Chapter 4

Nonexpansive Set-valued Mappings in Metric and Banach Spaces

We extend recent homotopy results of Sims, Xu, and Yuan for set-valued maps to a CAT(0) setting. We also introduce an ultrapower approach to proving fixed point theorems for nonexpansive set-valued mappings, both in this setting and in Banach spaces. This method provides an efficient way of recovering all of the classical Banach space results.

4.1 Introduction and Preliminaries

In [52], Sims, Xu and Yuan obtain homotopic invariance theorems for nonexpansive set-valued mappings in Banach spaces having Opial's property. They base their results on the fact that if T is a set-valued nonexpansive mapping having nonempty compact values, then the demiclosedness principle for $I-T$ is valid in such spaces. (If C is a nonempty closed convex subset of a Banach space X and if T maps points of C to nonempty closed subsets of X, then T is said to be demiclosed on C if the graph of T is closed in the product topology of $(X,\sigma)\times(X,\|\cdot\|)$ where σ and $\|\cdot\|$ denote the weak and strong topologies, respectively). One objective of this paper is to show that the results of [52] extend to CAT(0) spaces despite the fact that no weak topology is present. The results we obtain set-valued analogs of single-valued results found in [29].

We also introduce a new approach to the classical fixed point theorems for nonexpansive mappings in Banach spaces by reformulating the arguments in an ultrapower context. This approach seems to illuminate many underlying ideas.

Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X and for $x\in X$ set

$$r(x,\{x_n\})=\limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\})=\inf\{r(x,\{x_n\}):x\in X\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that every bounded sequence in a Banach space has a regular subsequence (see, e.g., [19], p. 166). The proof is metric in nature and carries over to the CAT(0) setting without change. We know from [11] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point. We will also need the following important fact about asymptotic centers.

Proposition 4.1.1. *If K is a closed convex subset of X and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Proof. Let $x \in X$ be the asymptotic center of (x_n). It is known that the nearest point projection $P : X \to K$ exists and is nonexpansive ([6], p. 177). If $x \notin K$ then $r(x, \{x_n\}) < r(P(x), \{x_n\})$, and we would have a contradiction. □

4.2 A fixed point theorem

A nonexpansive set-valued mapping $T : C \to FB(X)$ induces a nonexpansive set-valued mapping $\widetilde{T}$ defined on $\widetilde{C}$ as follows:

$$\widetilde{T}(\tilde{x}) = \left\{\tilde{u} \in \widetilde{X} : \exists \text{ a representative } \{u_n\} \text{ of } \tilde{u} \text{ with } u_n \in Tx_n \text{ for each } n\right\}.$$

To see that $\widetilde{T}$ is nonexpansive (and hence well-defined), let $\tilde{x}, \tilde{y} \in \widetilde{C}$, with $\tilde{x} = [\{x_n\}]$ and $\tilde{y} = [\{y_n\}]$. Then

$$\begin{aligned}\widetilde{D}\left(\widetilde{T}(\tilde{x}), \widetilde{T}(\tilde{y})\right) &\leq \lim_{\mathcal{U}} D(Tx_n, Ty_n) \\ &\leq \lim_{\mathcal{U}} d(x_n, y_n) \\ &= \tilde{d}(\tilde{x}, \tilde{y}).\end{aligned}$$

The following fact (see, e.g., [25], Proposition 1) will be needed.

$$\text{If } S \subseteq C \text{ is compact, then } \dot{S} = \widetilde{S}. \tag{4.13}$$

Next we have a result that is analogous to Proposition 7 of [32] for Banach spaces satisfying the Opial property. The proof is an adaptation of the one given in [32].

