Chapter 3

MAIN RESULTS

In this chapter we find the solution of nonlinear equations of product of the operators $\oplus^k$ and the operator $(\oplus+m^2)^k$, next we studied the relation or property of its solution.

Theorem 3.1 *Consider the nonlinear equation*

$$\oplus^k(\oplus+m^2)^k u(x) = f\big(x, \triangle^{k-1}\Box^k L_1^k L_2^k(\oplus+m^2)^k u(x)\big) \tag{3.1}$$

where the operator $\oplus^k$ *and* $\oplus^k(\oplus+m^2)^k$ *are defined by* (1.1)*and* (1.7) *respectively,* $\triangle^{k-1}$ *is the Laplace operator defined by* (1.6)*,* $\Box^k$ *is the ultra-hyperbolic operator defined by* (1.5)*, and the operator* L_1 *and* L_2 *are defined by* (1.3) *and* (1.4)*, respectively. Let* f *be defined and having continuous first derivative for all* $x \in \Omega \cup \partial\Omega$*, where* Ω *is an open subset of* $\mathbb{R}^n$ *and* $\partial\Omega$ *denotes the boundary of* Ω *and* n *is even with* $n \geq 4$*. Suppose* f *is bounded, that is*

$$|f\big(x, \triangle^{k-1}\Box^k L_1^k L_2^k(\oplus+m^2)^k u(x)\big)| \leq N \tag{3.2}$$

for all $x \in \Omega$ *and the boundary condition*

$$\triangle^{k-1}\Box^k L_1^k L_2^k(\oplus+m^2)^k u(x) = 0 \tag{3.3}$$

for all $x \in \partial\Omega$*. Then we obtain*

$$\begin{aligned} u(x) =& (-1)^{k-1} Y_{2k,2k,2k,2k}(u,v,w,z,m) * (i)^{q/2} T_{2k}(z) * (-i)^{q/2} S_{2k}(w) \\ & * R^H_{2k}(u) * R^e_{2(k-1)}(v) * W(x) \end{aligned} \tag{3.4}$$

as a solution of (3.1) *with the boundary condition*

$$u(x) = R^H_{2k}(u) * (-i)^{q/2} S_{2k}(w) * (i)^{q/2} T_{2k}(z) * Y_{2k,2k,2k,2k}(u,v,w,z,m) \\ * (-1)^{k-2} \big(R^e_{2(k-2)}(v)\big)^{(m)} \tag{3.5}$$

for $x \in \partial\Omega$, $m = \frac{(n-4)}{4}$, $k = 2,3,4,\ldots$, $W(x)$ *is a continuous function for* $x \in \Omega \cup \partial\Omega$.

Proof. The nonlinear equation (3.1) can be written in the form

$$\begin{aligned} \oplus^k(\oplus + m^2)^k u(x) &= \Diamond^k L_1^k L_2^k (\oplus + m^2)^k u(x) \\ &= \triangle(\triangle^{k-1} \Box^k L_1^k L_2^k (\oplus + m^2)^k u(x) \\ &= f\big(x, \triangle^{k-1} \Box^k L_1^k L_2^k (\oplus + m^2)^k u(x)\big) \end{aligned} \tag{3.6}$$

since $u(x)$ has continuous derivative up to order $8k$ for $k = 2,3,4,\ldots$ we can assume that

$$\triangle^{k-1} \Box^k L_1^k L_2^k (\oplus + m^2)^k u(x) = W(x) \tag{3.7}$$

for all $x \in \Omega$. Thus, (3.6) can be written in the form

$$\oplus^k(\oplus + m^2)^k u(x) = \triangle W(x) = f(x, W(x)) \tag{3.8}$$

by (3.2), we have the condition

$$|f(x, W(x))| \leq N \tag{3.9}$$

for all $x \in \Omega$, by (3.3) we have the condition

$$W(x) = \triangle^{k-1} \Box^k L_1^k L_2^k (\oplus + m^2)^k u(x) = 0, \tag{3.10}$$

for $x \in \partial\Omega$. Consider the equations (3.8) (3.9) and (3.10) by Lemma 2.3.12 there exists a unique solution $W(x)$ of (3.8) which satisfy the boundary condition (3.10). Now consider the equation (3.7), convolving both side of (3.7) by $(-1)^{k-1} R^e_{2(k-1)}(v)$, we obtain

$$(-1)^{k-1} R^e_{2(k-1)}(v) * \triangle^{k-1} \Box^k L_1^k L_2^k (\oplus + m^2)^k u(x) = (-1)^{k-1} R^e_{2(k-1)}(v) * w(x)$$

by the properties of convolution and (2.32) we have

$$\begin{aligned} (-1)^{k-1} R^e_{2(k-1)}(v) * w(x) &= \big[\triangle^{k-1}((-1)^{k-1} R^e_{2(k-1)}(v))\big] * \Box^k L_1^k L_2^k (\oplus + m^2)^k u(x) \\ &= \delta * \Box^k L_1^k L_2^k (\oplus + m^2)^k u(x) \\ &= \Box^k L_1^k L_2^k (\oplus + m^2)^k u(x). \end{aligned}$$

