Chapter 1

Introduction

In this chapter, we shall introduce the statement of the problem, the idea and the objectives of our research.

The operator $\oplus^k$ has been studied first by A.Kananthai, S.Suantai and V.Longani [6] and is defined by

$$\oplus^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \times \left[\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k$$
$$\times \left[\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k \quad (1.1)$$

where $p+q=n$ is the dimension of $\mathbb{R}^n$, $i=\sqrt{-1}$ and k is a nonnegative integer. The diamond operator is denoted by

$$\Diamond = \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2. \quad (1.2)$$

The operators L_1 and L_2 are defined by

$$L_1 = \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2} \quad (1.3)$$

and

$$L_2 = \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}. \quad (1.4)$$

Thus the equation (1.1) can be written as

$$\oplus^k = \Diamond^k L_1^k L_2^k.$$

The operator $\Diamond$ can also be expressed in the form $\Diamond = \Box\triangle = \triangle\Box$, where $\Box^k$ is the ultra-hyperbolic operator iterated k times defined by

$$\Box^k = \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \ldots - \frac{\partial^2}{\partial x_{p+q}^2}\right]^k \tag{1.5}$$

where $p+q=n$ and $\triangle^k$ is the Laplacian operator iterated k times defined by

$$\triangle^k = \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right]^k. \tag{1.6}$$

In 1994 Aguirre [9] studied the elementary solution of the ultra-hyperbolic and Laplacian operator, which iterates k-times. We obtain the elementary solution $R^H_{2k}(u)$ and $(-1)^k R^e_{2k}(v)$ defined by(2.25) and (2.26) respectively.

In 2002, A. Kananthai, S. Suantai and V. Longani [6] have studied the elementary solution or Green function of the operator $\oplus^k$ which is related to the solution of wave equation and Laplace equation. They found that the relationships of such solutions depending on the conditions of p, q and k.

In 2004, G. Sritanratana and A. Kananthai [7] have studied the solution of nonlinear equation

$$\Diamond^k_{c_1}\Diamond^k_{c_2} u(x) = f\left(x, \triangle^{k-1}_{c_1}\Box^k_{c_2}\Diamond^k_{c_2} u(x)\right)$$

where $\Diamond^k_{c_1}\Diamond^k_{c_2}$ is the product of the Diamond operators defined by

$$\Diamond^k_{c_1} = \left[\frac{1}{c_1^4}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k$$

and

$$\Diamond^k_{c_2} = \left[\frac{1}{c_2^4}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k$$

where c_1 and c_2 are positive constants. They found that the existence of the solution $u(x)$ of such equation depending on the conditions of f and $\triangle^{k-1}_{c_1}\Box^k_{c_2}\Diamond^k_{c_2} u(x)$. Moreover such solution $u(x)$ related to the elastic wave equation depending on the conditions of p, q and k.

Lastly, in 2006 J. Tariboon and A. Kananthai [10] have show that $Y_{2k,2k,2k,2k,}(u, v, w, z, m)$ defined by (2.29) is the Green function of the operator $(\oplus + m^2)^k$ and was defined by

$$(\oplus + m^2)^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4 + m^2\right]^k \tag{1.7}$$

where m is a positive real number and $p+q=n$ is the dimension of the n-dimensional Euclidean space $\mathbb{R}^n$.

In this research, we find the solution of the equation

$$\oplus^k(\oplus+m^2)^k u(x)=f\big(x,\triangle^{k-1}\square^k L_1^k L_2^k(\oplus+m^2)^k u(x)\big) \tag{1.8}$$

with f is continuous and bounded for all $x=(x_1,x_2,\ldots,x_n)\in\Omega\cup\partial\Omega$ where Ω is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω, that is $|f|\leq N$, N is constant. We can find the solution $u(x)$ of (1.8) and is unique under the boundary condition $\triangle^{k-1}\square^k L_1^k L_2^k(\oplus+m^2)^k u(x)=0$ for $x\in\partial\Omega$. By [1, *p*. 369] there exists the unique solution $W(x)$ of the equation $\triangle W(x)=f\big(x,W(x)\big)$ for all $x\in\Omega$ with the boundary condition $W(x)=0$ for all $x\in\partial\Omega$ where
$W(x)=\triangle^{k-1}\square^k L_1^k L_2^k(\oplus+m^2)^k u(x)$.