CHAPTER 1

INTRODUCTION

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and/or nonoscillation of solutions of second order linear and nonlinear neutral delay differential equations because of their importance in various applications. For instance, in biological applications, delay equations give better description of fluctuations in population than the ordinary ones. Also neutral delay differential equations have been used to describe various processes in physics and engineering sciences.

This nonlinear equations can be considered as a natural generalization of the half-linear differential equation

$$\big(r(t)|u'(t)|^{\alpha-1}u'(t)\big)' + q(t)|u(t)|^{\alpha-1}u(t) = 0, \quad \alpha > 0, \tag{1.1}$$

which has been the object of intensive studies by many authors (for example, see [1, 9, 15]). In 2000, the extension of the results of Philos [12] for differential equation

$$\big(r(t)\psi(u(t))|u'(t)|^{\alpha-1}u'(t)\big)' + q(t)f\big(u(t)\big) = 0, \quad \alpha > 0, \tag{1.2}$$

was obtained by Manojlović [10].

In order to present the oscillation criteria of Manojlović [10] we first introduce, following Philos [12], the class of functions $\mathcal{P}$ which will be extensively used in the sequel. Namely, let $D_0 = \{(t,s) : t > s \geq t_0\}$ and $D = \{(t,s) : t \geq s \geq t_0\}$. We will say that the function $H \in C\big(D; \mathbb{R}\big)$ belongs to the class $\mathcal{P}$ if it satisfies the following two conditions:

(1) $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ for $(t,s) \in D_0$,

(2) H has a continuous and nonpositive partial derivative in D_0 with respect to the second variable.

Theorem 1.1 (Manojlović [10, Theorem 2.4]) *Suppose that*

$$\frac{f'(u)}{\big(\psi(u)|f(u)|^{\alpha-1}\big)^{\frac{1}{\alpha}}} \geq K > 0 \quad \textit{for} \;\; u \neq 0, \tag{C_1}$$

and there exists the function $\rho \in C^1([t_0,\infty);(0,\infty))$ *and* $h, H \in C(D;\mathbb{R})$ *such that* H *belongs to the class* $\mathcal{P}$ *and*

$$h(t,s) = -\frac{\partial}{\partial s}H(t,s) \text{ is nonnegative continuous function on } D.$$

Then equation (1.2) *is oscillatory if*

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Big[\rho(s)q(s)H(t,s) - \beta\frac{\rho(s)r(s)}{H^{\alpha}(t,s)}G^{\alpha+1}(t,s)\Big]ds = \infty,$$

where $G(t,s) = h(t,s) + \dfrac{\rho'(s)}{\rho(s)}H(t,s)$ *and* $\beta = \dfrac{1}{\alpha K^{\alpha}}\Big(\dfrac{\alpha}{\alpha+1}\Big)^{\alpha+1}$.

Theorem 1.2 (Manojlović [10, Theorem 2.5]) *Let condition* (C_1) *holds and let the functions* H *and* h *be defined as in Theorem 1.1, and moreover, suppose that*

$$0 < \inf_{s\geq t_0}\Big[\lim_{t\to\infty}\inf\frac{H(t,s)}{H(t,t_0)}\Big] \leq \infty. \qquad (C_2)$$

Then equation (1.2) *is oscillatory if there exists* $\rho \in C^1([t_0,\infty);(0,\infty))$ *such that*

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{\rho(s)r(s)}{H^{\alpha}(t,s)}\Big(h(t,s) + \frac{\rho'(s)}{\rho(s)}H(t,s)\Big)^{\alpha+1}(t,s)ds < \infty,$$

and there exists a continuous function ϕ *on* $[t_0,\infty)$ *such that*

$$\lim_{t\to\infty}\sup\frac{1}{H(t,T)}\int_{T}^{t}\Big[\rho(s)q(s)H(t,s) - \beta\frac{\rho(s)r(s)}{H^{\alpha}(t,s)}G^{\alpha+1}(t,s)\Big]ds \geq \phi(T),$$

for $T \geq t_0$ *and*

$$\int_{t_0}^{\infty}\frac{\phi_{+}^{\frac{\alpha}{\alpha+1}}(s)}{r^{\frac{1}{\alpha}}(s)}ds = \infty, \qquad (C_3)$$

where $\phi_{+}(s) = \max\{\phi(s), 0\}$.

Theorem 1.3 (Manojlović [10, Theorem 2.6]) *Let condition* (C_1), (C_2) *hold and let the functions* H *and* h *be defined as in Theorem 1.1. Then equation* (1.2) *is oscillatory if there exists* $\rho \in C^1([t_0,\infty);(0,\infty))$ *such that*

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}\rho(s)q(s)H(t,s)ds < \infty,$$

and there exists a continuous function ϕ *on* $[t_0,\infty)$ *such that*

$$\lim_{t\to\infty}\inf\frac{1}{H(t,T)}\int_{T}^{t}\Big[\rho(s)q(s)H(t,s) - \beta\frac{\rho(s)r(s)}{H^{\alpha}(t,s)}G^{\alpha+1}(t,s)\Big]ds \geq \phi(T),$$

and (C_3) *holds.*

In recent years, Džurina and Stavroulakis [6], Sun and Meng [14], and Agarwal [1] considered the second order delay differential equation

$$\big(r(t)|u'(t)|^{\alpha-1}u'(t)\big)' + q(t)|u(\sigma(t))|^{\alpha-1}u(\sigma(t)) = 0, \quad \alpha > 0. \tag{1.3}$$

Džurina and Stavroulakis [6] are established some new oscillation criteria for the second order retarded differential equation (1.3).

