# Chapter 2

# PRELIMINARIES

In this chapter, we shall introduce some notations, definitions and theorems that will be used in our research. The first section is about distributions. Next section contains the tempered distributions, and the final section contains the basic concepts.

## 2.1 Distributions

### 2.1.1 The space $\mathcal{D}$ of Testing Functions

Before we describe distributions, we define the *testing functions* on which distributions operate. Throughout this and the next section, the independent real variable $t$ will be assumed to be one-dimentional. When a function has continuous derivatives of all orders on some set of points, we shall say that the function is *infinitely smooth on that set.* If this is true for all points, we shall merely say that the function is *infinitely smooth.* Moreover, whenever we refer to a *complex number* or a *complex-valued function,* it is understood that the number may be real or the function may be real-valued.

The space of testing functions, which is denoted by $\mathcal{D}$, consists of all complex-valued function $\phi(t)$ that are infinitely smooth with compact support, where the support of continuous function $\phi(t)$ is now defined as and open set $U = \{t \in \mathbb{R} : \phi(t) \neq 0\}$. The support of $\phi$ denoted by *supp* $\phi(t)$ and define *supp* $\phi = \overline{U}$ (the

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closure of $U$ ).

An example of a testing function in $\mathcal{D}$ is

$$\zeta(t) = \begin{cases} 0 & for|t| \geq 1 \\ \exp(\frac{1}{t^2-1}) & for|t| < 1 \end{cases}$$

It can be shown that every derivative of this function exists and is zero at $t = \pm 1$.

More generally, then, this function has continuous derivatives of all orders for every $t$,and they are all equal to zero for $|t| \geq 1$ and *supp* $\zeta(t) = [-1, 1]$.

### 2.1.2 Distributions

A *functional* is a rule that assigns a number to every member of a certain set of functions. For our purposes, the set of functions will be taken to be the space $\mathcal{D}$ and we shall consider functionals that assign a complex number to every member of $\mathcal{D}$. Denoting a functional by the symbol $f$, we designate the number that $f$ assign to a particular testing function $\phi$ by $\langle f, \phi \rangle$. *Distributions*, which we shall describe in this section, are functional on the space $\mathcal{D}$ that possess, in addition, two essential properties. The first of these is *linearity*. A functional $f$ on $\mathcal{D}$ is said to be *linearity* if, for any two testing functions $\phi_1$ and $\phi_2$ in $\mathcal{D}$ and any complex number $\alpha$, the following conditions are satisfied:

$$\langle f, \phi_1 + \phi_2 \rangle = \langle f, \phi_1 \rangle + \langle f, \phi_2 \rangle$$

$$\langle f, \alpha\phi \rangle = \alpha \langle f, \phi \rangle. \tag{2.1}$$

The second property is *continuity*. A functional $f$ on $\mathcal{D}$ is said to be continuous if, for any sequence of testing functions $\{\phi_\nu(t)\}_{\nu=1}^{\infty}$that converges in $\mathcal{D}$ to $\phi(t)$, the sequence of numbers $\{\langle f, \phi_\nu \rangle\}_{\nu=1}^{\infty}$ converges to the number $\langle f, \phi \rangle$ in the ordinary sense. If $f$ is known to be linear, the definition of continuity may be some what simplified. In this case, $f$ will be continuous if the numerical sequence

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$\{\langle f, \phi_\nu \rangle\}_{\nu=1}^{\infty}$ converges to zero whenever the sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{D}$ to zero.

Thus, we may state the following definition of a distribution defined over the one-dimensional real Euclidean $\mathbb{R}$:

*A continuous linear function on the space $\mathcal{D}$ is a distribution.*

The space of all such distributions is denoted by $\mathcal{D}'$ and $\mathcal{D}'$ is called the of *dual space* of $\mathcal{D}$.

We can generate distributions by the regular function as follows. Let $f(t)$ be a locally integrable function (i.e., a function that is integrable in the Lebesgue sense over every finite interval). Corresponding to such $f(t)$, we can define a distribution $f$ through the convergent integral

$$\langle f, \phi \rangle = \langle f(t), \phi(t) \rangle \triangleq \int_{-\infty}^{\infty} f(t)\phi(t)dt \tag{2.2}$$

where $\phi$ is any testing function with compact support.

An example of a distribution that is not a regular distribution is the so called *Dirac delta function* $\delta$, which is defined by the equation

$$\langle \delta, \phi \rangle \triangleq \phi(0). \tag{2.3}$$

Clearly, (2.3) is a continuous linear functional on $\mathcal{D}$. However, this distribution cannot be obtained from a locally integrable function through the use of (2.2). Indeed, if there were such a function $\delta(t)$, then we would have

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)dt = \phi(0) \tag{2.4}$$

for all $\phi(t)$ in $\mathcal{D}$. Moreover, we conject a new singular distribution, the first derivative $\delta'(t)$ of the delta functional, the following definition suggests itself:

$$\langle \delta'(t), \phi(t) \rangle \triangleq -\phi'(0).$$

Next, an example of a distribution is so called *Heaviside unit step function* *H(t)*:

$$H(t) = \begin{cases} 0 & for\ t < 0 \\ \frac{1}{2} & for\ t = 0 \\ 1 & for\ t > 0. \end{cases}$$

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Then for any continuous function $\phi$ with compact support we have the result

$$\int_{-\infty}^{\infty} H(t)\phi(0)dt = \int_{0}^{\infty} \phi(t)dt. \tag{2.5}$$

We use the symbol $H$ to represent the mapping

$$H : \phi \longrightarrow \int_{0}^{\infty} \phi(t)dt$$

which is well defined by (2.5) for all continuous testing functions of compact support. This means that $H$ represents something other than an ordinary function.

### 2.1.3 Multiplication of Distribution by Infinitely Smooth Function

An operation that would be useful in analysis involving distributions would be the multiplication of two arbitrary distributions. Unfortunately, it is not possible to define such an operation in general. It turns out that the product does not always exist within the system of distributions. As an example, for the one-dimensional variable $t$, let $f(t) = 1/\sqrt{|t|}$. Then, $f(t)$ represents a regular distribution as well as a locally integrable function. Now,$[f(t)]^2$ is a function of $t$ defined for all nonzero $t$. But it is not integrable over any interval that includes the origin. This means that it cannot define a distribution through the expression

$$\langle \frac{1}{|t|}, \phi \rangle = \int_{-\infty}^{\infty} \frac{\phi(t)}{|t|} dt$$

since the integral does not converge for every $\phi$ in $\mathcal{D}$ In short, the product of $1/\sqrt{|t|}$ with itself does not exist as a distribution.

It is, however, possible to define the product of distributions in special cases. For instance, if $f$ and $g$ are locally integrable functions over $\mathbb{R}^n$ and if their product $fg$ is also locally integrable, then the product of the corresponding regular distributions exists as a regular distribution defined by

$$\langle fg, \phi \rangle = \int_{\mathbb{R}^n} f(t)g(t)\phi(t)\, dt \qquad \phi \in \mathcal{D}.$$

A more important case arises when one of the distributions $\psi$ is a regular distribution corresponding to an infinitely smooth function. The product of $\psi$ with any distribution $f$ in $\mathcal{D}'$ exists and is defined by

$$\langle \psi f, \phi \rangle \triangleq \langle f, \psi\phi \rangle \qquad \phi \in \mathcal{D}. \tag{2.6}$$

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For every $\phi$ in $\mathcal{D}$ the function $\psi\phi$ is infinitely smooth everywhere and zero whenever $\phi$ is zero. Hence, $\psi\phi$ is also in $\mathcal{D}$.Thus (2.6) defines that functional on $\mathcal{D}$ which assigns to each $\phi$ in $\mathcal{D}$ the number $\langle f, \psi\phi \rangle$.

