CHAPTER 2

PRELIMINARIES

This chapter is essentially introductory in nature. Its main purpose is to present some basic concepts from the theory of delay differential equations and to sketch some preliminary results which will be used throughout the report. In section 2.1, we are concerned with the statement of the basic initial value problems and classification of equations with delays. In section 2.2, we provide definition of oscillation of solutions with or without delays. Section 2.3, 2.4, 2.5, 2.6 and 2.7 state increasing and decreasing functions, the completeness property of $\mathbb{R}$, the fundamental of calculus, Hölder's inequality and the Riccati equation which are important tools in oscillation theory, respectively.

2.1 Initial Value Problems

Let us consider the ordinary differential equation (ODE)

$$u'(t) = f(t, u) \tag{2.1}$$

together with the initial condition

$$u(t_0) = u_0. \tag{2.2}$$

It is well known that under certain assumptions on f the initial value problem (2.1) and (2.2) has a unique solution and is equivalent to the integral equation

$$u(t) = u(t_0) + \int_{t_0}^{t} f\big(s, u(s)\big)ds \;\text{ for }\; t \geq t_0.$$

Next, we consider a differential equation of the form

$$u'(t) = f\big(t, u(t), u(t-\tau)\big) \;\text{ with }\; \tau > 0 \;\text{ and }\; t \geq t_0 \tag{2.3}$$

in which the right-hand side depends not only on the instantaneous position $u(t)$, but also on $u(t-\tau)$, the position at τ units back, that is to say, the equation has past memory. Such an equation is called an *ordinary differential equation with delay* or *delay differential equation.* Whenever necessary, we shall consider the integral equation

$$u(t) = u(t_0) + \int_{t_0}^{t} f\big(s, u(s), u(s-\tau)\big)ds \quad \text{for} \quad t \geq t_0,$$

which is equivalent to (2.3), we need to have a known function φ on $[t_0 - \tau, t_0]$, instead of just the initial condition $u(t_0) = u_0$.

The basic initial value problem for a delay differential equation is posed as follows: On the interval $[t_0, T], T \leq \infty$, we seek a continuous function u that satisfies (2.3) and initial condition

$$u(t) = \varphi(t) \quad \text{for all} \quad t \in E_{t_0}, \tag{2.4}$$

where t_0 is an initial point, $E_{t_0} = [t_0 - \tau, t_0]$ is initial set; the known function φ on E_{t_0} is called the *initial function.* Usually, it is assumed that $\varphi(t_0 + 0) = \varphi(t_0)$. We always mean a one-sided derivative when we speak of the derivative at an endpoint of an interval.

Under general assumptions, the existence and uniqueness of solutions to the initial value problem (2.3) and (2.4) can be established. The solution sometimes is denoted by $u(t, \varphi)$. In the case of a variable delay $\tau = \tau(t) > 0$ in (2.3), it is also required to find a solution of this equation for $t > t_0$ such that on the initial set

$$E_{t_0} = t_0 \cup \{t - \tau(t) : t - \tau(t) < t_0,\ t \geq t_0\},$$

u coincides with the given initial function φ. If it is required to determine the solution on the interval $[t_0, T]$, then the initial set is

$$E_{t_0T} = \{t_0\} \cup \{t - \tau(t) : t - \tau(t) < t_0,\ t_0 \leq t \leq T\}.$$

Example 2.1.1 Consider the equation

$$u'(t) = f\big(t, u(t), u(t - \cos^2 t)\big),$$

$t_0 = 0, E_0 = [-1, 0]$, and the initial function φ must be given on the interval $[-1, 0]$.

The initial set E_{t_0} depends on the initial point t_0. Consider from the equation

$$y'(t) = ay(t/2),$$

we have $\tau(t) = t/2$ so that

$$E_0 = \{0\} \text{ and } E_1 = [1/2, 1].$$

2.2 Definition of Oscillation

Before we define oscillation of solutions, let us consider some examples.

Example 2.2.1 Consider the equation

$$u''(t) + u(t) = 0$$

has periodic solutions $u_1(t) = \cos t$ and $u_2(t) = \sin t$.

Example 2.2.2 Consider the equation

$$u''(t) - \frac{1}{t}u'(t) + 4t^2u(t) = 0,$$

whose solution is $u(t) = \sin t^2$. This solution is not periodic but has an oscillatory property.

Example 2.2.3 Consider the equation

$$u''(t) + \frac{1}{2}u'(t) - \frac{1}{2}u(t-\pi) = 0 \text{ for } t \geq 0,$$

whose solution $u(t) = 1 - \sin t$ has an infinite sequence of multiple zeros. This solution also has an oscillatory property.

