CHAPTER 3

OSCILLATION CRITERIA FOR SECOND-ORDER NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

In the general, the theory of neutral delay differential equations is more complicated than the theory of delay differential equations without neutral terms. In this chapter, we will present some results in the oscillation theory of second order neutral delay differential equations, and consequently this will be a useful source for researchers in this field. The study of oscillation for second order ordinary differential equations, we divided into 3 sections. In section 3.1, we shall establish some new oscillation criteria for second order nonlinear neutral delay differential equations. In section 3.2, we provide some sufficient conditions for second order nonlinear neutral delay differential equations by using Philos's class function idea. In section 3.3, we exemplify oscillation of second order nonlinear neutral delay differential equations.

Before we show detail of oscillation criteria for second order nonlinear neutral delay differential equations, let us state two sets of conditions commonly used in the literature which we rely on.

(S_1) f is a nondecreasing function and $f'(u)$ exists such that

$$\frac{f'(u)}{|f(u)|^{\frac{\alpha-1}{\alpha}}} \geq \gamma > 0$$

for some positive constant γ.

(S_2) $\dfrac{f(u)}{|u|^{\alpha-1}u} \geq \beta > 0$ for some positive constant β.

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3.1 Some New Oscillation Criteria

In order to prove our theorems we use the following well-known inequality which is due to Hardy, Littlewood, and Polya.

Lemma 3.1.1 ([7]) *If X and Y are nonnegative, then*

$$X^\lambda + (\lambda - 1)Y^\lambda - \lambda XY^{\lambda-1} \geq 0 \quad \textit{for} \quad \lambda > 1,$$

where the equality holds if and only if $X = Y$.

Lemma 3.1.2 *If $u(t)$ is an eventually positive solution of Eq.(E), let*

$$z(t) = u(t) + p(t)u(\tau(t)), \tag{3.1}$$

then $z(t)z'(t)$ is an eventually positive.

Proof. Suppose that $u(t)$ is an eventually positive solution of $Eq.(E)$ such that $u(t) > 0$, $u(\tau(t)) > 0$ and $u(\sigma(t)) > 0$ on $[T, \infty)$ for some sufficiently large $T \geq t_0$. Then $z(t) \geq u(t) > 0$ and moreover, Eq.(E) can be rewritten as

$$\Big(r(t)\psi(u(t))|z'(t)|^{\alpha-1}z'(t)\Big)' + q(t)f\big(u(\sigma(t))\big) = 0, \tag{3.2}$$

for $t \geq T$. It follows from (H_2) and (H_3) that

$$\Big(r(t)\psi(u(t))|z'(t)|^{\alpha-1}z'(t)\Big)' = -q(t)f\big(u(\sigma(t))\big) \leq 0,$$

for $t \geq T$. Hence, the function $r(t)\psi(u(t))|z'(t)|^{\alpha-1}z'(t)$ is decreasing and $z'(t)$ is eventually of one sign. We claim that

$$z'(t) > 0, \tag{3.3}$$

for $t \geq T$. Suppose that $z'(t) \leq 0$ for $t \geq T$. Then there is a positive constant N such that

$$r(t)\psi(u(t))|z'(t)|^{\alpha-1}z'(t) \leq r(T)\psi(u(T))|z'(T)|^{\alpha-1}z'(T) = -N \leq 0,$$

for $t \geq T$. That is

$$-r(t)\psi(u(t))\big(-z'(t)\big)^\alpha \leq -N,$$

for $t \geq T$. From (H_2), we have

$$z'(t) \leq -\Big(\frac{N}{r(t)\psi(u(t))}\Big)^{\frac{1}{\alpha}} \leq -\Big(\frac{N}{M}\Big)^{\frac{1}{\alpha}}\Big(\frac{1}{r(t)}\Big)^{\frac{1}{\alpha}},$$

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for $t \geq T$. Integrating the above inequality from T to t, we obtain

$$z(t) \leq z(T) - \Big(\frac{N}{M}\Big)^{\frac{1}{\alpha}} \int_T^t \Big(\frac{1}{r(s)}\Big)^{\frac{1}{\alpha}} ds = z(T) - \Big(\frac{N}{M}\Big)^{\frac{1}{\alpha}} \big(R(t) - R(T)\big).$$

Letting $t \to \infty$ in the above inequality, we get $z(t) \to -\infty$. This contradiction prove that (3.3) holds. Therefore. $z(t)z'(t)$ is an eventually positive. □

Now, the following theorems provide sufficient conditions for oscillation of all solutions of Eq.(E) with respect to properties of the function $f(u)$.

Theorem 3.1.1 *Let $(H_1) - (H_5)$ and (S_1) be satisfied. Suppose that there exists a constant $\bar{p}$ such that $0 \leq p(t) \leq \bar{p} < 1$. If*

$$\int^{\infty} \Big\{R^{\alpha}(\sigma(t))q(t) - M\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1} \frac{\sigma'(t)}{R(\sigma(t))r^{1/\alpha}(\sigma(t))}\Big\}dt = \infty, \qquad (3.4)$$

where $p^* = 1 - \bar{p}$, then Eq.(E) is oscillatory.