### 2.1.4 The Differentiation of Distributions

Distribution, on the other hand, always possess derivatives, and these derivative are again distributions. In order to explain this statement, we must, of course, define what we mean by the *derivative of a distribution.* Let us restrict ourselves for the moment to the case when the independent variable $t$ has only one dimension. An appropriate definition can be constructed by considering a regular distribution $f(t)$ generated by a function which is differentiable everywhere and whose derivative is continuous. The derivative again generates a regular distribution $f'(t)$ and, for each $\phi$ in $\mathcal{D}$, an integration by part yields

$$\begin{aligned} \langle f'(t), \phi \rangle &= \int_{-\infty}^{\infty} f'(t)\phi(t)dt \\ &= -\int_{-\infty}^{\infty} f(t)\phi'(t)dt = \langle f, -\phi' \rangle. \end{aligned} \tag{2.7}$$

Note that $\phi'$ is in $\mathcal{D}$ whenever $\phi$ is in $\mathcal{D}$. Thus, a knowledge of $f$ (and, therefore, of $\langle f, -\phi' \rangle$ )determines $\langle f', \phi \rangle$. In other words, (2.7) defines $f'$ as a functional on $\mathcal{D}$. This result is generalized in the following definition.

*The first derivative $f'(t)$ of any distribution f(t), where t is one-dimensional, is the functional on $\mathcal{D}$ given by*

$$\langle f'(t), \phi(t) \rangle = \langle f(t), -\phi'(t) \rangle \qquad \phi \in \mathcal{D}. \tag{2.8}$$

At times, the conventional notation $df/dt$ will also be used for the derivative of a distribution defined over $\mathbb{R}$.

A simple illustration is provided by the first derivative of the delta functional $\delta'$, which is defined by the equation

$$\langle \delta', \phi \rangle = \langle \delta, -\phi' \rangle = -\phi'(0)$$

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and in general the $p$ th derivative, $\delta^{(p)}$, of the delta distribution is given by the mapping

$$\phi \longrightarrow \langle \delta^{(p)}, \phi \rangle = (-1)^p \phi^{(p)}(0).$$

**Example 1** The unit step function *H(t)* function that equals zero for $t < 0$, $\frac{1}{2}$ for $t = 0$, and 1 for $t > 0$. Its first distributional derivative is $\delta(t)$. For, with $\phi$ in $\mathcal{D}$

$$\begin{aligned} \langle H'(t), \phi(t) \rangle &= \langle H(t), -\phi'(t) \rangle \\ &= -\int_0^\infty \phi'(t) dt \\ &= \phi(0) = \langle \delta(t), \phi(t) \rangle. \end{aligned}$$

On the other hand, the ordinary derivative of *H(t)* is the function that is zero everywhere except at the origin, where it does not exist.

When the independent variable $t$ has $n$ dimensions, it is the partial derivatives that are defined in a fashion analogous to (2.8).

*The first-order partial derivatives* $\partial f / \partial t_i \, (i = 1, 2, 3, \ldots, n)$ *of any distribution f defined over* $\mathbb{R}^n$ *are the functionals on* $\mathcal{D}$ *given by*

$$\langle \frac{\partial f}{\partial t_i}, \phi \rangle = \langle f, -\frac{\partial \phi}{\partial t_i} \rangle \qquad i = 1, 2, 3, \ldots, n \, ; \, \phi \in \mathcal{D}$$

**Theorem 2.1.1** *A first-order partial derivative of a distribution is again a distribution.*

**Proof.** See [10, p 48]. $\square$

**Theorem 2.1.2** *The order of differentiation of the higher-order partial derivatives of a distribution defined over* $\mathbb{R}^n$ *can be changed at random. For in stance,*

$$\frac{\partial^2 f}{\partial t_i \partial t_k} = \frac{\partial^2 f}{\partial t_k \partial t_i}.$$

**Proof.** See [10, p 48]. $\square$

Since the order of differentiation is of no consequence, a given partial differential operator $D^k$, when action on a distribution, is sufficiently specified by writing

$$D^k = \prod_{i=1}^{n} \left( \frac{\partial}{\partial t_i} \right)^{k_i}$$

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where the order in which the differentiations $\partial/\partial t_i$ are taken need not be stated. Letting $\widehat{k} \triangleq \sum_{i=1}^{n} k_i$, we have

$$\langle D^k f, \phi \rangle = \langle f, (-1)^{\widehat{k}} D^k \phi \rangle.$$

The delta functional, now defined over $\mathbb{R}^n$, provides a simple illustration:

$$\begin{aligned} \langle D^k \delta, \phi \rangle &= \langle \delta, (-1)^{\widehat{k}} D^k \phi \rangle \\ &= (-1)^{\widehat{k}} D^k \phi(t)|_{t=0}. \end{aligned}$$

The rule for the differentiation of the product of a distribution $f$ and a function $\psi$, which is infinitely smooth, is the same as that for the product of two differentiable functions:

$$\frac{\partial}{\partial t_i}(\psi f) = \psi \frac{\partial f}{\partial t_i} + f \frac{\partial \psi}{\partial t_i}. \tag{2.9}$$

This is established as follows. For any $\phi$ in $\mathcal{D}$.

$$\begin{aligned} \left\langle \frac{\partial}{\partial t_i}(\psi f), \phi \right\rangle &= \left\langle \psi f, -\frac{\partial \phi}{\partial t_i} \right\rangle \\ &= \left\langle f, -\psi \frac{\partial \phi}{\partial t_i} \right\rangle \\ &= \left\langle f, -\frac{\partial (\psi \phi)}{\partial t_i} \right\rangle + \left\langle f, \phi \frac{\partial \psi}{\partial t_i} \right\rangle \\ &= \left\langle \frac{\partial f}{\partial t_i}, \psi \phi \right\rangle + \left\langle f, \phi \frac{\partial \psi}{\partial t_i} \right\rangle \\ &= \left\langle \psi \frac{\partial f}{\partial t_i}, \phi \right\rangle + \left\langle f \frac{\partial \psi}{\partial t_i}, \phi \right\rangle. \end{aligned}$$

Two important properties of the differentiation of distributions are given by

**Theorem 2.1.3** *Differentiation is a continuous linear operation in the space $\mathcal{D}'$ in the following sense:*

*Linearity. For any two distributions $f$ and $g$ and for any number $\alpha$,*

$$D^k(f+g) = D^k f + D^k g$$

$$D^k(\alpha f) = \alpha D^k f.$$

*Continuity. For any sequence of distributions $\{f_\nu\}_{\nu=1}^{\infty}$ that converges in $\mathcal{D}'$ to a distribution $f$, the corresponding sequence of partial derivatives $\{D^k f_\nu\}_{\nu=1}^{\infty}$ also converges in $\mathcal{D}'$ to $D^k f$.*

**Proof.** See [10, p 50]. □

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### 2.1.5 The Convolution of Distributions

