Chapter 5
Fixed Point Theorems for Multivalued Mappings in Modular Function Spaces

The purpose of this chapter is to study the existence of fixed points for multivalued nonexpansive mappings in modular function spaces. We apply our main result to obtain fixed point theorems for multivalued mappings in the Banach spaces L_1 and l_1.

5.1 Introduction

The theory of modular spaces was initiated by Nakano [44] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [42] in 1959. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated and solved in modular spaces (see, for instance, [12, 13, 24, 26]). In particular, some fixed point theorems for (single-valued) nonexpansive mappings in modular function spaces are given in [26]. In 1969, Nadler [43] established the multivalued version of Banach's contraction principle in metric spaces. Since then the metric fixed point theory for multivalued mappings has been rapidly developed and many of papers have appeared proving the existence of fixed points for multivalued nonexpansive mappings in special classes of Banach spaces. In this chapter, we study similar problems in the setting of modular function spaces. Namely, we prove that every $\rho-$contraction $T : C \to F_\rho(C)$ has a fixed point where ρ is a convex function modular satisfying the Δ_2-type condition and C is a nonempty $\rho-$bounded $\rho-$closed subset of L_ρ. By using this result, we can assert the existence of fixed points for multivalued $\rho-$nonexpansive mappings. Finally, we apply our main result to obtain fixed point theorems in the Banach space L_1 (resp. l_1) for multivalued mappings whose domains are compact in the topology of the convergence locally in measure (resp. weak* topology).

5.2 Preliminaries

We start by recalling some basic concepts about modular function spaces. For more details and discussions the reader is referred to [33, 41].

Let Ω be a nonempty set and Σ be a nontrivial $\sigma-$algebra of subsets of Ω. Let $\mathcal{P}$ be a $\delta-$ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$ (for instance, $\mathcal{P}$ can be the class of sets of finite measure in a $\sigma-$finite measure space). By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}$ we will denote the space of all measurable functions, i.e., all functions $f : \Omega \to \mathbb{R}$ such that there exists a sequence $\{g_n\}$ in $\mathcal{E}$, $|g_n| \leq |f|$, and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

Let us recall that a set function $\mu : \Sigma \to [0, \infty]$ is called a $\sigma-$subadditive measure if $\mu(\emptyset) = 0$, $\mu(A) \leq \mu(B)$ for any $A \subset B$ and $\mu(\bigcup A_n) \leq \sum \mu(A_n)$ for any sequence of sets $\{A_n\} \subset \Sigma$. By 1_A we denote the characteristic function of the set A.

Definition 5.2.1. A functional $\rho : \mathcal{E} \times \Sigma \to [0, \infty]$ is called a function modular if :

(P_1) $\rho(0, E) = 0$ for any $E \in \Sigma$,

(P_2) $\rho(f, E) \leq \rho(g, E)$ whenever $|f(\omega)| \leq |g(\omega)|$ for any $\omega \in \Omega, f, g \in \mathcal{E}$, and $E \in \Sigma$,

(P_3) $\rho(f, .) : \Sigma \to [0, \infty]$ is a $\sigma-$subadditive measure for every $f \in \mathcal{E}$,

(P_4) $\rho(\alpha, A) \to 0$ as α decreases to 0 for every $A \in \mathcal{P}$, where $\rho(\alpha, A) = \rho(\alpha 1_A, A)$,

(P_5) if there exists $\alpha > 0$ such that $\rho(\alpha, A) = 0$, then $\rho(\beta, A) = 0$ for every $\beta > 0$,

(P_6) for any $\alpha > 0$, $\rho(\alpha, .)$ is order continuous on $\mathcal{P}$, that is, $\rho(\alpha, A_n) \to 0$ if $\{A_n\} \subset \mathcal{P}$ and decreases to $\emptyset$.

A $\sigma-$subadditive measure ρ is said to be additive if $\rho(f, A \cup B) = \rho(f, A) + \rho(f, B)$ whenever $A, B \in \Sigma$ such that $A \cap B = \emptyset$ and $f \in \mathcal{M}$.