Let us now restrict our discussion to those solutions u of the equation

$$u''(t) + q(t)u(t) = 0 \tag{2.5}$$

which exist on some ray $[T_u, \infty)$ and satisfy $\sup\{|u(t)| : t \geq T\} > 0$ for every $T \geq T_u$. In other words, $u(t) \neq 0$ for at least one $t \in [T, \infty)$. Such a solution sometimes is said to be a *regular solution*.

Definition 2.2.1 A nontrivial solution u (implying a regular solution always) is said to be *oscillatory* if it has arbitrarily large zeros for $t \geq t_0$, that is, there exists a sequence of zeros $\{t_n\}$ (i.e., $u(t_n) = 0$) of u such that $\lim_{n\to\infty} t_n = \infty$. Otherwise, u is said to be *nonoscillatory*.

For nonoscillatory solutions there exists t_1 such that

$$u(t) \neq 0 \quad \text{for all} \quad t \geq t_1.$$

Definition 2.2.2 Eq.(2.5) is said to be *oscillatory* if all of its solution are oscillatory.

2.3 Increasing and Decreasing Functions

In aim of this section is to give some definitions and properties of the increasing and decreasing functions which refer to book of R. G. Bartle [4].

Definition 2.3.1 Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be continuous function. Then, for $x_1, x_2 \in I$

(1) f is said to be *increasing* on I if $x_1 \leq x_2$, then $f(x_1) \leq f(x_2)$,

(2) f is said to be *strictly increasing* on I if $x_1 < x_2$, then $f(x_1) < f(x_2)$,

(3) f is said to be *decreasing* on I if $x_1 \leq x_2$, then $f(x_1) \geq f(x_2)$,

(4) f is said to be *strictly decreasing* on I if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Theorem 2.3.1 *Let $f : I \to \mathbb{R}$ be differentiable on the interval I. Then*

(1) *f is increasing on I if and only if $f'(x) \geq 0$ for all $x \in I$,*

(2) *f is decreasing on I if and only if $f'(x) \leq 0$ for all $x \in I$.*

Corollary 2.3.1 *Let $f : I \to \mathbb{R}$ be differentiable on the interval I. Then*

(1) *if f' is positive on I, then f is strictly increasing on I,*

(2) *if f' is negative on I, then f is strictly decreasing on I.*

2.4 The Completeness Property of $\mathbb{R}$

In aim of this section is to give the completeness property of $\mathbb{R}$.

Definition 2.4.1 Let S be a nonempty subset of $\mathbb{R}$.

(1) An element $u \in \mathbb{R}$ is said to be an *upper bound* of S if $s \leq u$ for all $s \in S$.

(2) An element $w \in \mathbb{R}$ is said to be a *lower bound* of S if $w \leq s$ for all $s \in S$.

Definition 2.4.2 Let S be a nonempty subset of $\mathbb{R}$.

(1) If S is bounded above, then an upper bound of S is said to be a *supremum* (or a *least upper bound*) of S if it is less than any other upper bound of S.

(2) If S is bounded below, then a lower bound of S is said to be a *infimum* (or a *greatest lower bound*) of S if it is greater than any other lower bound of S.

Theorem 2.4.1 (Supremum Property)
Every nonempty set of real numbers which has an upper bound has a supremum.

Theorem 2.4.2 (Infimum Property)
Every nonempty set of real numbers which has a lower bound has a infimum.

We say that a sequence $X = (x_n)$ is *bounded* if there exists $M > 0$ such that $\|x_n\| < M$ for all $n \in \mathbb{N}$.

Definition 2.4.3 Let $X = (x_n)$ be a bounded sequence in $\mathbb{R}$.

(1) The *limit superior* of X, which we denote by

$$\limsup X, \qquad \limsup\ (x_n), \qquad \text{or} \qquad \overline{\lim}\ (x_n),$$

is the infimum of the set V of $v \in \mathbb{R}$ such that there are at most a finite number of $n \in \mathbb{N}$ such that $v < x_n$.

(2) The *limit inferior* of X, which we denote by

$$\liminf X, \qquad \liminf\ (x_n), \qquad \text{or} \qquad \underline{\inf}\ (x_n),$$

is the supremum of the set W of $w \in \mathbb{R}$ such that there are at most a finite number of $m \in \mathbb{N}$ such that $x_m < w$.