Let $f(t)$ and $g(t)$ be two continuous functions with bounded support. Their convolution produces a third function $h(t)$, which is denoted by $f * g$ and defined by

$$h(t) \triangleq f(t) * g(t) \triangleq \int_{-\infty}^{\infty} f(t)g(t-\tau)d\tau.$$

Thus the rule that defines the convolution $f * g$ of two distributions $f(t)$ and $g(t)$ is suggested by this expression to be

$$\begin{aligned} \langle f * g, \phi \rangle &\triangleq \langle f(t) \times g(\tau), \phi(t+\tau) \rangle \\ &\triangleq \langle f(t), \langle g(\tau), \phi(t+\tau) \rangle \rangle. \end{aligned} \tag{2.10}$$

Even though the function $\phi(t+\tau)$ is infinitely smooth, it is not a testing function, since its support is not bounded in the $(t, \tau)$ plane. A meaning can still be assigned to the right-hand side of (2.10) if the supports of $f$ and $g$ are suitably restricted. In particular, if the support of $f(t) \times g(\tau)$ intersects the support of $\phi(t+\tau)$ in a bounded set, say $\Omega$, we can replace the right-hand side of (2.10) by

$$\langle f(t) \times g(\tau), \lambda(t,\tau)\phi(t+\tau) \rangle \tag{2.11}$$

where $\lambda(t, \tau)$ is some testing function in $\mathcal{D}_{t,\tau}$ (2.11) and, therefore,(2.10) serve to define $f * g$ in this case as a functional over all $\phi$ in $\mathcal{D}$.

**Theorem 2.1.4** *Let $f$ and $g$ be two distributions over $\mathbb{R}$ and let convolution $f * g$ be defined by (2.10), where the right-hand side of (2.10) is understood to be (2.11). Then, $f * g$ will exist as a distribution over $\mathbb{R}$ under any one of the following conditions:*

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*a. Either $f$ or $g$ have a bounded support.*

*b. Both $f$ and $g$ have supports bounded on the left(i.e.,there exists some constant $T_1$ such that $f(t) = g(t) = 0$ for $t < T_1$).*

*c. Both $f$ and $g$ have supports bounded on the right(i.e., there exists some constant $T_2$ such that $f(t) = g(t) = 0$ for $t > T_2$).*

**Proof.** See [10, p 124]. □

Since the direct product is commutative, it follows from (2.10) that the same property holds for convolution. Indeed,

$$\begin{aligned}\langle f * g, \phi\rangle &= \langle f(t) \times g(\tau), \phi(t+\tau)\rangle \\ &= \langle g(\tau) \times f(t), \phi(t+\tau)\rangle \\ &= \langle g * f, \phi\rangle \qquad (2.12)\end{aligned}$$

**Corollary 2.1.5** *The convolution of two distributions is commutative:*

$$f * g = g * f$$

*That is, for every* $\phi$ *in* $\mathcal{D}$,

$$\langle f(t), \langle g(\tau), \phi(t+\tau)\rangle\rangle = \langle g(\tau), \langle f(t), \phi(t+\tau)\rangle\rangle.$$

We mention in passing that the linearity of the direct product implies the linearity of the convolution process. That is, if $\alpha$ and $\beta$ are arbitrary constants and if $f$,$g$, and $h$ are distribution such that $f$ can be convolved with both $g$ and $h$ separately, then

$$f * (\alpha g + \beta h) = \alpha f * g + \beta f * h.$$

**Example 2** The convolution of the delta functional with any distribution yields that distribution again; the convolution of the $m$ th derivative of the delta functional with any distribution yields the $m$ th derivative of that distribution. In symbols,

$$\begin{aligned}\delta * f &= f \\ \delta^{(m)} * f &= f^{(m)}. \qquad (2.13)\end{aligned}$$

Note that these convolutions are valid for every distribution $f$ in $\mathcal{D}'$ because $\delta^{(m)}$ has a bounded support. The more general expression (2.13) may be justified as follows. For every $\phi$ in $\mathcal{D}$,

$$\begin{aligned}\langle \delta^{(m)} * f, \phi\rangle &= \langle f * \delta^{(m)}, \phi\rangle \\ &= \langle f(t), \langle \delta^{(m)}(\tau), \phi(t+\tau)\rangle\rangle \\ &= \langle f(t), (-1)^m \phi^{(m)}(t)\rangle \\ &= \langle f^{(m)}(t), \phi(t)\rangle.\end{aligned}$$

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An important consequence of (2.13) is that every linear differential operator with constant coefficients can be represented as a convolution. That is, with the $a_\nu$ being constants, we have

$$a_n f^{(n)} + a_{n-1} f^{(n-1)} + \ldots + a_0 f = (a_n \delta^{(n)} + a_{n-1}\delta^{(n-1)} + \ldots + a_0 \delta) * f.$$

Note that this statement could not be made if we restricted ourselves to the ordinary convolution of functions.

For the convolution equation

$$f * u = g \tag{2.14}$$

where $f$ and $g$ are known distributions in $\mathcal{D}'_R$ and $u$ is unknown but required to be in $\mathcal{D}'_R$, it is natural to solve (2.14) for $u$ by first finding a distribution that is an inverse of $f$ in the convolution algebra $\mathcal{D}'_R$. Such an inverse, which we denote by $f^{*-1}$, is any element of $\mathcal{D}'_R$ such that

$$f^{*-1} * f = \delta. \tag{2.15}$$

Then we may solve (2.14) by convolving both its sides by $f^{*-1}$ to obtain

$$u = \delta * u = f^{*-1} * f * u = f^{*-1} * g \tag{2.16}$$

**Theorem 2.1.6** *Let $f$ be a given distribution in $\mathcal{D}'_R$. A necessary and sufficient condition for (2.14) to have at least one solution in $\mathcal{D}'_R$ for every $g$ in $\mathcal{D}'_R$ is that $f$ possess an inverse $f^{*-1}$ in $\mathcal{D}'_R$. When $f$ does possess an inverse in $\mathcal{D}'_R$, this inverse in unique and (2.14) possess a unique solution in $\mathcal{D}'_R$, given by (2.16).*

**Proof.** See [10, p 151].

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## 2.2 Tempered Distributions

### 2.2.1 The Space $S$ Testing Functions of Rapid Descent

As usual, let $t \triangleq t_1, t_2, \ldots, t_n$ be the $n$-dimensional real variable and let $|t|$ denote

$$\sqrt{t_1^2 + t_2^2 + \ldots + t_n^2}$$

$\mathcal{S}$ is the space of all complex-valued functions $\phi(t)$ that are infinitely smooth and are such that, as $|t| \longrightarrow \infty$, they and all their partial derivatives decrease to zero faster than every power of $1/|t|$.

This required behavior as $|t| \longrightarrow \infty$ can also be stated in the following alternative way. When $t$ is one-dimensional, every function $\phi(t)$ in $\mathcal{S}$ satisfies the infinite set of inequalities

$$|t^m \phi^{(k)}(t)| \leq C_{mk}, \qquad -\infty < t < \infty \tag{2.17}$$

where $m$ and $k$ run through all nonnegative integers. Here the $C_{mk}$ are constants (with respect to $t$) which depend upon $m$ and $k$. When $t$ is $n$-dimensional, the requirement is that, for every set of nonnegative integers $m, k_1, k_2, \ldots, k_n$,

$$|t|^m \left| \frac{\partial^{k_1+k_2+\ldots+k_n}}{\partial t_1^{k_1} \partial t_2^{k_2} \ldots \partial t_n^{k_n}} \phi(t_1, t_2, \ldots, t_n) \right| \leq C_{m,k_1,k_2,\ldots,k_n} \tag{2.18}$$

over all of $\mathbb{R}^n$, where the quantity on the right-hand side of (2.18) is a constant with respect to $t$ but depends upon the choices of the $m, k_1, k_2, \ldots, k_n$. Because of the continuity of all the partial derivatives of $\phi(t)$, the order of differentiation in (2.18) may be changed in any fashion.