Proof. Suppose on the contrary that there is a nonoscillatory solution $u(t)$ of Eq.(E). Without loss of generality, we may assume that $u(t) > 0$, $u(\tau(t)) > 0$ and $u(\sigma(t)) > 0$ on $[T, \infty)$ for some sufficiently large $T \geq t_0$. From (3.1), by Lemma 3.1.2, we obtain

$$z(t) > 0 \quad \text{and} \quad z'(t) > 0,$$

for $t \geq T$. Since $r(t)\psi(u(t))|z'(t)|^{\alpha-1}z'(t)$ is decreasing function and $\sigma(t) \leq t$,

$$r(t)\psi(u(t))(z'(t))^{\alpha} \leq r(\sigma(t))\psi(u(\sigma(t)))(z'(\sigma(t)))^{\alpha} \leq Mr(\sigma(t))(z'(\sigma(t)))^{\alpha}.$$

That is

$$z'(\sigma(t)) \geq z'(t)\Big(\frac{r(t)\psi(u(t))}{Mr(\sigma(t))}\Big)^{\frac{1}{\alpha}}, \qquad (3.5)$$

for $t \geq T$. Since $z(t) \geq u(t)$ and $z'(t) > 0$, from (3.1) we have

$$\begin{aligned} u(t) &= z(t) - p(t)u(\tau(t)) \\ &\geq z(t) - p(t)z(\tau(t)) \\ &\geq z(t)\big(1 - p(t)\big) \\ &\geq z(t)\big(1 - \bar{p}\big) \\ &= p^* z(t), \end{aligned} \qquad (3.6)$$

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for $t \geq T$, where $p^* = 1 - \bar{p}$. From (S_1) and (3.6), we have

$$f\big(u(\sigma(t))\big) \geq f\big(p^* z(\sigma(t))\big)$$

and then (3.2) implies

$$\Big(r(t)\psi(u(t))(z'(t))^\alpha\Big)' + q(t) f\big(p^* z(\sigma(t))\big) \leq 0, \tag{3.7}$$

for $t \geq T$. Define

$$w(t) = R^\alpha(\sigma(t)) \frac{r(t)\psi(u(t))\big(z'(t)\big)^\alpha}{f\big(p^* z(\sigma(t))\big)}, \tag{3.8}$$

for $t \geq T$, then $w(t) > 0$. Using (S_1), (3.5), (3.7) and (3.8), we get

$$\begin{aligned}
w'(t) &= \frac{d}{dt}\Big(R^\alpha(\sigma(t)) \frac{r(t)\psi(u(t))\big(z'(t)\big)^\alpha}{f\big(p^* z(\sigma(t))\big)}\Big) \\
&= \frac{\alpha\sigma'(t) R^{\alpha-1}(\sigma(t))}{r^{1/\alpha}(\sigma(t))} \frac{r(t)\psi(u(t))\big(z'(t)\big)^\alpha}{f\big(p^* z(\sigma(t))\big)} + R^\alpha(\sigma(t)) \frac{\Big(r(t)\psi(u(t))\big(z'(t)\big)^\alpha\Big)'}{f\big(p^* z(\sigma(t))\big)} \\
&\quad - R^\alpha(\sigma(t)) \frac{r(t)\psi(u(t))\big(z'(t)\big)^\alpha f'\big(p^* z(\sigma(t))\big) p^* z'(\sigma(t))\sigma'(t)}{f^2\big(p^* z(\sigma(t))\big)} \\
&\leq \frac{\alpha\sigma'(t)}{R(\sigma(t)) r^{1/\alpha}(\sigma(t))} w(t) - R^\alpha(\sigma(t)) q(t) \\
&\quad - \gamma p^* \sigma'(t) R^\alpha(\sigma(t)) \frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha+1}}{f^2\big(p^* z(\sigma(t))\big)} \big[f\big(p^* z(\sigma(t))\big)\big]^{\frac{\alpha-1}{\alpha}} \Big(\frac{r(t)\psi(u(t))}{M r(\sigma(t))}\Big)^{\frac{1}{\alpha}} \\
&= \frac{\alpha\sigma'(t)}{R(\sigma(t)) r^{1/\alpha}(\sigma(t))} w(t) - R^\alpha(\sigma(t)) q(t) \\
&\quad - \frac{\gamma p^* \sigma'(t)}{M^{1/\alpha} R(\sigma(t)) r^{1/\alpha}(\sigma(t))} \Big(R^\alpha(\sigma(t)) \frac{r(t)\psi(u(t))\big(z'(t)\big)^\alpha}{f\big(p^* z(\sigma(t))\big)}\Big)^{\frac{\alpha+1}{\alpha}} \\
&= \frac{\alpha\sigma'(t)}{R(\sigma(t)) r^{1/\alpha}(\sigma(t))} w(t) - R^\alpha(\sigma(t)) q(t) - \frac{\gamma p^* \sigma'(t)}{M^{1/\alpha} R(\sigma(t)) r^{1/\alpha}(\sigma(t))} \big(w(t)\big)^{\frac{\alpha+1}{\alpha}} \\
&= \frac{\alpha\sigma'(t)}{R(\sigma(t)) r^{1/\alpha}(\sigma(t))} \Big[w(t) - \frac{\gamma p^*}{\alpha M^{1/\alpha}} \big(w(t)\big)^{\frac{\alpha+1}{\alpha}}\Big] - R^\alpha(\sigma(t)) q(t),
\end{aligned} \tag{3.9}$$