Proposition 4.2.1. *x is the asymptotic center of a regular sequence $(x_n) \subset X$ if and only if $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x} := [\{x_n\}]$ in the ultrapower $\widetilde{X}$.*

Proof. ($\Rightarrow$) Suppose x is the asymptotic center of $\{x_n\}$, and suppose $d_{\mathcal{U}}(\dot{y}, \tilde{x}) \leq d_{\mathcal{U}}(\dot{x}, \tilde{x})$ for some $y \in X$. Choose a subsequence $\{u_n\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty} d(y, u_n) = \liminf_{n\to\infty} d(y, x_n).$$

Using the fact that $\{x_n\}$ is regular we have

$$\begin{aligned}\lim_{n\to\infty} d(y, u_n) &\leq \lim_{\mathcal{U}} d(y, x_n) \\ &= \tilde{d}(\dot{y}, \tilde{x}) \\ &\leq \tilde{d}(\dot{x}, \tilde{x}) \\ &\leq \limsup_{n\to\infty} d(x, x_n) \\ &= r(\{x_n\}) \\ &= \limsup_{n\to\infty} d(x, u_n).\end{aligned}$$

Thus $\lim_{n\to\infty} d(y, u_n) \leq \limsup_{n\to\infty} d(x, u_n)$, and $y = x$ by uniqueness of the asymptotic center.

($\Leftarrow$) Suppose $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x} := [\{x_n\}]$, and suppose y is the asymptotic center of $\{x_n\}$. Then by the implication ($\Rightarrow$) $\dot{y}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x}$, whence $\dot{x} = \dot{y}$; thus $x = y$. $\square$

With the above observation, we are in a position to prove the fixed point theorem we will need in the next section. This result also extend Theorem 3.3.3 of the previous chapter.

Theorem 4.2.2. *Let K be a closed convex subset of a complete $CAT(0)$ space X, and let $T : K \to K(X)$ be a nonexpansive mapping. Suppose* $\operatorname{dist}(x_n, Tx_n) \to 0$ *for some bounded sequence $\{x_n\} \subset K$. Then T has a fixed point.*

Proof. By passing to a subsequence we may suppose $\{x_n\}$ is regular. Let x be the asymptotic center of $\{x_n\}$. By Proposition 4.2.1 $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x} := [\{x_n\}]$. By Proposition 4.1.1, $x \in K$ and also $\dot{x} \in \dot{K}$. Since $\tilde{x} \in \widetilde{T}(\tilde{x})$, $\tilde{x}$ must lie in a ρ-neighborhood of $\widetilde{T}(\dot{x})$ for $\rho = \widetilde{D}\left(\widetilde{T}(\tilde{x}), \widetilde{T}(\dot{x})\right)$. Since $\widetilde{T}(\dot{x})$ is compact, $\operatorname{dist}(\tilde{x}, \widetilde{T}(\dot{x})) = \tilde{d}(\tilde{x}, \dot{u})$ for some $\dot{u} \in \widetilde{T}(\dot{x})$. But since $\widetilde{T}(\dot{x}) \subset \dot{X}$, if $\dot{u} \neq \dot{x}$ we have the contradiction

$$\tilde{d}(\tilde{x}, \dot{u}) > \tilde{d}(\tilde{x}, \dot{x}) \geq \widetilde{D}\left(\widetilde{T}(\tilde{x}), \widetilde{T}(\dot{x})\right) = \rho.$$

Therefore $\dot{x} = \dot{u} \in \widetilde{T}(\dot{x})$. However $\widetilde{T}(\dot{x}) = \widetilde{Tx}$, so by (4.13) this in turn implies $x \in Tx$. □

Remark 4.2.3. Convexity of K is needed the the preceding argument only to assure that the asymptotic center of (x_n) lies in K. The theorem actually holds under the weaker assumption that K is closed and contains the asymptotic centers of all of its regular sequences.

4.3 Homotopic invariance

The following is an analog of Theorem 3.1 of [52].