For the sake of simplicity, we shall use symbolism

$$k \triangleq k_1, k_2, \ldots, k_n$$

and

$$D^k \triangleq \frac{\partial^{k_1+k_2+\ldots+k_n}}{\partial t_1 \partial t_2 \ldots \partial t_n}$$

and we shall replace (2.18) by the shorthand notation

$$|t|^m |D^k \phi(t)| \leq C_{mk}. \tag{2.19}$$

The elements of $\mathcal{S}$ are called *testing functions of rapid descent.* $\mathcal{S}$ is a linear space. If $\phi$ is in $\mathcal{S}$, every one of its partial derivatives is again in $\mathcal{S}$. Furthermore, all the testing functions in $\mathcal{D}$ are also in $\mathcal{S}$. However, there are testing functions in $\mathcal{S}$ that are not in $\mathcal{D}$ as, for example,

$$\exp(-t_1^2 - t_2^2 - \ldots - t_n^2).$$

Thus, $\mathcal{D}$ is a proper subspace of $\mathcal{S}$.

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### 2.2.2 The Space $\mathcal{S}'$ of Distributions of Slow Growth

A distribution $f$ is said to be *of slow growth* if is a continuous linear functional on the space $\mathcal{S}$ of testing functions of rapid descent. (Such distributions are also called *temperate or tempered* distributions.) That is, a distribution $f$ of slow growth is a rule that assigns a number $\langle f, \phi \rangle$ to each $\phi$ in $\mathcal{S}$ in such away that the following conditions are fulfilled.

*Linearity*: If $\phi_1$ and $\phi_2$ are in $\mathcal{S}$ and if $\alpha$ is a number, then

$$\langle f, \phi_1 + \phi_2 \rangle = \langle f, \phi_1 \rangle + \langle f, \phi_2 \rangle$$

and

$$\langle f, \alpha\phi_1 \rangle = \alpha \langle f, \phi_1 \rangle.$$

*Continuity*: If $\{\phi_r\}_{r=1}^{\infty}$ is any sequence that converges in $\mathcal{S}$ to zero, then

$$\lim_{r \to \infty} \langle f, \phi_r \rangle = 0.$$

(As usual, in this continuity requirement we may replace the sequences $\{\phi_r\}_{r=1}^{\infty}$ by nondenumerable directed sets that converge in $\mathcal{S}$ to zero.)

The space of all distributions of slow growth is denoted by $\mathcal{S}'$. $\mathcal{S}'$ is also called the *dual* (or *conjugate*) *space of* $\mathcal{S}$.

Assume still that $f$ denotes a distribution of slow growth. Since $\mathcal{D}$ is a subspace of $\mathcal{S}$, $\langle f, \phi \rangle$ is defined whenever $\phi$ is in $\mathcal{D}$. $f$ is clearly linear as a functional on $\mathcal{D}$. Also, convergence in $\mathcal{D}$ implies converges in $\mathcal{S}$ and $\{\langle f, \phi_r \rangle\}_{r=1}^{\infty}$ therefore converges to zero whenever $\{\phi_r\}_{r=1}^{\infty}$ converges in $\mathcal{D}$ to zero. Thus, $f$ is also a distribution in $\mathcal{D}'$. Furthermore, a knowledge of $\langle f, \phi \rangle$, as $\phi$ traverses only the space $\mathcal{D}$, uniquely determines $\langle f, \phi \rangle$ for all $\phi$ in $\mathcal{S}$. This is because $\mathcal{D}$ is dense in $\mathcal{S}$ and each $\langle f, \phi \rangle$ ($\phi \in \mathcal{S}$) is the limit of ever sequence $\{\langle f, \phi_r \rangle\}_{r=1}^{\infty}$ where the $\phi$ are all in $\mathcal{D}$ and converge in $\mathcal{S}$ to $\phi$. In summary, $\mathcal{S}'$ is subspace of $\mathcal{D}'$; moreover, if $f_1$ and $f_2$ are both in$\mathcal{S}'$ and if $\langle f_1, \phi \rangle = \langle f_2, \phi \rangle$ for every $\phi$ in $\mathcal{D}$, then $\langle f_1, \phi \rangle = \langle f_2, \phi \rangle$ for every $\phi$ in $\mathcal{S}$.

$\mathcal{S}'$ is a *proper* subspace of $\mathcal{D}'$; that is, there are distributions in $\mathcal{D}'$ that are not in $\mathcal{S}'$. For in stance, the series

$$g(t) = \sum_{\mu=1}^{\infty} e^{\mu^2} \delta(t - \mu) \tag{2.20}$$

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defines a distribution in $\mathcal{D}'$. Indeed, given any $\phi$ in $\mathcal{D}$,

$$\langle g, \phi \rangle = \sum_{\mu=1}^{\infty} e^{\mu^2} \phi(\mu). \tag{2.21}$$

The last series possesses only a finite number of nonzero terms and therefore converges. On the other hand, there are testing functions in $\mathcal{S}$, such as $\phi(t) = \exp(-t^2)$, for which the series (2.21) does not converge. Hence, (2.20) is not a distribution of slow growth.

In order for a locally integrable function $f(t)$ to assign a finite number $\langle f, \phi \rangle$ to every testing function $\phi$ in $\mathcal{S}$ through the expression

$$\langle f, \phi \rangle \triangleq \int_{-\infty}^{\infty} f(t)\phi(t)dt \tag{2.22}$$

the behavior of $f(t)$ as $|t| \to \infty$ must be restricted that the integral converges for all $\phi$ in $\mathcal{S}$. This is certainly assured if $f(t)$ satisfies the condition

$$\lim_{t\to\infty} |t|^{-N} f(t) = 0 \tag{2.23}$$

for some integer $N$. Functions that satisfy (2.23) are said to be *functions of slow growth. Every locally integrable function of slow growth defines a regular distribution of slow growth through* (2.22).

Since each testing function in $\mathcal{S}$ certainly satisfies (2.23), it generates a regular distribution of slow growth.

Another fact, which can be readily proved, is that *every distribution in $\mathcal{D}'$ with a bounded support is of slow growth.* Thus, the delta functional and its derivatives are distributions of slow growth.

### 2.2.3 A Boundedness Property for Distributions of Slow Growth

For any given finite closed interval $I$ there exist a nonnegative integer $r$ and a constant $C$ such that for all testing functions $\phi$ in $\mathcal{D}_r$

$$|\langle f, \phi \rangle| \leq C \sup_t |\phi^{(r)}(t)|.$$

Here, $C$ and $r$ depend only on $f$ and $I$ and not on $\phi$. This result in no longer true for all $f$ in $\mathcal{D}'$ if we allow $I$ to be an infinite interval. However, all distributions

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of slow growth do possess a boundedness property of this sort that holds over the infinite interval, moreover, in this case $\phi$ may be a testing function in $\mathcal{S}$ and not merely in $\mathcal{D}$. Namely, for a given $f$ in $\mathcal{S}'$ and for all $\phi$ in $\mathcal{S}$

$$|\langle f, \phi\rangle| \leq C\sup_t |(1+t^2)^r \phi^{(r)}(t)| \tag{2.24}$$

where the constant $C$ and the nonnegative integer $r$ depend only on $f$.