for $t \geq T$. Set

$$X = \Big(\frac{\gamma p^*}{\alpha M^{1/\alpha}}\Big)^{\frac{\alpha}{\alpha+1}} w(t), \quad Y = \Big(\frac{\alpha M^{1/\alpha}}{\gamma p^*}\Big)^{\frac{\alpha^2}{\alpha+1}} \Big(\frac{\alpha}{\alpha+1}\Big)^\alpha, \quad \text{and} \quad \lambda = \frac{\alpha+1}{\alpha} > 1.$$

From Lemma 3.1.1, we then obtain for $t \geq T$

$$\begin{aligned}
w(t) - \frac{\gamma p^*}{\alpha M^{1/\alpha}} \big(w(t)\big)^{\frac{\alpha+1}{\alpha}} &\leq \Big(\frac{\alpha+1}{\alpha} - 1\Big) \Big[\Big(\frac{\alpha M^{1/\alpha}}{\gamma p^*}\Big)^{\frac{\alpha^2}{\alpha+1}} \Big(\frac{\alpha}{\alpha+1}\Big)^\alpha\Big]^{\frac{\alpha+1}{\alpha}} \\
&= \frac{M}{\alpha} \Big(\frac{\alpha}{\gamma p^*}\Big)^\alpha \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}.
\end{aligned}$$

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Hence, (3.9) implies for $t \geq T$

$$w'(t) \leq M\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{\sigma'(t)}{R(\sigma(t))r^{1/\alpha}(\sigma(t))} - R^{\alpha}(\sigma(t))q(t).$$

Integrating the above inequality from T to t , we have

$$w(t) \leq w(T) - \int_T^t \Big\{R^{\alpha}(\sigma(s))q(s) - M\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{\sigma'(s)}{R(\sigma(s))r^{1/\alpha}(\sigma(s))}\Big\}ds.$$

Letting $t \to \infty$ in the above inequality, we get $w(t) \to -\infty$. This contradiction completes the proof of Theorem 3.1.1. □

From Theorem 3.1.1, we get the following Corollaries.

Corollary 3.1.1 *Let $(H_1) - (H_5)$ and (S_1) be satisfied. Suppose that there exists a constant $\bar{p}$ such that $0 \leq p(t) \leq \bar{p} < 1$. If there exists a number $T \geq t_0$ such that*

$$\liminf_{t\to\infty} \frac{1}{\ln R(\sigma(t))}\int_T^t R^{\alpha}(\sigma(s))q(s)ds > M\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}, \qquad (3.10)$$

where $p^* = 1 - \bar{p}$, then Eq.(E) is oscillatory.

Proof. It is easy to see that (3.10) yields the existence $\epsilon > 0$ such that for sufficiently large t,

$$\frac{1}{\ln R(\sigma(t))}\int_T^t R^{\alpha}(\sigma(s))q(s)ds \geq M\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1} + \epsilon,$$

which follows that

$$\int_T^t R^{\alpha}(\sigma(s))q(s)ds - M\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\ln R(\sigma(t)) \geq \epsilon \ln R(\sigma(t)).$$

Next, we consider

$$\begin{aligned}
\ln R(\sigma(t)) &= \int_{t_0}^t d\big(\ln R(\sigma(s))\big)\\
&= \int_{t_0}^t \frac{1}{R(\sigma(s))}d\big(R(\sigma(s))\big)\\
&= \int_{t_0}^t \frac{\sigma'(s)}{R(\sigma(s))r^{1/\alpha}(\sigma(s))}ds\\
&= \int_T^t \frac{\sigma'(s)}{R(\sigma(s))r^{1/\alpha}(\sigma(s))}ds + \int_{t_0}^T \frac{\sigma'(s)}{R(\sigma(s))r^{1/\alpha}(\sigma(s))}ds\\
&= \int_T^t \frac{\sigma'(s)}{R(\sigma(s))r^{1/\alpha}(\sigma(s))}ds + \ln R(\sigma(T)).
\end{aligned}$$