Theorem 4.3.1. *Let C be a nonempty closed convex subset of a complete CAT(0) space X, with int(C) $\neq \emptyset$, let $\{T_t\}_{0 \leq t \leq 1}$ be a family of λ-contractions from C to $K(X)$ which is equi-continuous in $t \in [0,1]$ over C. Assume that some T_t has a fixed point in C, and assume every T_t is fixed point free on ∂C. Then T_t has a fixed point in C for each $t \in [0,1]$.*

Proof. Let $V = \{t \in [0,1] : T_t \text{ has a fixed point in } C\}$. Then V is nonempty by assumption. We show that V is both open and closed in $[0,1]$ and therefore conclude that $V = [0,1]$. The proof that V is open in $[0,1]$ is identical to the one given in the proof of Lemma 3.1 of [52]. To show that V is closed, assume $\{t_n\} \subset V$ is such that $t_n \to t_0$. Then for each n there exists $x_n \in C$ such that $x_n \in T_{t_n}(x_n)$. We note that the sequence $\{x_n\}$ is bounded (the proof is similar to the one given in Theorem 3.1 of [52]). By equi-continuity of T_{t_0} we have

$$\operatorname{dist}(x_n, T_{t_0}(x_n)) \leq D(T_{t_n}(x_n), T_{t_0}(x_n)) \to 0.$$

By Theorem 4.2.2, T_{t_0} has a fixed point in C, so $t_0 \in V$. □

We now turn to an analog of Theorem 4.1 of [52].

Theorem 4.3.2. *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X. Suppose $T, G : C \to K(X)$ are two set-valued nonexpansive mappings and suppose there exists a homotopy $H : [0,1] \times C \to K(X)$ such that*

(1) $H(0,\cdot) = T(\cdot)$ *and* $H(1,\cdot) = G(\cdot)$;

(2) *for each* $t \in [0,1]$, $H(t,\cdot)$ *is a set-valued nonexpansive mapping from* C *to* $K(X)$;

(3) $H(t,x)$ *is equi-continuous in* $t \in [0,1]$ *over* C;

(4) *for each sequence* $\{t_n\}$ *in* $[0,1]$ *with*

$$\inf_{x\in C} \operatorname{dist}(x, H(t_n, x)) > 0,$$

$\lim_{n\to\infty} t_n = t_0$ *implies* $\inf_{x\in C} \operatorname{dist}(x, H(t_0, x)) > 0$.

Then T *has a fixed point in* C *if and only if* G *has a fixed point in* C.

Proof. Assume T has a fixed point in C, and let

$$V = \{t \in [0,1] : \text{ there exists } x \in C \text{ such that } x \in H(t,x)\}.$$

We can show that V is closed as in the proof of Theorem 4.3.1. Suppose V is not open. Then there exists $t_0 \in V$ and a sequence $\{t_n\} \subset [0,1]\backslash V$ such that $\lim_{n\to\infty} t_n = t_0$. Since $t_n \notin V$, $\operatorname{dist}(x, H(t_n, x)) > 0$ for all $n \in \mathbb{N}$ and $x \in C$. We claim that

$$\inf_{x\in C} \operatorname{dist}(x, H(t_n, x)) > 0 \text{ for all } n \in \mathbb{N}.$$

Otherwise, there exists a sequence $\{x_m\} \subset C$ such that

$$\lim_{m\to\infty} \operatorname{dist}(x_m, H(t_n, x_m)) = 0,$$

and by Theorem 4.2.2 $H(t_n, \cdot)$ has a fixed point. But this contradicts $t_n \notin V$, so we have the claim. Condition (4) now implies

$$\inf_{x\in C} \operatorname{dist}(x, H(t_0, x)) > 0,$$

which in turn implies $t_0 \notin V$ and this is a contradiction. Therefore V is open in $[0,1]$, and hence $V = [0,1]$, from which the conclusion follows. □

The other results of [52], including the alternative principles, carry over the present setting as well.

Remark 4.3.3. In view of Remark 4.2.3, in both Theorems 4.3.1 and 4.3.2 the assumption of convexity can be replaced by the assumption that C contains the asymptotic center of each of its regular sequences.

4.4 An ultrapower approach in Banach spaces

As we shall see, the ultrapower approach used in proving Theorem 4.2.2 also provides a very efficient method for proving the classical Banach space fixed point theorems for nonexpansive set-valued mappings.