**Theorem 2.2.1** *For each distribution $f$ of slow growth there exist a constant $C$ and a nonnegative integer $r$ such that the in equality (2.24) is fulfilled for every $\phi$ in $\mathcal{S}$. $C$ and $r$ depend only on $f$ and not on $\phi$.*

**Proof.** See [10]. □

## 2.3 Basic Concepts

**Definition 2.3.1** Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and write

$$u = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \ldots - x_{p+q}^2, \qquad p + q = n.$$

Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \textit{ and } u > 0\}$ the interior of forward cone and $\overline{\Gamma}_+$ denote its closure.

For any complex number $\alpha$, we define the function

$$R_\alpha^H(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+ \\ 0, & \text{for } x \notin \Gamma_+ \end{cases} \tag{2.25}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})}.$$

The function $R_\alpha^H$ is called the ultra-hyperbolic Kernel of Marcel Riesz and was introduced by Y. Nozaki [ 8, p72 ].

It is well known that $R_\alpha^H$ is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of $\alpha$ if $Re(\alpha) < n$. Let supp $R_\alpha^H(u)$ denote the support of $R_\alpha^H(u)$ and suppose supp $R_\alpha^H(u) \subset \overline{\Gamma}_+$, that is supp $R_\alpha^H(u)$ is compact.

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**Definition 2.3.2** Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and write

$$v = x_1^2 + x_2^2 + ... + x_n^2.$$

For any complex number $\beta$, define

$$R^e_\beta(v) = 2^{-\beta}\pi^{\frac{-n}{2}}\Gamma(\frac{n-\beta}{2})\frac{v^{\frac{\beta-n}{2}}}{\Gamma(\frac{\beta}{2})}. \tag{2.26}$$

The function $R^e_\beta(v)$ is called the elliptic Kernel of Marcel Riesz and is ordinary function if $Re(\beta) \geq n$ and is a distribution of $\beta$ if $Re(\beta) < n$.

**Definition 2.3.3** Let $x = (x_1, x_2, ..., x_n)$ be a point of the space $\mathbb{R}^n$ of the n-dimensional complex space and write

$$w = x_1^2 + x_2^2 + ... + x_p^2 - i\left(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2\right)$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$ and $i = \sqrt{-1}$.
For any complex number $\gamma$, define the function

$$S_\gamma(w) = 2^{-\gamma}\pi^{\frac{-n}{2}}\Gamma(\frac{n-\gamma}{2})\frac{w^{\frac{\gamma-n}{2}}}{\Gamma(\frac{\gamma}{2})}. \tag{2.27}$$

The function $S_\gamma(w)$ is an ordinary function if $Re(\gamma) \geq n$ and is a distribution of $\gamma$ if $Re(\gamma) < n$.

**Definition 2.3.4** For any complex number $\nu$, define the function

$$T_\nu(z) = 2^{-\nu}\pi^{\frac{-n}{2}}\Gamma(\frac{n-\nu}{2})\frac{z^{\frac{\nu-n}{2}}}{\Gamma(\frac{\nu}{2})} \tag{2.28}$$

where

$$z = x_1^2 + x_2^2 + ... + x_p^2 + i\left(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2\right),$$

$x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, p + q = n$ is the dimension of $\mathbb{R}^n$ and $i = \sqrt{-1}$.

**Definition 2.3.5** Let $x = (x_1, x_2, ..., x_n)$ be a point of the space $\mathbb{R}^n$ and the function $Y_{\alpha,\beta,\gamma,\nu}(u, v, w, z, m)$ is defined by

$$Y_{\alpha,\beta,\gamma,\nu}(u, v, w, z, m) = \sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(\eta/2 + r)}{r!\Gamma(\eta/2)}(m^2)^r K_{\alpha+2r,\beta+2r,\gamma+2r,\nu+2r}(u, v, w, z), \tag{2.29}$$

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where $K_{\alpha+2r,\beta+2r,\gamma+2r,\nu+2r}(u,v,w,z)$ is defined by

$$K_{\alpha+2r,\beta+2r,\gamma+2r,\nu+2r}(u,v,w,z) = (-1)^{\beta/2+r} R^H_{\alpha+2r}(u) * R^e_{\beta+2r}(v) * S_{\gamma+2r}(w) * T_{\nu+2r}(z) \tag{2.30}$$

$\eta$ is any complex number and $m$ is a nonnegative real number.

**Definition 2.3.6** Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$ and the function $R^e_\alpha(x)$ be defined by

$$R^e_\alpha(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)}$$

where $W_n(\alpha) = \frac{\pi^{n/2}2^\alpha\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}$, $\alpha$ is a complex parameter and $|x| = (x_1^2+x_2^2+\ldots+x_n^2)^{1/2}$.

**Lemma 2.3.7** *The functions $R^H_{2k}(u)$ and $(-1)^k R^e_{2k}(v)$ are the elementary solutions of the operators $\Box^k$ and $\triangle^k$, respectively, where the operators $\Box^k$ and $\triangle^k$ are defined by* (1.5) *and* (1.6) *respectively , $R^H_{2k}(u)$ and $R^e_{2k}(v)$ are defined by* (2.25) *and* (2.26), *respectively, with $\alpha = \beta = 2k$. That is,*

$$\Box^k\big(R^H_{2k}(u)\big) = \delta \tag{2.31}$$

*and*

$$\triangle^k\big((-1)^k R^e_{2k}(v)\big) = \delta \tag{2.32}$$

*where $\delta$ is a Dirac-delta distribution.*

**Proof.** See [9, $p$.147; 4, $p$.31] □

**Lemma 2.3.8** *The convolution $R^H_{2k}(u) * (-1)^k R^e_{2k}(v)$ is an elementary solution of the operator $\Diamond^k$ iterated k-times and is defined by equation* (1.2).

**Proof.** See [4, $p$. 33] □

**Lemma 2.3.9** 1. *The functions $(-1)^k(-i)^{q/2}S_{2k}(w)$ and $(-1)^k(i)^{q/2}T_{2k}(z)$ are the elementary solutions of the operators $L_1^k$ and $L_2^k$, respectively, where $S_{2k}(w)$ and $T_{2k}(z)$ are defined by* (2.27) *and* (2.28), *respectively, with $\gamma = \eta = 2k$. The operators $L_1^k$ and $L_2^k$ are defined by* (1.3) *and* (1.4), *respectively. That is,*

$$L_1^k\big((-1)^k(-i)^{q/2}S_{2k}(w)\big) = \delta \tag{2.33}$$

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*and*

$$L_2^k\big((-1)^k(i)^{q/2}T_{2k}(z)\big) = \delta \tag{2.34}$$

*where $\delta$ is a Dirac-delta distribution.*

2. *The functions $(-1)^k(-i)^{q/2}S_{-2k}(w)$ and $(-1)^k(i)^{q/2}T_{-2k}(z)$ are the inverses in the convolution algebras of $(-1)^k(-i)^{q/2}S_{2k}(w)$ and $(-1)^k(i)^{q/2}T_{2k}(z)$ respectively.*