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That is

$$\int_T^t \Big(R^\alpha(\sigma(s))q(s) - M\Big(\frac{\alpha}{\gamma p^*}\Big)^\alpha\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{\sigma'(s)}{R(\sigma(s))r^{1/\alpha}(\sigma(s))}\Big)ds$$
$$\geq \epsilon \ln R(\sigma(t)) + M\Big(\frac{\alpha}{\gamma p^*}\Big)^\alpha\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\ln R(\sigma(T)). \qquad (3.11)$$

Simple calculation shows that (3.11) implies (3.4) and the statement follows from Theorem 3.1.1. □

Corollary 3.1.2 *Let* $(H_1) - (H_5)$ *and* (S_1) *be satisfied. Suppose that there exists a constant* $\bar{p}$ *such that* $0 \leq p(t) \leq \bar{p} < 1$ *and* $\sigma'(t) > 0$. *If there exists a number* $T \geq t_0$ *such that*

$$\liminf_{t\to\infty} q(t)\frac{r^{1/\alpha}(\sigma(t))R^{\alpha+1}(\sigma(t))}{\sigma'(t)} > M\Big(\frac{\alpha}{\gamma p^*}\Big)^\alpha\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}, \qquad (3.12)$$

where $p^* = 1 - \bar{p}$, *then Eq.(E) is oscillatory.*

Proof. It is easy to see that (3.12) yields the existence $\epsilon > 0$ such that for sufficiently large t,

$$q(t)\frac{r^{1/\alpha}(\sigma(t))R^{\alpha+1}(\sigma(t))}{\sigma'(t)} \geq M\Big(\frac{\alpha}{\gamma p^*}\Big)^\alpha\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1} + \epsilon.$$

Multiplying $\dfrac{\sigma'(t)}{r^{1/\alpha}(\sigma(t))R(\sigma(t))}$ on both side of the above inequality, we obtain

$$R^\alpha(\sigma(t))q(t) - M\Big(\frac{\alpha}{\gamma p^*}\Big)^\alpha\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{\sigma'(t)}{r^{1/\alpha}(\sigma(t))R(\sigma(t))} \geq \frac{\epsilon\sigma'(t)}{r^{1/\alpha}(\sigma(t))R(\sigma(t))}. \qquad (3.13)$$

Simple calculation shows that (3.13) implies (3.4) and the statement follows from Theorem 3.1.1. □

Theorem 3.1.2 *Let* $(H_1) - (H_5)$ *and* (S_2) *be satisfied. If*

$$\int^\infty \Big(\beta R^\alpha(\sigma(t))Q(t) - M\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{\sigma'(t)}{R(\sigma(t))r^{1/\alpha}(\sigma(t))}\Big)dt = \infty, \qquad (3.14)$$

where $Q(t) = q(t)\big(1 - p(\sigma(t))\big)^\alpha$, *then Eq.(E) is oscillatory.*

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Proof. On the contrary, we may assume that $u(t)$ is an eventually positive solution of Eq.(E). From the proof of Theorem 3.1.1, we conclude that $u(t) \geq z(t)\big(1-p(t)\big)$, $z'(t) > 0$ and $r(t)\psi(u(t))|z'(t)|^{\alpha-1}z'(t)$ is decreasing function on $[T,\infty)$ for some sufficiently large $T \geq t_0$. From Eq.(E), we have

$$\Big(r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}\Big)' + \beta Q(t)z^{\alpha}(\sigma(t)) \leq 0, \tag{3.15}$$

where $Q(t) = q(t)\big(1-p(\sigma(t))\big)^{\alpha}$. Define

$$w(t) = R^{\alpha}(\sigma(t))\frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}}{\big(z(\sigma(t))\big)^{\alpha}}, \quad t \geq T, \tag{3.16}$$