The definition of ρ is then extended to $f \in \mathcal{M}$ by

$$\rho(f, E) = \sup \{\rho(g, E) : g \in \mathcal{E}, |g(\omega)| \leq |f(\omega)| \text{ for every } \omega \in \Omega\}.$$

Definition 5.2.2. A set E is said to be $\rho-$null if $\rho(\alpha, E) = 0$ for every $\alpha > 0$. A property $p(\omega)$ is said to hold $\rho-$almost everywhere ($\rho-$a.e.) if the set $\{\omega \in \Omega : p(\omega)$ does not hold$\}$ is $\rho-$null. For example, we will say frequently $f_n \to f$ $\rho-$a.e.

Note that a countable union of $\rho-$null sets is still $\rho-$null. In the sequel we will identify sets A and B whose symmetric difference $A\Delta B$ is $\rho-$null, similarly we will identify measurable functions which differ only on a $\rho-$null set.

In the above condition, we define the function $\rho : \mathcal{M} \to [0,\infty]$ by $\rho(f) = \rho(f,\Omega)$. We know from [33] that ρ satisfies the following properties :

(i) $\rho(f) = 0$ if and only if $f = 0$ $\rho-$a.e.

(ii) $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$ and $f \in \mathcal{M}$.

(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ if $\alpha + \beta = 1,\ \alpha, \beta \geq 0$ and $f, g \in \mathcal{M}$.
In addition, if the following property is satisfied

(iii)' $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1,\ \alpha, \beta \geq 0$ and $f, g \in \mathcal{M}$,
we say that ρ is a convex modular.

A function modular ρ is called $\sigma-$finite if there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $0 < \rho(1_{K_n}) < \infty$ and $\Omega = \bigcup K_n$.

The modular ρ defines a corresponding modular space L_ρ, which is given by

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

In general the modular ρ is not subadditive and therefore does not behave as a norm or a distance. But one can associate to a modular an $F-$norm. Recall that a functional $\|\cdot\| : X \to [0,\infty]$ defines an $F-$norm on a linear space X if and only if

(1) $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\alpha x\| = \|x\|$ whenever $|\alpha| = 1$,

(3) $\|x + y\| \leq \|x\| + \|y\|$,

(4) $\|\alpha_n x_n - \alpha x\| \to 0$ if $\alpha_n \to \alpha$ and $\|x_n - x\| \to 0$.

The modular space L_ρ can be equipped with an $F-$norm defined by

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \leq \alpha\right\}.$$

We know from [33] that the linear space $(L_\rho, \|\cdot\|_\rho)$ is a complete metric space. If ρ is convex, the formula

$$\|f\|_l = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \leq 1\right\}$$

defines a norm which is frequently called the Luxemburg norm. The formula

$$\|f\|_a = \inf\left\{\frac{1}{k}(1+\rho(kf)) : k > 0\right\}$$

defines a different norm which is called the Amemiya norm. Moreover, $\|\cdot\|_l$ and $\|\cdot\|_a$ are equivalent norms. We can also consider the space

$$E_\rho = \{f \in \mathcal{M} : \rho(\alpha f, \cdot) \text{ is order continuous for all } \alpha > 0\}.$$

Definition 5.2.3. A function modular ρ is said to satisfy the Δ_2−condition if

$$\sup_{n\geq 1} \rho(2f_n, D_k) \to 0 \text{ as } k \to \infty \text{ whenever } \{f_n\} \subset \mathcal{M},\ D_k \in \Sigma$$

$$\text{decreases to } \emptyset \text{ and } \sup_{n\geq 1} \rho(f_n, D_k) \to 0 \text{ as } k \to \infty.$$

It is known that the Δ_2−condition is equivalent to $E_\rho = L_\rho$.

Definition 5.2.4. A function modular ρ is said to satisfy the Δ_2−type condition if there exists $K > 0$ such that for any $f \in L_\rho$ we have $\rho(2f) \leq K\rho(f)$.

In general, the Δ_2−type condition and Δ_2−condition are not equivalent, even though it is obvious that the Δ_2− type condition implies the Δ_2−condition.

Definition 5.2.5. Let L_ρ be a modular space.

(1) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ−convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$.

(2) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ−a.e. convergent to $f \in L_\rho$ if the set $\{\omega \in \Omega : f_n(\omega) \nrightarrow f(\omega)\}$ is ρ−null.

(3) A subset C of L_ρ is called ρ−closed if the ρ−limit of a ρ−convergent sequence of C always belongs to C.

(4) A subset C of L_ρ is called ρ−a.e. closed if the ρ−a.e. limit of a ρ−a.e. convergent sequence of C always belongs to C.