Theorem 1.4 *Let* $\alpha \geq 1$. *Suppose that*

$$\lim_{t\to\infty} R(t) = \lim_{t\to\infty} \int_{t_0}^{t} \Big(\frac{1}{r(s)}\Big)^{\frac{1}{\alpha}} ds = \infty \tag{C_4}$$

and for some $k \in (0,1)$,

$$\int^{\infty} \Big(R^{\alpha}(\sigma(t))q(t) - \frac{\alpha\sigma'(t)}{4kR(\sigma(t))r^{\frac{1}{\alpha}}(\sigma(t))}\Big)dt = \infty.$$

Then Eq.(1.3) *is oscillatory.*

Theorem 1.5 *Let* $0 < \alpha < 1$. *Assume that* (C_4) *holds and*

$$\int^{\infty} \Big(R^{\alpha}(\sigma(t))q(t) - \frac{\alpha\sigma'(t)}{4R^{2-\alpha}(\sigma(t))r^{\frac{2}{\alpha}-1}(\sigma(t))\hat{p}(t)}\Big)dt = \infty,$$

where

$$\hat{p}(t) = \Big(\frac{1}{r(\sigma(t))}\int_t^{\infty} q(s)ds\Big)^{\frac{1-\alpha}{\alpha}}.$$

Then Eq.(1.3) *is oscillatory.*

In 2006, Xu and Meng [16] concerned with the oscillatory behavior of solutions of the second order quasi-linear neutral delay differential equations of the form

$$\Big(r(t)\big|\big(u(t)+p(t)u(t-\tau)\big)'\big|^{\alpha-1}\big(u(t)+p(t)u(t-\tau)\big)'\Big)' + q(t)f\big(u(\sigma(t))\big) = 0. \tag{1.4}$$

Xu and Meng [16] proved the following oscillation criteria for equation (1.4).

Theorem 1.6 *Assume that* (C_4) *holds and*

$$\frac{f(u)}{|u|^{\alpha-1}u} \geq \beta > 0 \ \ \text{for } u \neq 0, \ \ \beta \text{ is a constant.}$$

Then Eq.(1.4) *is oscillatory if*

$$\int^{\infty} \Big(\beta R^{\alpha}(\sigma(t))\bar{p}(t) - \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{\sigma'(t)}{R(\sigma(t))r^{\frac{1}{\alpha}}(\sigma(t))}\Big)dt = \infty,$$

where $\bar{p}(t) = q(t)\big(1 - p(\sigma(t))\big)^{\alpha}$.

The results of Džurina and Stavroulakis [6] have been improved and extended to neutral delay differential equations by Xu and Meng [16].

Motivated by the work of the above results, we are concerned with the oscillatory behavior problem of the solutions of the follwing second order nonlinear neutral delay differential equations of the form

$$\Big(r(t)\psi(u(t))\big|\big(u(t)+p(t)u(\tau(t))\big)'\big|^{\alpha-1}\big(u(t)+p(t)u(\tau(t))\big)'\Big)'+q(t)f\big(u(\sigma(t))\big)=0 \tag{E}$$

for $t \geq t_0$, where $\alpha > 0$ is a constant. Throughout this thesis we suppose that the following conditions hold:

(H_1) $r \in C^1\big([t_0,\infty);\mathbb{R}\big)$, $r(t) > 0$ and $R(t) := \int_{t_0}^{t} \Big(\frac{1}{r(s)}\Big)^{\frac{1}{\alpha}} ds \to \infty$ as $t \to \infty$,

(H_2) $\psi, f \in C\big(\mathbb{R};\mathbb{R}\big)$, $0 < \psi(u) \leq M$ and $uf(u) > 0$ for $u \neq 0$, M is a constant,

(H_3) $p, q \in C\big([t_0,\infty);\mathbb{R}\big)$, $0 \leq p(t) \leq 1$ and $q(t) \geq 0$,

(H_4) $\tau \in C\big([t_0,\infty);\mathbb{R}\big)$, $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$,

(H_5) $\sigma \in C^1\big([t_0,\infty);\mathbb{R}\big)$, $\sigma'(t) \geq 0$, $\sigma(t) \leq t$ and $\lim_{t\to\infty} \sigma(t) = \infty$.

In this thesis, we shall continue in this direction the study of oscillatory properties of Eq.(E). The purpose of this thesis is to improve, generalize and extend the above-mentioned results. We shall further the investigation and offer some criteria for oscillation of Eq.(E).

This thesis is organized as follows. In next Chapter, we present some notations and preliminaries. The main results of oscillation criteria and applications for neutral differential equations are given in Chapter 3. The final Chapter provides conclusion of oscillation criteria for second order nonlinear neutral delay differential equations.