then $w(t) > 0$. Using (3.5), (3.15) and (3.16), we get

$$\begin{aligned}
w'(t) &= \frac{d}{dt}\Big(R^{\alpha}(\sigma(t))\frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}}{\big(z(\sigma(t))\big)^{\alpha}}\Big)\\
&= \frac{\alpha\sigma'(t)R^{\alpha-1}(\sigma(t))}{r^{1/\alpha}(\sigma(t))}\frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}}{\big(z(\sigma(t))\big)^{\alpha}} + R^{\alpha}(\sigma(t))\frac{\Big(r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}\Big)'}{\big(z(\sigma(t))\big)^{\alpha}}\\
&\quad - R^{\alpha}(\sigma(t))\frac{\alpha\sigma'(t)r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}z'(\sigma(t))}{\big(z(\sigma(t))\big)^{\alpha+1}}\\
&\leq \frac{\alpha\sigma'(t)}{R(\sigma(t))r^{1/\alpha}(\sigma(t))}w(t) - \beta R^{\alpha}(\sigma(t))Q(t)\\
&\quad - R^{\alpha}(\sigma(t))\frac{\alpha\sigma'(t)r(t)\psi(u(t))\big(z'(t)\big)^{\alpha+1}}{\big(z(\sigma(t))\big)^{\alpha+1}}\Big(\frac{r(t)\psi(u(t))}{Mr(\sigma(t))}\Big)^{\frac{1}{\alpha}}\\
&= \frac{\alpha\sigma'(t)}{R(\sigma(t))r^{1/\alpha}(\sigma(t))}w(t) - \beta R^{\alpha}(\sigma(t))Q(t)\\
&\quad - \frac{\alpha\sigma'(t)}{M^{1/\alpha}R(\sigma(t))r^{1/\alpha}(\sigma(t))}\Big(R^{\alpha}(\sigma(t))\frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}}{\big(z(\sigma(t))\big)^{\alpha}}\Big)^{\frac{\alpha+1}{\alpha}}\\
&= \frac{\alpha\sigma'(t)}{R(\sigma(t))r^{1/\alpha}(\sigma(t))}w(t) - \beta R^{\alpha}(\sigma(t))Q(t) - \frac{\alpha\sigma'(t)\big(w(t)\big)^{\frac{\alpha+1}{\alpha}}}{M^{1/\alpha}R(\sigma(t))r^{1/\alpha}(\sigma(t))}\\
&= \frac{\alpha\sigma'(t)}{R(\sigma(t))r^{1/\alpha}(\sigma(t))}\Big[w(t) - \Big(\frac{w(t)}{M^{1/(\alpha+1)}}\Big)^{\frac{\alpha+1}{\alpha}}\Big] - \beta R^{\alpha}(\sigma(t))Q(t),
\end{aligned} \tag{3.17}$$

for $t \geq T$. Set

$$X = \Big(\frac{1}{M}\Big)^{\frac{1}{\alpha+1}}w(t), \quad Y = M^{\frac{\alpha}{\alpha+1}}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha} \quad \text{and} \quad \lambda = \frac{\alpha+1}{\alpha} > 1.$$

From Lemma 3.1.1, we then obtain for $t \geq T$

$$\begin{aligned}
w(t) - \Big(\frac{w(t)}{M^{\alpha+1}}\Big)^{\frac{\alpha+1}{\alpha}} &\leq \Big(\frac{\alpha+1}{\alpha} - 1\Big)\Big[M^{\frac{\alpha}{\alpha+1}}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha}\Big]^{\frac{\alpha+1}{\alpha}}\\
&= \frac{M}{\alpha}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}.
\end{aligned}$$

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Hence, (3.17) implies for $t \geq T$

$$w'(t) \leq M\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{\sigma'(t)}{R(\sigma(t))r^{1/\alpha}(\sigma(t))} - \beta R^{\alpha}(\sigma(t))Q(t).$$

Integrating the above inequality from T to t , we have

$$w(t) \leq w(T) - \int_T^t \Big\{\beta R^{\alpha}(\sigma(s))Q(s) - M\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{\sigma'(s)}{R(\sigma(s))r^{1/\alpha}(\sigma(s))}\Big\}ds.$$

Letting $t \to \infty$, we get $w(t) \to -\infty$. This contradiction completes the proof of Theorem 3.1.2. □

Theorem 3.1.2 provides the following Corollaries.

Corollary 3.1.3 *Let $(H_1)-(H_5)$ and (S_2) be satisfied. Assume that there exists a number $T \geq t_0$ such that*

$$\liminf_{t\to\infty} \frac{1}{\ln R(\sigma(t))}\int_T^t \beta R^{\alpha}(\sigma(s))Q(s)ds > M\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}. \tag{3.18}$$

Then Eq.(E) is oscillatory.

Proof. It is easy to see that (3.18) yields the existence $\epsilon > 0$ such that for sufficiently large t,

$$\frac{1}{\ln R(\sigma(t))}\int_T^t \beta R^{\alpha}(\sigma(s))Q(s)ds \geq M\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1} + \epsilon,$$

which follows that

$$\int_T^t \beta R^{\alpha}(\sigma(s))Q(s)ds - M\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\ln R(\sigma(t)) \geq \epsilon \ln R(\sigma(t)).$$

That is

$$\int_T^t \Big(\beta R^{\alpha}(\sigma(s))Q(s) - M\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{\sigma'(s)}{R(\sigma(s))r^{1/\alpha}(\sigma(s))}\Big)ds$$
$$\geq \epsilon \ln R(\sigma(t)) + M\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\ln R(\sigma(T)). \tag{3.19}$$

Simple calculation shows that (3.19) implies (3.14) and the statement follows from Theorem 3.1.2. □

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Corollary 3.1.4 *Let* $(H_1) - (H_5)$ *and* (S_2) *be satisfied. Assume that* $\sigma'(t) > 0$ *and there exists a number* $T \geq t_0$ *such that*

$$\liminf_{t\to\infty} \beta Q(t) \frac{r^{1/\alpha}(\sigma(t)) R^{\alpha+1}(\sigma(t))}{\sigma'(t)} > M\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{3.20}$$