Recall that a Banach space is said to have the Opial's property ([47]) if given whenever (x_n) converges weakly to $x \in X$,

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \text{ for each } y \in X \text{ with } y \neq x.$$

As in the CAT(0) case, a nonexpansive set-valued mapping $T : C \to FB(X)$ induces a nonexpansive set-valued mapping $\widetilde{T}$ defined on $\widetilde{C}$ as follows:

$$\widetilde{T}(\tilde{x}) = \left\{\tilde{u} \in \widetilde{X} : \exists \text{ a representative } (u_n) \text{ of } \tilde{u} \text{ with } u_n \in Tx_n \text{ for each } n\right\}.$$

The following simple idea, which is extracted from the proof of Theorem 4.2.2, is the basis for all of our Banach space results. Recall that a set C is said to be (uniquely) proximinal if each point $x \in X$ has a (unique) nearest point in C.

Lemma 4.4.1. *Let K be a subset of a Banach space X, suppose $T : K \to 2^X\backslash\emptyset$ is nonexpansive, and suppose there exists $x_0 \in K$ such that $x_0 \in Tx_0$. Suppose C is a subset of K for which $T : C \to K(C)$, and suppose C is uniquely proximinal in K. Then T has a fixed point in C. Indeed, the point of C which is nearest to x_0 is a fixed point of T.*

Proof. If $x_0 \in C$ we are finished. Otherwise let x be the unique point of C nearest to x_0. We assert that $x \in Tx$. Since $x_0 \in Tx_0$, x_0 must lie in a ρ-neighborhood of Tx for $\rho = D(Tx_0, Tx)$. Therefore, since Tx is compact, $\text{dist}(x_0, Tx) = \|x_0 - u\| \leq \rho$ for some $u \in Tx$. But since $Tx \subset C$, if $u \neq x$,

$$\|x_0 - u\| > \|x_0 - x\| \geq D(Tx, Tx_0) = \rho,$$

and we have a contradiction. Therefore $x = u \in Tx$. □

The preceding lemma quickly yields the following result. Notice that boundedness of C is not needed. This observation may be known, but we are not aware of an explicit citation.

Theorem 4.4.2. *Let X be a uniformly convex Banach space, and let C be a closed convex subset of X. If $T : C \to K(C)$ is a nonexpansive mapping that satisfies*

$$\text{dist}(x_n, Tx_n) \to 0 \qquad \text{as } n \to \infty \tag{4.14}$$

for some bounded sequence $\{x_n\}$ in C, then T has a fixed point.

Proof. Let $\tilde{x} = [\{x_n\}] \in \widetilde{C}$. As we have observed, $\widetilde{T} : \widetilde{C} \to 2^{\widetilde{C}} \backslash \emptyset$ is nonexpansive. Also (4.14) implies $\tilde{x} \in \widetilde{T}(\tilde{x})$. Since uniform convexity is a super property, $\widetilde{X}$ is uniformly convex and then $\tilde{x}$ has a unique nearest point $\dot{x} \in \dot{C}$. Since $\widetilde{T} : \dot{C} \to K(\dot{C})$, Lemma 4.4.1 implies there exists $\dot{x} \in \dot{C}$ such that $\dot{x} \in \widetilde{T}(\dot{x})$. However by (4.13) $\widetilde{T}(\dot{x}) = \dot{T}x$, and this in turn implies that $x \in Tx$. □

If X has the Opial's property, the assumption that $T : C \to K(C)$ can be weakened to $T : C \to K(X)$. For this we will make use of the following fact.

Proposition 4.4.3. (Kirk and Sims [32]) *Let X be a Banach space that has the Opial property. Then $x \in X$ is the weak limit of a regular sequence $\{x_n\} \subset X$ if and only if $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x} := [\{x_n\}]$ in the ultrapower $\widetilde{X}$.*

Theorem 4.4.4. *Let X be a Banach space that has the Opial's property, and let C be a weakly compact subset of X. If $T : C \to K(X)$ is a nonexpansive mapping that satisfies*

$$\operatorname{dist}(x_n, Tx_n) \to 0 \qquad \text{as } n \to \infty$$

for some bounded sequence $\{x_n\}$ in C, then T has a fixed point.