(5) A subset C of L_ρ is called ρ−compact if every sequence in C has a ρ−convergent subsequence in C.

(6) A subset C of L_ρ is called ρ−a.e. compact if every sequence in C has a ρ−a.e. convergent subsequence in C.

(7) A subset C of L_ρ is called $\rho-$bounded if

$$\text{diam}_\rho(C) = \sup\{\rho(f-g) : f, g \in C\} < \infty.$$

We know by [33] that under the Δ_2-condition the norm convergence and modular convergence are equivalent, which implies that the norm and modular convergence are also the same when we deal with the Δ_2-type condition. In the sequel we will assume that the modular function ρ is convex and satisfies the Δ_2-type condition.

Definition 5.2.6. Let ρ be as above. We define a growth function ω by

$$\omega(t) = \sup\left\{\frac{\rho(tf)}{\rho(f)} : f \in L_\rho, 0 < \rho(f) < \infty\right\} \text{ for all } 0 \le t < \infty.$$

The following properties of the growth function can be found in [13].

Lemma 5.2.7. *Let ρ be as above. Then the growth function ω has the following properties :*

(1) $\omega(t) < \infty,\ \forall t \in [0, \infty)$.

(2) $\omega : [0,\infty) \to [0,\infty)$ *is a convex, strictly increasing function. So, it is continuous.*

(3) $\omega(\alpha\beta) \le \omega(\alpha)\omega(\beta); \forall \alpha, \beta \in [0, \infty)$.

(4) $\omega^{-1}(\alpha)\omega^{-1}(\beta) \le \omega^{-1}(\alpha\beta); \forall \alpha, \beta \in [0,\infty)$, *where ω^{-1} is the function inverse of ω.*

The following lemma shows that the growth function can be used to give an upper bound for the norm of a function.

Lemma 5.2.8. (Domínguez Benavides et al. [13]) *Let ρ be as above. Then*

$$\|f\|_\rho \le \frac{1}{\omega^{-1}(1/\rho(f))} \text{ whenever } f \in L_\rho \backslash \{0\}.$$

The following lemma is a technical lemma which will be need because of lack of the triangular inequality.

Lemma 5.2.9. (Domínguez Benavides et al. [13]) *Let ρ be as above, $\{f_n\}$ and $\{g_n\}$ be two sequences in L_ρ. Then*

$$\lim_{n\to\infty} \rho(g_n) = 0 \Longrightarrow \limsup_{n\to\infty} \rho(f_n + g_n) = \limsup_{n\to\infty} \rho(f_n)$$

and

$$\lim_{n\to\infty} \rho(g_n) = 0 \Longrightarrow \liminf_{n\to\infty} \rho(f_n + g_n) = \liminf_{n\to\infty} \rho(f_n).$$

In the same way as the Hausdorff distance defined on the family of bounded closed subsets of a metric space, we can define the analogue to the Hausdorff distance for modular function spaces. We will call $\rho-$Hausdorff distance even though it is not a metric.

Definition 5.2.10. Let C be a nonempty subset of L_ρ. We shall denote by $F_\rho(C)$ the family of nonempty $\rho-$closed subsets of C and by $K_\rho(C)$ the family of nonempty $\rho-$compact subsets of C. Let $H_\rho(\cdot,\cdot)$ be the $\rho-$Hausdorff distance on $F_\rho(L_\rho)$, i.e.,

$$H_\rho(A,B) = \max\left\{\sup_{f\in A}\operatorname{dist}_\rho(f,B), \sup_{g\in B}\operatorname{dist}_\rho(g,A)\right\}, \quad A,B \in F_\rho(L_\rho),$$

where $\operatorname{dist}_\rho(f,B) = \inf\{\rho(f-g) : g \in B\}$ is the $\rho-$distance between f and B. A multivalued mapping $T : C \to F_\rho(L_\rho)$ is said to be a $\rho-$contraction if there exists a constant $k \in [0,1)$ such that

$$H_\rho(Tf,Tg) \le k\rho(f-g), \;\; f,\, g \in C. \tag{5.15}$$

If (5.15) is valid when $k = 1$, then T is called $\rho-$nonexpansive. A function $f \in C$ is called a fixed point for a multivalued mapping T if $f \in Tf$.

5.3 Main results

We begin stating the Banach Contraction Principle for multivalued mappings in modular function spaces.