Then Eq.(E) *is oscillatory.*

Proof. It is easy to see that (3.20) yields the existence $\epsilon > 0$ such that for sufficiently large t,

$$\beta Q(t) \frac{r^{1/\alpha}(\sigma(t)) R^{\alpha+1}(\sigma(t))}{\sigma'(t)} \geq M\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \epsilon.$$

Multiplying $\dfrac{\sigma'(t)}{r^{1/\alpha}(\sigma(t))R(\sigma(t))}$ on both side of the above inequality, we obtain

$$\beta R^{\alpha}(\sigma(t))Q(t) - M\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\sigma'(t)}{r^{1/\alpha}(\sigma(t))R(\sigma(t))} \geq \frac{\epsilon\sigma'(t)}{r^{1/\alpha}(\sigma(t))R(\sigma(t))}. \tag{3.21}$$

Simple calculation shows that (3.21) implies (3.14) and the statement follows from Theorem 3.1.2. □

3.2 Philos's Type Oscillation Criteria

We present some oscillation results for Eq.(E), by using integral averages condition of Philos-type. Following Philos [12], we introduce a class of function $\mathcal{P}$. Let

$$D_0 = \{(t,s) : t > s \geq t_0\} \quad \text{and} \quad D = \{(t,s) : t \geq s \geq t_0\}.$$

The function $H \in C(D, \mathbb{R})$ is said to belongs to the class $\mathcal{P}$ if

(i) $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ for $(t,s) \in D_0$,

(ii) H has a continuous nonpositive partial derivative on D_0 with respect to the second variable such that

$$\frac{\partial}{\partial s} H(t,s) = -h(t,s)\sqrt{H(t,s)} \quad \text{for } (t,s) \in D_0 \tag{3.22}$$

where h is a nonnegative and continuous function on D.

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Theorem 3.2.1 *Let $(H_1)-(H_5)$ and (S_1) be satisfied. Suppose that there exists a constant $\bar{p}$ such that $0 \le p(t) \le \bar{p} < 1$. If there exist function $H \in \mathcal{P}$ and positive function $\rho \in C^1([t_0,\infty);\mathbb{R}^+)$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big\{ H(t,s)\rho(s)q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s)\Big\} ds = \infty, \tag{3.23}$$

then Eq.(E) is oscillatory, where $p^* = 1-\bar{p}$ and

$$G(t,s) = \frac{\rho(s)r(\sigma(s))}{(\sigma'(s))^{\alpha}\big(\sqrt{H(t,s)}\big)^{\alpha-1}}\Big(h(t,s) + \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\Big)^{\alpha+1}.$$

Proof. Suppose on contrary that there exists a nonoscillatory solution $u(t)$ of Eq.(E). Without loss of generality, we may assume that $u(t) > 0, u(\tau(t)) > 0$, and $u(\sigma(t)) > 0$ on $[T,\infty)$ for some sufficiently large $T \ge t_0$. From Lemma 3.1.2 and Theorem 3.1.1, we get for $t \ge T$

$$z'(\sigma(t)) \ge z'(t)\Big(\frac{r(t)\psi(u(t))}{Mr(\sigma(t))}\Big)^{\frac{1}{\alpha}} \text{ and } \Big(r(t)\psi(u(t))(z'(t))^{\alpha}\Big)' + q(t)f\big(p^*z(\sigma(t))\big) \le 0.$$

Define the function

$$w(t) = \rho(t)\frac{r(t)\psi(u(t))(z'(t))^{\alpha}}{f\big(p^*z(\sigma(t))\big)} \quad \text{for } t \ge T. \tag{3.24}$$

Then $w(t) > 0$ and

$$\begin{aligned} w'(t) &= \frac{d}{dt}\Big(\rho(t)\frac{r(t)\psi(u(t))(z'(t))^{\alpha}}{f(p^*z(\sigma(t)))}\Big) \\ &= \rho'(t)\frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}}{f\big(p^*z(\sigma(t))\big)} + \rho(t)\frac{\Big(r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}\Big)'}{f\big(p^*z(\sigma(t))\big)} \\ &\quad - \rho(t)\frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}f'\big(p^*z(\sigma(t))\big)p^*z'(\sigma(t))\sigma'(t)}{f^2\big(p^*z(\sigma(t))\big)} \\ &\le \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)q(t) \\ &\quad - \gamma p^*\sigma'(t)\rho(t)\frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha+1}}{f^2\big(p^*z(\sigma(t))\big)}\big[f\big(p^*z(\sigma(t))\big)\big]^{\frac{\alpha-1}{\alpha}}\Big(\frac{r(t)\psi(u(t))}{Mr(\sigma(t))}\Big)^{\frac{1}{\alpha}} \\ &= \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)q(t) - \frac{\gamma p^*\sigma'(t)}{\big(M\rho(t)r(\sigma(t))\big)^{1/\alpha}}\Big(\rho(t)\frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}}{f\big(p^*z(\sigma(t))\big)}\Big)^{\frac{\alpha+1}{\alpha}} \\ &= \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)q(t) - \frac{\gamma p^*\sigma'(t)}{\big(M\rho(t)r(\sigma(t))\big)^{1/\alpha}}\big(w(t)\big)^{\frac{\alpha+1}{\alpha}}. \end{aligned}$$