**Proof.** $(i)$ [cf.[6]] We need to show that

$$L_1^k\big((-1)^k(-i)^{q/2}S_{2k}(w)\big) = \delta$$

and

$$L_2^k\big((-1)^k(i)^{q/2}T_{2k}(z)\big) = \delta$$

At first we have to show that

$$L_1^k S_\gamma(w) = (-1)^k S_{\gamma-2k}(w), \tag{2.35}$$

$$L_2^k T_\nu(z) = (-1)^k T_{\nu-2k}(z), \tag{2.36}$$

and also

$$S_{-2k}(w) = (-1)^k(i)^{q/2}L_1^k\delta, \tag{2.37}$$

$$T_{-2k}(z) = -1)^k(-i)^{q/2}L_2^k\delta. \tag{2.38}$$

Now, for $k=1$,

$$L_1 S_\gamma(w) = \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)S_\gamma(w),$$

where $S_\gamma(w)$ is defined by (2.27) By computing directly, we obtain

$$\begin{aligned} L_1 S_\gamma(w) &= 2^{-\gamma}\Pi^{-n/2}\frac{\Gamma(\frac{n-\gamma}{2})}{\Gamma(\frac{\gamma}{2})}(\gamma-n)(\gamma-2)w^{(\gamma-2-n)/2} \\ &= (-1)2^{-(\gamma-2)}\frac{\Gamma(\frac{n-(\gamma-2)}{2})}{\Gamma(\frac{\gamma-2}{2})}w^{(\gamma-2-n)/2} \end{aligned}$$

by the properties of Gamma function. Thus $L_1S_\gamma(w) = -S_{\gamma-2}(w)$. By repeating $k$-times in operating $L_1$ to $S_{\gamma-2k}(w)$, we obtain

$$L_1^k S_\gamma(w) = (-1)^k S_{\gamma-2k}(w).$$

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Similarly,

$$L_2^k T_\nu(z) = (-1)^k T_{\nu-2k}(z).$$

Thus we obtain (2.35) and (2.36) as required. Now consider

$$w = x_1^2 + x_2^2 + ... + x_p^2 - i\left(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2\right), \qquad p + q = n$$

by changing the variable

$$x_1 = y_1, x_2 = y_2, \ldots, x_p = y_p, x_{p+1} = \frac{y_{p+1}}{\sqrt{-i}}, x_{p+2} = \frac{y_{p+2}}{\sqrt{-i}}, \ldots, x_{p+q} = \frac{y_{p+q}}{\sqrt{-i}}.$$

Thus we have

$$w = y_1^2 + y_2^2 + \ldots + y_p^2 + y_{p+1}^2 + \ldots + y_{p+q}^2.$$

Denote $w = r^2 = y_1^2 + y_2^2 + \ldots + y_n^2$ and consider the generalized function $w^\lambda = r^{2\lambda}$ where $\lambda$ is any complex number. Now $\langle w^\lambda, \varphi\rangle = \int_{\mathbb{R}^n} w^\lambda \varphi(x)dx$, where $\varphi \in \mathcal{D}$ the space of infinitely differentiable functions with compact supports. Thus

$$\begin{aligned}
\langle w^\lambda, \varphi\rangle &= \int_{\mathbb{R}^n} r^{2\lambda}\varphi \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)} dy_1 dy_2 \ldots dy_n \\
&= \frac{1}{(-i)^{q/2}} \int_{\mathbb{R}^n} r^{2\lambda}\varphi dy \\
&= \frac{1}{(-i)^{q/2}} \langle r^{2\lambda}, \varphi\rangle.
\end{aligned}$$

By Gelfand and Shilov $[3, p.271]$, the functional $r^{2\lambda}$ have simple poles at $\lambda = (-n/2) - k, k$ is nonnegative and for $k = 0$ we can find the residue of $r^{2\lambda}$ at $\lambda = -n/2$ and by $[6, p.73]$, we obtain

$$\underset{\lambda=-n/2}{res}\, r^{2\lambda} = \frac{2\pi^{-n/2}}{\Gamma(n/2)}\delta(x).$$

Thus

$$\underset{\lambda=-n/2}{res}\, w^\lambda = (i)^{2(q/2)} \frac{\pi^{n/2}}{\Gamma(n/2)}\delta(x). \tag{2.39}$$

we next find the residues of $w^\lambda$ at $\lambda = (-n/2) - k$. Now, by computing directly we have

$$L_1 w^{\lambda+1} = 2(\lambda+1)(2\lambda+n)w^\lambda,$$

where $w$ is defined by Definition 2.3.3 and $L_1$ is defined by (1.3). By $k$-fold iteration, we obtain

$$L_1^k w^{\lambda+k} = 4^k(\lambda+1)(\lambda+2)\ldots(\lambda+k)(\lambda+\frac{n}{2})(\lambda+\frac{n}{2}+1)\ldots(\lambda+\frac{n}{2}+k-1)w^\lambda$$

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or

$$w^{\lambda} = \frac{1}{4^k(\lambda+1)(\lambda+2)\ldots(\lambda+k)(\lambda+\frac{n}{2})(\lambda+\frac{n}{2}+1)\ldots(\lambda+\frac{n}{2}+k-1)} L_1^k w^{\lambda+k}.$$

Thus

$$\underset{\lambda=(-n/2)-k}{res} w^{\lambda} = \frac{1}{4^k k(\frac{n}{2}+k-1)(\frac{n}{2}+k-2)\ldots\frac{n}{2}} \underset{\lambda=-n/2}{res} L_1^k w^{\lambda}.$$

By (2.39) and the properties of Gamma functions, we obtain

$$\underset{\lambda=(-n/2)-k}{res} w^{\lambda} = \frac{2(i)^{q/2}\pi^{n/2}}{4^k\Gamma(\frac{n}{2}+k)} L_1^k\delta(x). \tag{2.40}$$

Now we consider $S_{-2k}(w)$ we have

$$\begin{aligned} S_{-2k}(w) &= \lim_{\gamma\to-2k} S(w) \\ &= \pi^{-n/2}\frac{\lim_{\gamma\to-2k} w^{(\gamma-n)/2}}{\lim_{\gamma\to-2k}\Gamma(\frac{\gamma}{2})} \lim_{\gamma\to-2k}(2^{-\gamma}\Gamma(\frac{n-\gamma}{2})) \\ &= \pi^{-n/2}\frac{\lim_{\gamma\to-2k}(\gamma+2k)w^{(\gamma-n)/2}}{\lim_{\gamma\to-2k}(\gamma+2k)\Gamma(\frac{\gamma}{2})} 2^{2k}\Gamma(\frac{n+2k}{2}) \\ &= 4^k\pi^{-n/2}\frac{\underset{\gamma=-2k}{res}\, w^{(\gamma-n)/2}}{\underset{\gamma=-2k}{res}\,\Gamma(\frac{\gamma}{2})}\Gamma(\frac{n+2k}{2}). \end{aligned}$$

Since

$$\underset{\lambda=(-n/2)-k}{res} w^{\lambda} = \underset{\gamma=-2k}{res}\, w^{(\gamma-n)/2} \qquad and \qquad \underset{\gamma=-2k}{res}\,\Gamma(\frac{\gamma}{2}) = \frac{2(-1)^k}{k!},$$

by (2.40) and the properties of Gamma function we obtain

$$S_{-2k}(w) = (-1)^k(i)^{q/2}L_1^k\delta(x).$$

Similarly

$$T_{-2k}(z) = (-1)^k(-i)^{q/2}L_2^k\delta(x).$$

Thus we have

$$\begin{aligned} S_0(w) &= (i)^{q/2}L_1^k\delta(x), \\ T_0(w) &= (-i)^{q/2}L_1^k\delta(x). \end{aligned} \tag{2.41}$$

Now, from (2.35) $L_1^k S_{2k}(w) = (-1)^k S_0(w)$ for $\gamma = 2k$. Thus, by (2.41) we obtain $L_1^k(-1)^k(-i)^{q/2}S_{2k}(w) = \delta(x)$. It follows that $(-1)^k(-i)^{q/2}S_{2k}(w)$ is an elementary

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solution of the operator $L_1^k$. Similarly $(-1)^k(i)^{q/2}T_{2k}(z)$ is also an elementary solution of $L_2^k$.