Proof. By passing to a subsequence if necessary we may suppose that $\{x_n\}$ is regular and converges weakly, say to $x \in C$. By Proposition 4.4.3 $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x}$. The proof is now identical to the proof of Theorem 4.2.2 upon replacing $\tilde{d}$ with $\|\cdot\|_{\mathcal{U}}$. □

As a corollary of the preceding results we have the classical results of both Lim and Lami Dozo.

Theorem 4.4.5. ([37], [36]) *Suppose X is either a uniformly convex Banach space, or a reflexive Banach space that has the Opial's property. Let C be a bounded closed convex subset of X, and suppose $T : C \to K(C)$ is nonexpansive. Then T has a fixed point.*

Proof. Fix $z \in C$, and for each $n \geq 1$, consider the contraction mapping $T_n : C \to K(C)$ defined by

$$T_n(x) = \frac{1}{n}z + \left(1 - \frac{1}{n}\right)Tx, \qquad x \in C.$$

Then by Nadler's theorem [43], for each $n \geq 1$ there exists $x_n \in C$ such that $x_n \in T_n(x_n)$. Moreover

$$\operatorname{dist}(x_n, Tx_n) \leq \frac{1}{n}\operatorname{diam}(C) \to 0 \text{ as } n \to \infty.$$

□

In order to extend Lim's theorem to nonself mappings. The following two facts will be needed.

Lemma 4.4.6. ([15, Corollary 2]) *Suppose C is a closed convex subset of a Banach space X and suppose $T : C \to K(X)$ is a weakly inward contraction on C. Then T has a fixed point in C.*

Lemma 4.4.7. *Let X be a uniformly convex Banach space, let C be a closed convex subset of X, and suppose $x_0 \in X$. Let x be the unique point of C which is nearest to x_0. Then x is the unique point of $\overline{I_C(x)}$ which is nearest to x_0.*

Proof. Suppose not, and let y be the unique point of $\overline{I_C(x)}$ which is nearest to x_0. Then, since $C \subseteq \overline{I_C(x)}$ and $y \in \overline{I_C(x)} \backslash C$, it must be the case that

$$\|y - x_0\| < \|x - x_0\|.$$

By the continuity of $\|\cdot\|$ there exists $z \in I_C(x) \backslash C$ such that $\|z - x_0\| < \|x - x_0\|$. This implies $z = (1-\alpha)x + \alpha w$ for some $w \in C$ and $\alpha > 1$. Hence

$$\|w - x_0\| \leq \frac{1}{\alpha}\|z - x_0\| + \left(1 - \frac{1}{\alpha}\right)\|x - x_0\| < \|x - x_0\|,$$

a contradiction. □

The following theorem was first proved for inward mappings independently by Downing and Kirk [15] and by Reich [48]. The slightly more general formulation below is due to T. C. Lim [39] and H. K. Xu [54]. Our proof is much shorter than the proofs of Lim and Xu (although it depends on deeper facts).

Theorem 4.4.8. *Let C be a bounded closed convex subset of a uniformly convex Banach space X, and suppose $T : C \to K(X)$ is nonexpansive and weakly inward on C. Then T has a fixed point.*

Proof. As in the proof of Theorem 4.4.5, approximate T with the contraction mappings T_n. Each of the mapping T_n is also weakly inward and by Lemma 4.4.6 has a fixed point x_n. Since the sequence $\{x_n\}$ satisfies $\operatorname{dist}(x_n, Tx_n) \to 0$. Let

$\tilde{x} = [\{x_n\}]$, and let $\dot{x}$ be the unique point of $\dot{C}$ which is nearest $\tilde{x}$. Since $\widetilde{T}$ is nonexpansive there exists a point $\tilde{y} \in \widetilde{T}(\dot{x})$ such that $\|\tilde{y} - \tilde{x}\|_{\mathcal{U}} \leq \|\dot{x} - \tilde{x}\|_{\mathcal{U}}$, and since $\widetilde{T}$ is weakly inward on $\dot{C}$, $\tilde{y} \in \overline{I_{\dot{C}}(\dot{x})}$. Lemma 4.4.7 implies $\tilde{y} = \dot{x}$. Thus $\dot{x} \in \widetilde{T}(\dot{x})$ and the conclusion follows. □