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Consequently, we have

$$\int_T^t H(t,s)\rho(s)q(s)ds \le -\int_T^t H(t,s)w'(s)ds + \int_T^t H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds - \int_T^t H(t,s)\frac{\gamma p^*\sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}}\big(w(s)\big)^{\frac{\alpha+1}{\alpha}}ds.$$

Since

$$\int_T^t H(t,s)w'(s)ds = -H(t,T)w(T) - \int_T^t w(s)\frac{\partial}{\partial s}H(t,s)ds$$

and in view of (3.22), the previous inequality becomes

$$\begin{aligned}
&\int_T^t H(t,s)\rho(s)q(s)ds \\
&\le H(t,T)w(T) + \int_T^t w(s)\frac{\partial}{\partial s}H(t,s)ds + \int_T^t H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds \\
&\quad - \int_T^t H(t,s)\frac{\gamma p^*\sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}}\big(w(s)\big)^{\frac{\alpha+1}{\alpha}}ds \\
&= H(t,T)w(T) - \int_T^t w(s)h(t,s)\sqrt{H(t,s)}ds + \int_T^t H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds \\
&\quad - \int_T^t H(t,s)\frac{\gamma p^*\sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}}\big(w(s)\big)^{\frac{\alpha+1}{\alpha}}ds \\
&\le H(t,T)w(T) + \int_T^t w(s)h(t,s)\sqrt{H(t,s)}ds + \int_T^t H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds \\
&\quad - \int_T^t H(t,s)\frac{\gamma p^*\sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}}\big(w(s)\big)^{\frac{\alpha+1}{\alpha}}ds \\
&= H(t,T)w(T) + \int_T^t \Big[h(t,s)\sqrt{H(t,s)} + H(t,s)\frac{\rho'(s)}{\rho(s)}\Big]w(s)ds \\
&\quad - \int_T^t H(t,s)\frac{\gamma p^*\sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}}\big(w(s)\big)^{\frac{\alpha+1}{\alpha}}ds \qquad (3.25)
\end{aligned}$$

In Lemma 3.1.1, we let

$$\lambda = \frac{\alpha+1}{\alpha} > 1, \qquad X = \Big(\frac{\big(\gamma p^*\sigma'(s)H(t,s)\big)^{\alpha}}{M\rho(s)r(\sigma(s))}\Big)^{\frac{1}{\alpha+1}}w(t)$$

and

$$Y = \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha}\Big(h(t,s)\sqrt{H(t,s)} + H(t,s)\frac{\rho'(s)}{\rho(s)}\Big)^{\alpha}\Big(\frac{M\rho(s)r(\sigma(s))}{\big(\gamma p^*\sigma'(s)H(t,s)\big)^{\alpha}}\Big)^{\frac{\alpha}{\alpha+1}}.$$

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From Lemma 3.1.1, we then obtain for $t > s \geq T$

$$
\begin{aligned}
&\Big[h(t,s)\sqrt{H(t,s)} + H(t,s)\frac{\rho'(s)}{\rho(s)}\Big]w(s) - H(t,s)\frac{\gamma p^* \sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}}\big(w(s)\big)^{\frac{\alpha+1}{\alpha}} \\
&\leq \frac{1}{\alpha}\Big[\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha}\Big(h(t,s)\sqrt{H(t,s)} + H(t,s)\frac{\rho'(s)}{\rho(s)}\Big)^{\alpha}\Big(\frac{M\rho(s)r(\sigma(s))}{\big(\gamma p^* \sigma'(s) H(t,s)\big)^{\alpha}}\Big)^{\frac{\alpha}{\alpha+1}}\Big]^{\frac{\alpha+1}{\alpha}} \\
&= \frac{1}{\alpha}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\Big(h(t,s)\sqrt{H(t,s)} + H(t,s)\frac{\rho'(s)}{\rho(s)}\Big)^{\alpha+1}\frac{M\rho(s)r(\sigma(s))}{\big(\gamma p^* \sigma'(s) H(t,s)\big)^{\alpha}} \\
&= M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha}\frac{\rho(s)r(\sigma(s))}{(\sigma'(s))^{\alpha}\big(\sqrt{H(t,s)}\big)^{\alpha-1}}\Big(h(t,s) + \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\Big)^{\alpha+1} \\
&= M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha}G(t,s).
\end{aligned}
$$

