$

It follows from the nonexpansiveness of T that

$$d(y_n, v_n) \leq d(x_n, v).$$

Let $\tilde{v} = [\{v_n\}]$, then

$$\tilde{d}(\tilde{y}, \tilde{v}) \leq \tilde{d}(\tilde{x}, \dot{v}).$$

Since $\tilde{x} = \tilde{y}$, we have

$$\tilde{d}(\tilde{x}, \tilde{v}) \leq \tilde{d}(\tilde{x}, \dot{v}). \tag{3.9}$$

Because of the compactness of Tv, there exists $w \in Tv$ such that $w = \lim_{\mathcal{U}} v_n$. It follows that $\tilde{v} = \dot{w}$. This fact and (3.9) imply

$$\tilde{d}(\tilde{x}, \dot{w}) \leq \tilde{d}(\tilde{x}, \dot{v}). \tag{3.10}$$

Since $\dot{w} \in \overline{I_{\dot{C}}(\dot{v})}$ as $w \in \overline{I_C(v)}$, (3.10), and Proposition 3.3.1 then imply that $\dot{w} = \dot{v}$. So $v = w \in Tv$ which then completes the proof. □

As an immediate consequence of Theorem 3.3.3, we obtain

Corollary 3.3.5. *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X and $T : C \to K(C)$ a nonexpansive mapping. Then T has a fixed point.*

As we have observed at the end of Definition 3.2.1, we can restate Theorem 3.3.3 for $\mathbb{R}-$trees as follow:

Corollary 3.3.6. *Let X be a complete $\mathbb{R}-$tree, C a nonempty bounded closed convex subset of X, and $T : C \to K(X)$ a nonexpansive mapping. Assume that $Tx \subset \overline{I_C(x)}$, $\forall x \in C$. Then T has a fixed point.*

Finally, as a consequence of Theorem 4.3 of [17] and the idea given in the proof of Theorem 3.3.3, we can relax the boundedness condition and the compactness of the values of a multivalued self mapping T for $\mathbb{R}-$trees:

Corollary 3.3.7. *Let (X, d) be a complete $\mathbb{R}-$tree, and suppose C is a closed convex subset of X which does not contain a geodesic ray, and $T : C \to FC(C)$ a nonexpansive mapping. Then T has a fixed point.*

Proof. By [6, p. 176], for each $x \in C$, there exists a unique point $p(x) \in Tx$ such that

$$d(x, p(x)) = \text{dist}(x, Tx).$$

So we have defined a mapping $p : C \to C$. The nonexpansiveness of T and the convexity of Tx imply that p is a nonexpansive mapping. By [17, Theorem 4.3], there exists $z \in C$ such that $z = p(z) \in Tz$ which then completes the proof. □

3.4 A common fixed point theorem

We consider in this section a common fixed point of nonexpansive mappings. Let $t : C \to C$ and $T : C \to 2^X \backslash \emptyset$. t and T are said to be commuting if $ty \in Ttx$ $\forall y \in Tx \cap C$, $\forall x \in C$. If C is a nonempty bounded closed convex subset of X and t is nonexpansive, we know that $\text{Fix}(t)$ is a nonempty bounded closed convex subset of C (see [30, Theorem 12]).

Theorem 3.4.1. *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X, and let $t : C \to C$ and $T : C \to KC(C)$ be nonexpansive. Assume that for some $p \in Fix(t)$,*

$$\alpha p \oplus (1 - \alpha)Tx \ \text{ convex } \forall x \in C, \forall \alpha \in [0, 1]. \tag{3.11}$$

If t and T are commuting, then there exists a point $z \in C$ such that $tz = z \in Tz$.

Proof. Let $A = \text{Fix}(t)$. Since $ty \in Ttx = Tx$ for each $x \in A$ and $y \in Tx$, Tx is invariant under t for each $x \in A$, and again by [30, Theorem 12], $Tx \cap A \neq \emptyset$.