Theorem 2.4.3 *If $X = (x_n)$ is bounded sequence in $\mathbb{R}$, then the following statements are equivalent for a real number x^*.*

(1) $x^* = \limsup(x_n)$.

(2) *If $\epsilon > 0$, then there are at most a finite number of $n \in \mathbb{N}$ such that $x^* + \epsilon < x_n$, but there are an finite number such that $x^* - \epsilon < x_n$.*

(3) *If $v_m = \sup\{x_n : n \geq m\}$, then $x^* = \inf\{v_m : n \in \mathbb{N}\}$.*

(4) *If $v_m = \sup\{x_n : n \geq m\}$, then $x^* = \lim(v_m)$.*

(5) *If L is the set of $v \in \mathbb{R}$ such that there exists a subsequence of X which converges to v, then $x^* = \sup L$.*

Theorem 2.4.4 *Let $X = (x_n)$ and $Y = (y_n)$ be bounded sequences of real numbers. Then the following relations holds.*

(1) $\liminf(x_n) \leq \limsup(x_n)$.

(2) *If $c \geq 0$, then $\liminf(cx_n) = c \liminf(x_n)$ and $\limsup(cx_n) = c \limsup(x_n)$.*

(3) *If $c \leq 0$, then $\liminf(cx_n) = c \limsup(x_n)$ and $\limsup(cx_n) = c \liminf(x_n)$.*

(4) $\liminf(x_n) + \liminf(y_n) \leq \liminf(x_n + y_n)$.

(5) $\limsup(x_n + y_n) \leq \limsup(x_n) + \limsup(y_n)$.

(6) *If $x_n \leq y_n$, then $\liminf(x_n) \leq \liminf(y_n)$ and $\limsup(x_n) \leq \limsup(y_n)$.*

Theorem 2.4.5 (Monotone Convergence Theorem)

(1) *Let $X = (x_n)$ be a sequence of real numbers which is monotone increasing in the sense that*

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots .$$

Then the sequence X converges if and only if it is bounded, in which case

$$\lim(x_n) = \sup\{x_n\}.$$

(2) *Let $X = (x_n)$ be a sequence of real numbers which is monotone decreasing in the sense that*

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq \cdots .$$

Then the sequence X converges if and only if it is bounded, in which case

$$\lim(x_n) = \inf\{x_n\}.$$

2.5 The Fundamental Theorem of Calculus

Theorem 2.5.1 (The First Fundamental Theorem of Calculus)
Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and let $F : [a, b] \to \mathbb{R}$ satisfy the conditions:

(1) *F is continuous on $[a, b]$,*

(2) *the derivative F' exists and $F'(x) = f(x)$ for all $x \in (a, b)$.*

Then

$$\int_a^b f \, dx = F(b) - F(a). \tag{2.6}$$

Corollary 2.5.1 *Let $F : [a, b] \to \mathbb{R}$ satisfy the conditions:*

(1) *the derivative F' exists on $[a, b]$,*

(2) *the function F' is integrable on $[a, b]$.*

Then equation (2.6) holds with $f = F'$.

Theorem 2.5.2 (The Second Fundamental Theorem of Calculus)
Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and let

$$F(x) = \int_a^x f\ dx \qquad \text{for } x \in [a, b];$$

then F is continuous on $[a, b]$. Moreover, if f is continuous at a point $c \in [a, b]$, then F is differentiable at c and

$$F'(c) = f(c).$$

Corollary 2.5.2 *Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and let*

$$F(x) = \int_a^x f\ dx \qquad \text{for } x \in [a, b].$$

Then F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.

Theorem 2.5.3 (The Combined Fundamental Theorem of Calculus)
Let F and f be continuous functions on $[a, b]$ and let $F(a) = 0$. Then the following statements are equivalent:

(1) *$F'(x) = f(x)$ for all $x \in [a, b]$,*

(2) *$F(x) = \int_a^x f\ dx$ for all $x \in [a, b]$.*

Definition 2.5.1 Let $I = [a, b]$ be an interval in $\mathbb{R}$.

(1) If $f : I \to \mathbb{R}$, then an *antiderivative* of f on I is a function $F : I \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in I$.

(2) If $f : I \to \mathbb{R}$ is *integrable* on I, then the function $F : I \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f\ dx \qquad \text{for } x \in I$$

is called the *indefinite integral* of f on I.