Theorem 5.3.1. *Let ρ be a convex function modular satisfying the Δ_2-type condition, C a nonempty $\rho-$bounded $\rho-$closed subset of L_ρ, and $T : C \to F_\rho(C)$ a $\rho-$contraction mapping, i.e., there exists a constant $k \in [0,1)$ such that*

$$H_\rho(Tf,Tg) \le k\rho(f-g), \;\; f,\, g \in C. \tag{5.16}$$

Then T has a fixed point.

Proof. Let $f_0 \in C$ and $\alpha \in (k,1)$. Since Tf_0 is nonempty, there exists $f_1 \in Tf_0$ such that $\rho(f_0 - f_1) > 0$ (otherwise f_0 is a fixed point of T). In view of (5.16), we have

$$\operatorname{dist}_\rho(f_1,Tf_1) \le H_\rho(Tf_0,Tf_1) \le k\rho(f_0-f_1) < \alpha\rho(f_0-f_1).$$

Since $\operatorname{dist}_\rho(f_1,Tf_1) = \inf\{\rho(f_1-g) : g \in Tf_1\}$, it follows that there exists $f_2 \in Tf_1$ such that

$$\rho(f_1-f_2) < \alpha\rho(f_0-f_1).$$

Similarly, there exists $f_3 \in Tf_2$ such that

$$\rho(f_2 - f_3) < \alpha\rho(f_1 - f_2).$$

Continuing in this way, there exists a sequence $\{f_n\}$ in C satisfying $f_{n+1} \in Tf_n$ and

$$\begin{aligned}\rho(f_n - f_{n+1}) &< \alpha\rho(f_{n-1} - f_n)\\ &< \alpha^2(\rho(f_{n-2} - f_{n-1}))\\ &< \dots\\ &< \alpha^{n-1}(\rho(f_1 - f_2))\\ &< \alpha^n(\rho(f_0 - f_1))\\ &\le \alpha^n \mathrm{diam}_\rho(C).\end{aligned}$$

Let $M = \mathrm{diam}_\rho(C)$, then

$$\frac{1}{\alpha^n M} < \frac{1}{\rho(f_n - f_{n+1})}.$$

By Lemma 5.2.7, we have

$$\Big(\omega^{-1}\big(\frac{1}{\alpha}\big)\Big)^n \omega^{-1}\Big(\frac{1}{M}\Big) < \omega^{-1}\Big(\frac{1}{\rho(f_n - f_{n+1})}\Big).$$

It follows that

$$\frac{1}{\omega^{-1}\Big(\frac{1}{\rho(f_n - f_{n+1})}\Big)} < \frac{1}{\Big(\omega^{-1}\big(\frac{1}{\alpha}\big)\Big)^n \omega^{-1}\Big(\frac{1}{M}\Big)}.$$

By Lemma 5.2.8, we obtain

$$\|f_n - f_{n+1}\|_\rho < \Big(\frac{1}{\omega^{-1}\big(\frac{1}{\alpha}\big)}\Big)^n \cdot \frac{1}{\omega^{-1}\Big(\frac{1}{M}\Big)}.$$

Since ω^{-1} is strictly increasing, we have $\frac{1}{\omega^{-1}\left(\frac{1}{\alpha}\right)} < 1$. This implies that $\{f_n\}$ is a Cauchy sequence in $(L_\rho, \|\cdot\|_\rho)$. Since $(L_\rho, \|\cdot\|_\rho)$ is a complete metric space, there exists $f \in L_\rho$ such that $\{f_n\}$ is $\|\cdot\|_\rho$−convergent to f. Since under the Δ_2−type condition, norm convergence and modular convergence are identical, $\{f_n\}$ is ρ−convergent to f and $f \in C$ because C is ρ−closed. Since $f_n \in Tf_{n-1}$, we have

$$\mathrm{dist}_\rho(f_n, Tf) \le H_\rho(Tf_{n-1}, Tf) \le k\rho(f_{n-1} - f) \longrightarrow 0. \tag{5.17}$$

We observe that, for each n, there exists $g_n \in Tf$ such that

$$\rho(f_n - g_n) \le \mathrm{dist}_\rho(f_n, Tf) + \frac{1}{n}. \tag{5.18}$$

Thus, (5.17) and (5.18) imply that $\lim_{n\to\infty} \rho(f_n - g_n) = 0$. By Lemma 5.2.9,

$$\limsup_{n\to\infty} \rho(g_n - f) = \limsup_{n\to\infty} \rho(g_n - f_n + f_n - f) = \limsup_{n\to\infty} \rho(f_n - f) = 0.$$

Since Tf is $\rho-$closed, we can conclude that $f \in Tf$. □

The following results will be very useful in the proof of our main theorem.