Convolving both sides of above equation by $R^H_{2k}(u)$, $(-1)^k(-i)^{q/2}S_{2k}(w)$, $(-1)^k(i)^{q/2}T_{2k}(z)$ and $Y_{2k,2k,2k,2k}(u,v,w,z,m)$ respectively, then by Lemma 2.3.7, Lemma 2.3.9 and Theorem 2.3.13 we obtain

$$\begin{aligned} u(x) =& Y_{2k,2k,2k,2k}(u,v,w,z,m) * (-1)^k(i)^{q/2}T_{2k}(z) * (-1)^k(-i)^{q/2}S_{2k}(w) \\ & * R^H_{2k}(u) * (-1)^{k-1}R^e_{2(k-1)}(v) * W(x) \\ =& (-1)^{k-1}Y_{2k,2k,2k,2k}(u,v,w,z,m) * (i)^{q/2}T_{2k}(z) * (-i)^{q/2}S_{2k}(w) \\ & * R^H_{2k}(u) * R^e_{2(k-1)}(v) * W(x) \end{aligned} \tag{3.11}$$

as a solution of (3.1) as required.

Next, consider the equation

$$\triangle^{k-1}\square^k L_1^k L_2^k(\oplus + m^2)^k u(x) = 0$$

for all $x \in \partial\Omega$. By Lemma 2.3.11 we have

$$\square^k L_1^k L_2^k(\oplus + m^2)^k u(x) = \left(R^e_{2(k-2)}(v)\right)^{(m)} \tag{3.12}$$

where $m = \frac{(n-4)}{2}, n \geq 4$ and n is even. Convolving both sides of above equation by $R^H_{2k}(u) * (-i)^{q/2}S_{2k}(w) * (i)^{q/2}T_{2k}(z) * Y_{2k,2k,2k,2k}(u,v,w,z,m)$, we obtain

$$\begin{aligned} u(x) =& R^H_{2k}(u) * (-i)^{q/2}S_{2k}(w) * (i)^{q/2}T_{2k}(z) * Y_{2k,2k,2k,2k}(u,v,w,z,m) \\ & * (-1)^{k-2}\left(R^e_{2(k-2)}(v)\right)^{(m)} \end{aligned} \tag{3.13}$$

for $x \in \partial\Omega$ and $k = 2, 3, 4, \ldots$. □

In particular, convolving both side of equation (3.11) by $(-1)^{k-1}R^e_{2(1-k)}(v) * (i)^{q/2}T_{-2k}(z) * (-i)^{q/2}S_{-2k}(w)$ we obtain

$$\begin{aligned} & (-1)^{k-1}R^e_{2(1-k)}(v) * (i)^{q/2}T_{-2k}(z) * (-i)^{q/2}S_{-2k}(w) * u(x) \\ & = R^H_{2k}(u) * Y_{2k,2k,2k,2k}(u,v,w,z,m) * W(x) \end{aligned}$$

by Lemma 2.3.14 and Lemma 2.3.9. If we put $\alpha = \gamma = \nu = -2r$ and $\beta = \eta = 2k$ in equation (2.29) we obtain $Y_{-2r,2k,-2r,-2r}(u,v,w,z,m)$ is an elementary solution of $(\triangle + m^2)^k$ operator, see [10]. Thus by Lemma 2.3.7 and Theorem 2.3.13 we obtain

$$V(x) = (-1)^{k-1}R^e_{2(1-k)}(v) * (i)^{q/2}T_{-2k}(z) * (-i)^{q/2}S_{-2k}(w) * u(x)$$

as a solution of the equation $\Box^k(\triangle + m^2)^k V(x) = W(x)$. If we put $k = 1, p = n, q = 0$ then the solution $V(x)$ is the solution of the inhomogeneous biharmonic equation

$$\triangle^2 V(x) = g(x, \triangle V(x))$$

where $g(x, \triangle V(x)) = W(x) - m^2 \triangle V(x)$.