Theorem 2.5.4 (Integration by Parts)

If $f, g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$ and have antiderivatives F, G on $[a, b]$, then

$$\int_a^b F(x)g(x)\,dx = \Big(F(b)G(b) - F(a)G(a)\Big) - \int_a^b f(x)G(x)\,dx.$$

Theorem 2.5.5 (First Substitution Theorem)

Let $J = [\alpha, \beta]$ and let $\varphi : J \to \mathbb{R}$ have a continuous derivative on J. If f is continuous on an interval I containing $\varphi(J)$, then

$$\int_\alpha^\beta f\big(\varphi(t)\big)\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx.$$

Theorem 2.5.6 (Second Substitution Theorem)

Let $J = [\alpha, \beta]$ and let $\varphi : J \to \mathbb{R}$ have a continuous derivative such that $\varphi'(t) \neq 0$ for $t \in J$. Let I be an interval containing $\varphi(J)$, and let $\psi : I \to \mathbb{R}$ be the function inverse to φ. If $f : I \to \mathbb{R}$ is continuous on I, then

$$\int_\alpha^\beta f\big(\varphi(t)\big)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\psi'(x)\,dx.$$

Theorem 2.5.7 *Let $a < b$ and let f and g be two real and piecewise continuous functions on $[a, b]$ such that $f(x) \leq g(x)$ for all the points of continuity of f and g (except, perhaps, in a finite number of points). Then*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Equality holds if and only if $f(x) = g(x)$ in all the points of continuity of f and g.

2.6 Hölder's Inequality

Theorem 2.6.1 *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are functions defined on $[a, b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a, b]$. Then*

$$\int_a^b |f(x)g(x)|dx \leq \Big(\int_a^b |f(x)|^p dx\Big)^{\frac{1}{p}}\Big(\int_a^b |g(x)|^q dx\Big)^{\frac{1}{q}},$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$ almost everywhere, where A and B are constants.

2.7 The Riccati Equation

If the substitution $z = \dfrac{r(t)u'(t)}{u(t)}$ is made in the self-adjoint differential equation

$$\big(r(t)u'(t)\big)' + q(t)u(t) = 0 \tag{2.7}$$

where $r(t)$ and $q(t)$ are continuous on an interval $[a,b]$, we obtain

$$\big(z(t)u(t)\big)' + q(t)u(t) = 0,$$

or

$$z'(t) + \frac{1}{r(t)}z^2(t) + q(t) = 0. \tag{2.8}$$

Equation (2.8) is a *Riccati Equation*. The general Riccati equation is usually written as

$$z'(t) + a(t)z(t) + b(t)z^2(t) + c(t) = 0, \tag{2.9}$$

where we shall suppose $a(t)$, $b(t)$, and $c(t)$ are continuous on the interval $[a,b]$. Equation (2.9) is only apparently more general than equation (2.8), since the substitution in (2.9) of

$$w(t) = e^{\int_a^t a(s)ds} z(t) \tag{2.10}$$

reduces this equation to

$$w'(t) + q(t)w^2(t) + p(t) = 0, \tag{2.11}$$

where $\quad q(t) = b(t)e^{-\int_a^t a(s)ds}$, and $\quad p(t) = c(t)e^{\int_a^t a(s)ds}$.

If $b(t) = 0$, equation (2.9) is, of course, linear and it is immediately integrable. If $b(t) \neq 0$ on any subinteval of $[a,b]$, to study the solutions of (2.9) we may employ the substitution (2.10) to reduce (2.9) to the form (2.11). The substitution $q(t)w(t) = \dfrac{u'(t)}{u(t)}$ then reduces (2.11) to the form (2.7), where $r(t) = \dfrac{1}{q(t)}$. The zeros of $q(t)$ are then singular points of the differential equation (2.7). It will be observed that these successive substitutions may be replaced by the substitution $b(t)z(t) = \dfrac{u'(t)}{u(t)}$.

Example 2.7.1 Study the solutions of the Riccati equation

$$w'(t) - w^2(t) - 1 = 0. \tag{2.12}$$

This equation is already in the form (2.11), where $q(t) = -1$ and $p(t) = -1$. The substitution $-w(t) = \dfrac{u'(t)}{u(t)}$ leads then to the linear self-adjoint differential equation.

$$u''(t) + u(t) = 0,$$

the general solution of which is $c_1 \sin t + c_2 \cos t$. The null solution $(c_1 = c_2 = 0)$ leads to no solution w. All other solutions u provide solutions

$$w(t) = \frac{c_1 \cos t - c_2 \sin t}{c_1 \sin t + c_2 \cos t} \quad (c_1 \text{ and } c_2 \text{ not both zero}).$$

of (2.12). The choice $c_1 = 0$ leads to the particular solution $w(t) = \tan t$.