$(ii)$ We need to show that

$$[(-1)^k(-i)^{q/2}S_{2k}(w)] * [(-1)^k(-i)^{q/2}S_{-2k}(w)] = \delta$$

and

$$[(-1)^k(i)^{q/2}T_{2k}(z)] * [(-1)^k(i)^{q/2}T_{-2k}(z)] = \delta.$$

Now, from (2.37) we have $(-1)^k(-i)^{q/2}S_{-2k}(w) = L_1^k\delta$ convolving both side by $(-1)^k(-i)^{q/2}S_{2k}(w)$ we obtain

$$\begin{aligned} [(-1)^k(-i)^{q/2}S_{2k}(w)] * [(-1)^k(-i)^{q/2}S_{-2k}(w)] &= [(-1)^k(-i)^{q/2}S_{2k}(w)] * L_1^k\delta \\ &= L_1^k[(-1)^k(-i)^{q/2}S_{2k}(w)] * \delta \\ &= \delta * \delta \\ &= \delta \end{aligned}$$

by Lemma 2.3.9(1).

Similarly $[(-1)^k(i)^{q/2}T_{2k}(z)] * [(-1)^k(i)^{q/2}T_{-2k}(z)] = \delta.$ □

**Lemma 2.3.10** *Given $P$ is a hyper-surface then*

$$P\delta^{(k)}(P) + k\delta^{(k-1)}(P) = 0$$

*where $\delta^{(k)}$ is the Dirac-delta distribution with $k$ derivatives.*

**Proof.** See [2, $p.$ 233] □

**Lemma 2.3.11** *Given the equation*

$$\triangle^k u(x) = 0 \tag{2.42}$$

*where $\triangle^k$ is defined by* (1.6) *and* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *then* $u(x) = (-1)^{(k-1)}\left(R^e_{2(k-1)}(x)\right)^{(m)}$ *is a solution of (2.42) where $m$ is a nonnegative integer with* $m = \frac{n-4}{2}, n \geq 4$ *and $n$ is even and* $\left(R^e_{2(k-1)}(x)\right)^{(m)}$ *is a function defined by Definition* 2.3.6 *or* (2.26) *with $m$ derivatives and* $\alpha = 2(k-1)$

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**Proof.** [cf.[5]] We first show that the generalized function $u(x) = \delta^{(m)}(r^2)$ where $r^2 = |x|^2 = x_1^2 + x_2^2 + \ldots + x_n^2$ is a solution of

$$\triangle u(x) = 0 \tag{2.43}$$

where $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is a Laplace operator. Now

$$\frac{\partial}{\partial x_i}\delta^{(m)}(r^2) = 2x_i\delta^{(m+1)}(r^2)$$

$$\frac{\partial^2}{\partial^2 x_i^2}\delta^{(m)}(r^2) = 2\delta^{(m+1)}(r^2) + 4x_i^2\delta^{(m+2)}(r^2).$$

Thus

$$\begin{aligned}\triangle\delta^{(m)}(r^2) &= \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2)\\ &= 2n\delta^{(m+1)}(r^2) + 4r^2\delta^{(m+2)}(r^2)\\ &= 2n\delta^{(m+1)}(r^2) - 4(m+2)\delta^{(m+1)}(r^2)\end{aligned}$$

by Lemma 2.3.10 with $P = r^2$. We have

$$\begin{aligned}\triangle\delta^{(m)}(r^2) &= \left[2n - 4(m+2)\right]\delta^{(m+1)}(r^2)\\ &= 0 \qquad if\ 2n - 4(m+2) = 0\end{aligned}$$

or $m = \frac{n-4}{2}, n \geq 4$ and n is even. Thus $\delta^{(m)}(r^2)$ is a solution of (2.43) with $m = \frac{n-4}{2}, n \geq 4$ and n is even. Now $\triangle^k u(x) = \triangle(\triangle^{k-1}u(x)) = 0$ then from the above proof $\triangle^{k-1}u(x) = \delta^{(m)}(r^2)$ with $m = \frac{n-4}{2}, n \geq 4$ and n is even.

Convolving both sides of the above equation by the function $(-1)^{(k-1)}\left(R^e_{2(k-1)}(x)\right)$, we obtain

$$\begin{aligned}(-1)^{(k-1)}R^e_{2(k-1)}(x) * \triangle^{k-1}u(x) &= (-1)^{(k-1)}R^e_{2(k-1)}(x) * \delta^{(m)}(r^2)\\ or\ \ \triangle^{k-1}(-1)^{k-1}R^e_{2(k-1)}(x) * u(x) &= (-1)^{(k-1)}R^e_{2(k-1)}(x) * \delta^{(m)}(r^2)\end{aligned}$$

or $\delta * u(x) = u(x) = (-1)^{(k-1)}R^e_{2(k-1)}(x) * \delta^{(m)}(r^2)$ by (2.32). Now from *Definition* 2.3.6

$$\begin{aligned}R^e_{2(k-1)}(x) &= \frac{|x|^{2(k-1)-n}}{W_n(\alpha)}\\ &= \frac{\left(|x|^2\right)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)}\\ &= \frac{\left(r^2\right)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)}\end{aligned}$$

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where $r = |x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$. Hence

$$\begin{aligned} R^e_{2(k-1)}(x) * \delta^{(m)}(r^2) &= \frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} * \delta^{(m)}(r^2) \\ &= \left[ \frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} \right]^{(m)} \\ &= \left[ R^e_{2(k-1)}(x) \right]^{(m)}. \end{aligned}$$

It follows that $u(x) = (-1)^{k-1} \left[ R^e_{2(k-1)}(x) \right]^{(m)}$ is a solution of (2.42) with $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension of $\mathbb{R}^n$. □

**Lemma 2.3.12** *Given the equation*

$$\triangle u(x) = f(x, u(x)) \tag{2.44}$$

*where $f$ is defined and has continuous first derivatives for all $x \in \Omega \cup \partial\Omega, \Omega$ is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ is the boundary of $\Omega$. Assume that $f$ is bounded, that is $|f(x, u(x)| \leq N$ and the boundary condition $u(x) = 0$ for $x \in \partial\Omega$. Then the equation (2.44) has a unique solution u(x).*