Finally, we remark that it is possible to use this approach to prove the following theorem of Kirk and Massa ([31]; also see [28]). We omit the details because the ultrapower proof is not appreciably shorter than the one given in [31] (which also uses nonstandard techniques). Indeed, this result has recently been extended to spaces X for which $\varepsilon_\beta(X) < 1$, where $\varepsilon_\beta(X)$ denotes the characteristic of noncompact convexity for the separation measure of noncompactness (see [14]).

Theorem 4.4.9. *Suppose C is a nonempty bounded closed convex subset of a Banach space X, and suppose $T : C \to KC(C)$ is nonexpansive. Suppose also that the asymptotic center in C of each bounded sequence in X is nonempty and compact. Then T has a fixed point.*.

Remark 4.4.10. It might be worth noting that Lemma 4.4.1 holds for mappings taking only closed values if it is assumed that the space is uniformly convex.

Lemma 4.4.11. *Let K be a subset of a uniformly convex Banach space X, suppose $T : K \to 2^X \backslash \emptyset$ is nonexpansive, and suppose there exists $x_0 \in K$ such that $x_0 \in Tx_0$. Suppose C is a closed convex subset of K for which $T : C \to F(C)$. Then the point of C which is nearest to x_0 is a fixed point of T.*

Proof. If $x_0 \in C$ we are finished. Otherwise let x be unique the point of C nearest to x_0. We assert that $x \in Tx$. Suppose not. Since $x_0 \in Tx_0$, x_0 must lie in a ρ-neighborhood of Tx for $\rho = D(Tx_0, Tx)$. If $\operatorname{dist}(x_0, Tx) > \|x_0 - x\|$ we have a contradiction as in the proof of Lemma 4.4.1. On the other hand, if $dist(x_0, Tx) = \|x - x_0\|$, then there exists a sequence $\{u_n\} \subset Tx$ such that $\|x_0 - u_n\| \to \|x_0 - x\|$ as $n \to \infty$. Since $\left\| x_0 - \frac{x + u_n}{2} \right\| > \|x_0 - x\|$, the uniform convexity of X yields $\|x - u_n\| \to 0$ as $n \to \infty$. Since Tx is closed, $x \in Tx$. □

Remark 4.4.12. In Theorems 3.1 and 4.1 of [52] the domain of the mappings is assumed to be weakly compact and convex. However weak compactness suffices – the convexity assumption may be dropped. To see this one could either use Theorem 4.4.4 in lieu of the demiclosedness principle in the proofs of those theorems, or observe that convexity is not needed in the proof of the demiclosedness principle itself (Lemma 2.1 of [52]).

4.5 Weak convergence in CAT(0) spaces

We conclude with a question. A comparison of Propositions 4.2.1 and 4.4.3 clearly suggests that the following would be a reasonable way to define weak convergence in a CAT(0) space, especially since it does indeed coincide with weak convergence in a Hilbert space.

Definition 4.5.1. A sequence [net] $\{x_n\}$ in $X$is said to converge weakly to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence [subnet] $\{u_n\}$ of $\{x_n\}$.

This notion of convergence was first introduced in metric spaces by T. C. Lim [38], who called it Δ-convergence. (T. Kuczumow [34] introduced a similar notion of convergence in Banach spaces which he called 'almost convergence'.)

This raises a very fundamental question: For what CAT(0) spaces, aside from Hilbert space, does the notion convergence in Definition 4.5.1 actually correspond to convergence relative to some topology? Specifically, when is there a topology τ on X such that a net $\{x_\alpha\}$ converges to x in the sense of Definition 4.5.1 if and only if $\{x_\alpha\}$ is τ-convergent to x?