Let $\widetilde{X}$ be an ultrapower of X and let $p \in A$ satisfying (3.11). As before we define for each $n \geq 1$ the contraction $T_n : A \to KC(C)$ by

$$T_n(x) := \frac{1}{n}p \oplus (1 - \frac{1}{n})Tx,\ x \in A.$$

Convexity of A implies $T_n(x) \cap A \neq \emptyset$. Lemma 3.4.2 below shows that T_n has a fixed point $x_n \in A$. Let y_n be the unique point in Tx_n such that $d(x_n, y_n) = \text{dist}(x_n, Tx_n)$. Let $\tilde{x} = [\{x_n\}]$ and $\tilde{y} = [\{y_n\}]$, then $\tilde{x} = \tilde{y}$ since $\text{dist}(x_n, Tx_n) \to 0$ as $n \to \infty$. Now,

$$d(x_n, ty_n) = d(tx_n, ty_n) \leq d(x_n, y_n) = \text{dist}(x_n, Tx_n).$$

Since $y_n \in Tx_n$, we have $ty_n \in Ttx_n = Tx_n$ and thus the uniqueness of y_n implies that $ty_n = y_n$. So $y_n \in Tx_n \cap A$. Since $\dot{A}$ is a closed convex subset of the complete CAT(0) space $\widetilde{X}$, there exists a unique point $\dot{z} \in \dot{A}$ such that

$$\tilde{d}(\tilde{x}, \dot{z}) = \text{dist}(\tilde{x}, \dot{A}).$$

For each n there exists a unique point $z_n \in Tz$ such that

$$d(y_n, z_n) = \text{dist}(y_n, Tz).$$

As before we see that $z_n \in Tz \cap A$. By the compactness of $Tz \cap A$, we can find $w \in Tz \cap A$ such that $\lim_{\mathcal{U}} z_n = w$. Let $\tilde{z} = [\{z_n\}]$, then $\tilde{z} = \dot{w}$.
Observe that

$$d(y_n, z_n) = \text{dist}(y_n, Tz) \leq D(Tx_n, Tz) \leq d(x_n, z).$$

Therefore $\tilde{d}(\tilde{y}, \tilde{z}) \leq \tilde{d}(\tilde{x}, \dot{z})$. Since $\tilde{y} = \tilde{x}$ and $\tilde{z} = \dot{w}$,

$$\tilde{d}(\tilde{x}, \dot{w}) \leq \tilde{d}(\tilde{x}, \dot{z}) = \text{dist}(\tilde{x}, \dot{A}).$$

The uniqueness of $\dot{z}$ implies that $\dot{w} = \dot{z}$. Therefore $tz = z = w \in Tz$. □

It remains to prove our Lemma.

Lemma 3.4.2. *Let A be as above and $T : A \to FC(C)$ be $k-$contractive for some $k \in [0, 1)$. Assume that T satisfies, for all $x \in A$,*

$$Tx \cap A \neq \emptyset.$$

Then T has a fixed point.

Proof. The proof is similar to the proof of Lemma 3.3.4. Let $M = \{(x, z) : z \in Tx \cap A,\ x \in A\}$ and define a metric ρ on M by $\rho((x,z),(u,v)) = \max\{d(x,u), d(z,v)\}$. Again (M, ρ) is a complete metric space. Choose $\varepsilon > 0$ so that $\varepsilon + k < 1$.

Define $\psi : M \to [0, \infty)$ by $\psi(x,z) = \frac{d(x,z)}{\varepsilon}$. Suppose that $x \neq z$ for all $(x,z) \in M$. Since Tz is a closed convex subset of X, there exists a unique point $v \in Tz$ such that

$$d(z,v) = \operatorname{dist}(z, Tz).$$

Bearing in mind that $A = \operatorname{Fix}(t)$, thus by the commuting assumption and the uniqueness of v, we have $v \in Tz \cap A$.

Now we define a mapping $g : M \to M$ by $g(x,z) = (z,v)$ for each $(x,z) \in M$.
We claim that g satisfies

$$\rho((x,z), g(x,z)) < \psi(x,z) - \psi(g(x,z)) \quad \forall (x,z) \in M. \tag{3.12}$$

Again by applying the Caristi's theorem we obtain a contradiction. Thus T has a fixed point.

So it remains to prove (3.12).
The fact that $d(z,v) = \operatorname{dist}(z, Tz) \leq D(Tx, Tz) \leq kd(x,z)$, we have

$$\begin{aligned} \varepsilon d(x,z) + d(z,v) &\leq \varepsilon d(x,z) + kd(x,z) \\ &= (\varepsilon + k)d(x,z) \\ &< d(x,z). \end{aligned}$$

Therefore $\rho((x,z),(z,v)) < \frac{1}{\varepsilon}(d(x,z) - d(z,v))$, and (3.12) is verified. □