**Proof.** See [1 , P. 369 – 372] □

**Theorem 2.3.13** *Given the equation*

$$(\oplus + m^2)^k G(x) = \delta(x) \tag{2.45}$$

*where $(\oplus + m^2)^k$ is the operator iterated k-times defined by* (1.7), *$\delta$ is the Dirac-delta distribution, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $k$ is a nonnegative integer. Then we obtain $G(x) = Y_{2k,2k,2k,2k}(u, v, w, z, m)$,*

$$Y_{2k,2k,2k,2k}(u, v, w, z, m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r! \Gamma(k)} (m^2)^r K_{2k+2r,2k+2r,2k+2r,2k+2r}(u, v, w, z) \tag{2.46}$$

*as an elementary solution of* (2.45) *where $m$ is a nonnegative real number and $K_{2k+2r,2k+2r,2k+2r,2k+2r}(u, v, w, z)$ is defined by* (2.30) *with $\alpha = \beta = \gamma = \nu = \eta = 2k$.*

**Proof.**$[cf.[10]]$ At first,

$$\Gamma\left(\frac{\eta}{2} + r\right) = \frac{\eta}{2}\left(\frac{\eta}{2} + 1\right) \cdots \left(\frac{\eta}{2} + r - 1\right) \Gamma\left(\frac{\eta}{2}\right).$$

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Equivalently,

$$(-1)^r \frac{1}{r!}\Gamma\left(\frac{\eta}{2}+r\right) = \frac{(-1)^r \frac{\eta}{2}\left(\frac{\eta}{2}+1\right)\cdots\left(\frac{\eta}{2}+r-1\right)\Gamma\left(\frac{\eta}{2}\right)}{r!}$$
$$= \frac{\left(-\frac{\eta}{2}\right)\left(-\frac{\eta}{2}-1\right)\cdots\left[-\left(\frac{\eta}{2}+r-1\right)\right]}{r!}\Gamma\left(\frac{\eta}{2}\right).$$

Now we put, by definition,

$$(-1)^r \frac{1}{r!}\Gamma\left(\frac{\eta}{2}+r\right) = \binom{-\frac{\eta}{2}}{r}\Gamma\left(\frac{\eta}{2}\right).$$

Then, we obtain the function $Y_{\alpha,\beta,\gamma,\nu}(u,v,w,z,m)$ is defined by (2.29) become

$$Y_{\alpha,\beta,\gamma,\nu}(u,v,w,z,m) = \sum_{r=0}^{\infty}\binom{-\frac{\eta}{2}}{r}(m^2)^r K_{\alpha+2r,\beta+2r,\gamma+2r,\nu+2r}(u,v,w,z). \tag{2.47}$$

Putting $\eta = 2k$ and $\alpha=\beta=\gamma=\nu=2k$ in (2.47), we have

$$Y_{2k,2k,2k,2k}(u,v,w,z,m) = \sum_{r=0}^{\infty}\binom{-k}{r}(m^2)^r K_{2k+2r,2k+2r,2k+2r,2k+2r}(u,v,w,z)$$
$$= \sum_{r=0}^{\infty}\binom{-k}{r}(m^2)^r(-1)^{k+r}R^H_{2k+2r}(u) * R^e_{2k+2r}(v)$$
$$* S_{2k+2r}(w) * T_{2k+2r}(z). \tag{2.48}$$

Since, the operator $\diamondsuit, L_1, L_2, \Box$ and $\triangle$ are defined by (1.2), (1.3), (1.4), (1.5) and (1.6) respectively, are linearly continuous and 1–1 mapping. Then all of them possess their own inverses. From Lemma 2.3.7,Lemma 2.3.8 and Lemma 2.3.9, we obtain

$$Y_{2k,2k,2k,2k}(u,v,w,z,m) = \sum_{r=0}^{\infty}\binom{-k}{r}(m^2)^r\Box^{-k-r}\delta * \triangle^{-k-r}\delta * L_1^{-k-r}\delta * L_2^{-k-r}\delta$$
$$= \sum_{r=0}^{\infty}\binom{-k}{r}(m^2)^r\diamondsuit^{-k-r}L^{-k-r}\delta$$
$$= \sum_{r=0}^{\infty}\binom{-k}{r}(m^2)^r \oplus^{-k-r}\delta$$
$$= (\oplus + m^2)^{-k}\delta. \tag{2.49}$$

By applying the operator $(\oplus + m^2)^k$ to both sides of (2.49), we obtain

$$(\oplus + m^2)^k Y_{2k,2k,2k,2k}(u,v,w,z,m) = (\oplus + m^2)^k.(\oplus + m^2)^{-k}\delta. \tag{2.50}$$

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Thus

$$(\oplus + m^2)^k Y_{2k,2k,2k,2k}(u,v,w,z,m) = \delta.$$

Moreover, by putting $\beta = \gamma = \nu = -2r$ in equation (2.30) we obtain

$$\begin{aligned} K_{\alpha+2r,0,0,0}(u,v,w,z) &= R^H_{\alpha+2r}(u) * R^e_0(v) * S_0(w) * T_0(z) \\ &= R^H_{\alpha+2r}(u) * \delta * \delta * \delta = R^H_{\alpha+2r}(u). \end{aligned}$$

Then (2.47) become

$$Y_{\alpha,-2r,-2r,-2r}(u,v,w,z,m) = \sum_{r=0}^{\infty} \binom{-\frac{\eta}{2}}{r} (m^2)^r R^H_{\alpha+2r}(u). \tag{2.51}$$

Now, putting $\alpha = \eta = 2k$ to obtain

$$\begin{aligned} Y_{2k,-2r,-2r,-2r}(u,v,w,z,m) &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r R^H_{2k+2r}(u) \\ &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \Box^{-k-r} \delta \\ &= (\Box + m^2)^{-k} \delta. \end{aligned} \tag{2.52}$$

By applying the operator $(\Box + m^2)^k$ to both sides of (2.52), we obtain

$$(\Box + m^2)^k Y_{2k,-2r,-2r,-2r}(u,v,w,z,m) = (\Box + m^2)^k (\Box + m^2)^{-k} \delta = \delta. \tag{2.53}$$

Then $Y_{2k,-2r,-2r,-2r}(u,v,w,z,m) = W^H_{2k}(u,m)$ as an elementary solution of the ultra-hyperbolic Klein-Gordon operator iterated $k$-times defined by $W^H_{2k}(u,m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r! \Gamma(k)} (m^2)^r R^H_{2k+2r}(u)$. In particular, putting $\alpha = \gamma = \nu = -2r$ and $\beta = \eta = 2k$ of (2.47), we obtain

$$\begin{aligned} Y_{-2r,2k,-2r,-2r}(u,v,w,z,m) &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R^e_{2k+2r}(v) \\ &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \triangle^{-k-r} \delta \\ &= (\triangle + m^2)^{-k} \delta. \end{aligned} \tag{2.54}$$

By applying the operator $(\triangle + m^2)^k$ to both sides of (2.54), we obtain that $Y_{-2r,2k,-2r,-2r}(u,v,w,z,m)$ is an elementary solution of Helmholtz operator $k$-times.

Similarly,

$$Y_{2k,2k,-2r,-2r} = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R^H_{\alpha+2r}(u) * R^e_{2k+2r}(v)$$

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is the Green function of the operator $(\Diamond + m^2)^k$. □

**Lemma 2.3.14** *The function* $R^e_{-2k}(v)$ *is the inverse of the convolution algebra of* $R^e_{2k}(v)$*, that is*

$$R^e_{-2k}(v) * R^e_{2k}(v) = R^e_{-2k+2k}(v) = R^e_0(v) = \delta$$

**Proof.** See [2 , *P.* 158]. □

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