Theorem 5.3.2. (Khamsi [24]) *Let $\{f_n\} \subset L_\rho$ be $\rho-$a.e. convergent to 0. Assume there exists $k > 1$ such that*

$$\sup_{n\geq 1} \rho(kf_n) = M < \infty.$$

Let $g \in E_\rho$, then we have

$$\liminf_{n\to\infty} \rho(f_n + g) = \liminf_{n\to\infty} \rho(f_n) + \rho(g).$$

The following lemma guarantees that every nonempty $\rho-$compact subset of L_ρ attains a nearest point.

Lemma 5.3.3. *Let ρ be a convex function modular satisfying the Δ_2-type condition, $f \in L_\rho$, and K a nonempty $\rho-$compact subset of L_ρ. Then there exists $g_0 \in K$ such that*

$$\rho(f - g_0) = \text{dist}_\rho(f, K).$$

Proof. Let $m = \text{dist}_\rho(f, K)$. For each $n \in \mathbb{N}$, there exists $g_n \in K$ such that

$$m - \frac{1}{n} \leq \rho(f - g_n) \leq m + \frac{1}{n}.$$

By the $\rho-$compactness of K, we can assume, by passing through a subsequence, that $g_n \xrightarrow{\rho} g_0 \in K$. By Lemma 5.2.9, we obtain

$$\begin{aligned} m = \limsup_{n\to\infty} \rho(g_n - f) &= \limsup_{n\to\infty} \rho(g_n - g_0 + g_0 - f) \\ &= \limsup_{n\to\infty} \rho(g_0 - f) \\ &= \rho(g_0 - f). \end{aligned}$$

□

We can now state our main theorem.

Theorem 5.3.4. *Let ρ be a convex function modular satisfying the Δ_2-type condition, C a nonempty $\rho-$a.e. compact $\rho-$bounded convex subset of L_ρ, and $T : C \to K_\rho(C)$ a $\rho-$nonexpansive mapping. Then T has a fixed point.*

Proof. Fix $f_0 \in C$. For each $n \in \mathbb{N}$, the ρ−contraction $T_n : C \to F_\rho(C)$ is defined by

$$T_n(f) = \frac{1}{n} f_0 + (1 - \frac{1}{n}) Tf, \quad f \in C.$$

By Theorem 5.3.1, we can conclude that T_n has a fixed point, say f_n. It is easy to see that

$$\text{dist}_\rho(f_n, Tf_n) \le \frac{1}{n} \text{diam}_\rho(C) \longrightarrow 0.$$

Because of ρ−a.e. compactness of C, we can assume, by passing through a subsequence, that $f_n \xrightarrow{\rho-a.e.} f$ for some $f \in C$. By Lemma 5.3.3, for each $n \in \mathbb{N}$, there exists $g_n \in Tf_n$ and $h_n \in Tf$ such that

$$\rho(f_n - g_n) = \text{dist}_\rho(f_n, Tf_n)$$

and

$$\rho(g_n - h_n) = \text{dist}_\rho(g_n, Tf) \le H_\rho(Tf_n, Tf) \le \rho(f_n - f).$$

Because of ρ−compactness of Tf, we can assume, by passing through a subsequence, that $h_n \xrightarrow{\rho} h \in Tf$. Since ρ satisfies the Δ_2−type condition, there exists $K > 0$ such that $\rho(2(f_n - f)) \le K\rho(f_n - f)$ for all $n \in \mathbb{N}$.
This implies that

$$\sup_{n\ge1} \rho(2(f_n - f)) \le K \sup_{n\ge1} \rho(f_n - f) < \infty.$$

By Theorem 5.3.2 and Lemma 5.2.9, we obtain

$$\begin{aligned}
\liminf_{n\to\infty} \rho(f_n - f) + \rho(f - h) &= \liminf_{n\to\infty} \rho(f_n - f + f - h) \\
&= \liminf_{n\to\infty} \rho(f_n - h) \\
&= \liminf_{n\to\infty} \rho(f_n - g_n + g_n - h_n + h_n - h) \\
&= \liminf_{n\to\infty} \rho(g_n - h_n) \\
&\le \liminf_{n\to\infty} \rho(f_n - f).
\end{aligned}$$

It follows that $\rho(f - h) = 0$ and then we have $f = h \in Tf$. □

Consider the space $L_p(\Omega, \mu)$ for a σ−finite measure μ with the usual norm. Let C be a bounded closed convex subset of L_p for $1 < p < \infty$ and $T : C \to K(C)$ a multivalued nonexpansive mapping. Because of uniform convexity of L_p, it is known that T has a fixed point. For $p = 1$, T can fail to have a fixed point even in the singlevalued case for a weakly compact convex set C (see [2]). However, since L_1 is a modular space where $\rho(f) = \int_\Omega |f|d\mu = \|f\|$ for all $f \in L_1$, Theorem 5.3.4 implies the existence of a fixed point when we define mappings on a ρ−a.e. compact ρ−bounded convex subset of L_1. Thus we can state :

Corollary 5.3.5. *Let (Ω, μ) be as above, $C \subset L_1(\Omega, \mu)$ a nonempty bounded convex set which is compact for the topology of the convergence locally in measure, and $T : C \to K(C)$ a nonexpansive mapping. Then T has a fixed point.*

Proof. Under the above hypothesis, ρ−a.e. compact sets and compact sets in the topology of the convergence locally in measure are identical (see [12]). Consequently, Theorem 5.3.4 can be applied to obtain a fixed point for T. □

In the case of the space l_1 we also can obtain a bounded closed convex set C and a nonexpansive mapping $T : C \to C$ which is fixed point free. Indeed, consider the following easy and well known example:
Let

$$C = \Big\{\{x_n\} \in l_1 : 0 \le x_n \le 1 \text{ and } \sum_{n=1}^{\infty} x_n = 1\Big\}.$$

Define a nonexpansive mapping $T : C \to C$ by

$$T(x) = (0, x_1, x_2, x_3, \ldots) \quad \text{where } x = \{x_n\}.$$

Then T is a fixed point free. However, if we consider $L_\rho = l_1$ where $\rho(x) = \|x\|$, $\forall x \in l_1$. Then ρ−a.e. convergence and weak* convergence are identical on bounded subsets of l_1 (see [13]). This fact leads us to obtain the following corollary:

Corollary 5.3.6. *Let C be a nonempty weak* compact convex subset of l_1 and $T : C \to K(C)$ a nonexpansive mapping. Then T has a fixed point.*

Proof. By the above argument, we know that ρ−a.e. compact bounded sets and weak* compact sets are identical. Then we can apply Theorem 5.3.4 to assert the existence of a fixed point of T. □

In fact Corollary 5.3.5 and 5.3.6 are consequences of a general result: Assume that X is a linear normed space and τ is a Hausdorff topology on X. We say that X satisfies the strict τ-Opial property if

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for each sequence $\{x_n\}$ in X which converges to x for the topology τ and each $y \neq x$. Following the same argument as in [37] it is easy to prove the following theorem:

Theorem 5.3.7. *Let X be a Banach space, C a convex bounded sequentially τ-compact subset of X, and $T : C \to K(C)$ a nonexpansive mapping. If X satisfies the strict τ-Opial property, then T has a fixed point.*

When X is a modular function space equipped with either Luxemburg or Amemiya norm, we can consider the topology τ of convergence ρ-a.e. In this case, Theorem 5.3.7 yields to the following:

Theorem 5.3.8. *Let ρ be a convex additive σ-finite function modular satisfying the Δ_2-type condition. Assume that L_ρ is equipped either with Luxemburg or Amemiya norm. Let C be a nonempty ρ-a.e. compact ρ-bounded convex subset of L_ρ, and $T : C \to K(C)$ a nonexpansive mapping. Then T has a fixed point.*

Proof. From [22] (Theorem 4.1 and 4.3), X satisfies the uniform Opial property with respect to the topology of ρ-a.e. convergence. Since ρ-a.e. compact sets and ρ-a.e. sequentially compact sets are identical for this topology (see [12]), we can deduce the result from Theorem 5.3.7 □

Remark 5.3.9. In the case of the space $L^1(\Omega)$ we have

$$\rho(f) = \int_\Omega |f| d\mu = \|f\|_l = \|f\|_a$$

and we can deduce Corollary 5.3.5 and 5.3.6 from Theorem 5.3.8.