Hence, (3.25) implies for $t > s \geq T$

$$
\begin{aligned}
\int_T^t \Big\{H(t,s)\rho(s)q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha}G(t,s)\Big\}ds& \\
\leq H(t,T)w(T)& \qquad (3.26)\\
\leq H(t,t_0)w(T).&
\end{aligned}
$$

Thereby, including (ii), we conclude that for $t > t_0$

$$
\begin{aligned}
&\int_{t_0}^t \Big\{H(t,s)\rho(s)q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha}G(t,s)\Big\}ds \\
&\quad = \int_{t_0}^T \Big\{H(t,s)\rho(s)q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha}G(t,s)\Big\}ds \\
&\quad + \int_{T}^t \Big\{H(t,s)\rho(s)q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha}G(t,s)\Big\}ds \\
&\quad \leq \int_{t_0}^T H(t,s)\rho(s)q(s)ds + H(t,t_0)w(T) \\
&\quad \leq H(t,t_0)\Big[\int_{t_0}^T \rho(s)q(s)ds + w(T)\Big].
\end{aligned}
$$

Accordingly, we obtain

$$
\begin{aligned}
\frac{1}{H(t,t_0)}\int_{t_0}^t \Big\{H(t,s)\rho(s)q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha}G(t,s)\Big\}ds& \\
\leq \int_{t_0}^T \rho(s)q(s)ds + w(T) < \infty&
\end{aligned}
$$

for $t \geq t_0$. Taking the limit superior as $t \to \infty$ in the above inequality, we obtain a contradiction to (3.23), which completes the proof. □

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As immediate consequences of Theorem 3.2.1, we obtain the following Corollaries.

Corollary 3.2.1 *Let* $(H_1)-(H_5)$ *and* (S_1) *be satisfied. If there exist function* $H \in \mathcal{P}$ *and positive function* $\rho \in C^1\big([t_0,\infty);\mathbb{R}^+\big)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s)ds = \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s)ds < \infty,$$

then Eq.(E) *is oscillatory, where* $p^* = 1-\bar{p}$ *and*

$$G(t,s) = \frac{\rho(s)r(\sigma(s))}{\big(\sigma'(s)\big)^{\alpha}\big(\sqrt{H(t,s)}\big)^{\alpha-1}}\Big(h(t,s)+\frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\Big)^{\alpha+1}.$$

Corollary 3.2.2 *Let* $(H_1)-(H_5)$ *and* (S_1) *be satisfied and let* $\rho(t)=1$. *Suppose that there exists a constant* $\bar{p}$ *such that* $0 \le p(t) \le \bar{p} < 1$. *If there exists function* $H \in \mathcal{P}$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big\{H(t,s)q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s)\Big\}ds = \infty,$$

then Eq.(E) *is oscillatory, where* $p^* = 1-\bar{p}$ *and*

$$G(t,s) = \frac{r(\sigma(s))\big(h(t,s)\big)^{\alpha+1}}{\big(\sigma'(s)\big)^{\alpha}\big(\sqrt{H(t,s)}\big)^{\alpha-1}}.$$

Corollary 3.2.3 *Let* $(H_1)-(H_5)$ *and* (S_1) *be satisfied and let* $\rho(t)=1$ *and* $\alpha=1$. *Suppose that there exists a constant* $\bar{p}$ *such that* $0 \le p(t) \le \bar{p} < 1$. *If there exists function* $H \in \mathcal{P}$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big\{H(t,s)q(s) - \frac{M}{4\gamma p^*}\frac{r(\sigma(s))\big(h(t,s)\big)^2}{\sigma'(s)}\Big\}ds = \infty,$$

then Eq.(E) *is oscillatory, where* $p^* = 1-\bar{p}$.

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With an appropriate choice of the functions H and h, we can derive from Theorem 3.2.1 a number of oscillation criteria for Eq.(E). Let us consider, for example, the function $H(t,s)$ defined by

$$H(t,s) = (t-s)^{\lambda}, \quad \text{for } (t,s) \in D,$$

where $\lambda > \alpha$ is a constant. Clearly, H belongs to the class $\mathcal{P}$. Furthermore, the function

$$h(t,s) = \frac{-\frac{\partial}{\partial s}H(t,s)}{\sqrt{H(t,s)}} = \frac{\lambda(t-s)^{\lambda-1}}{(t-s)^{\frac{\lambda}{2}}} = \lambda(t-s)^{(\lambda-2)/2}, \quad \text{for } (t,s) \in D,$$

is continuous on $[t_0,\infty)$ and satisfies condition (3.22). Then, by Theorem 3.2.1, we obtain the following oscillation criteria.

Corollary 3.2.4 *Let $(H_1)-(H_5)$ and (S_1) be satisfied. Suppose that there exists a constant $\bar{p}$ such that $0 \le p(t) \le \bar{p} < 1$. If there exists a positive function $\rho \in C^1\big([t_0,\infty);\mathbb{R}^+\big)$ such that*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda}} \int_{t_0}^{t} \Big\{ (t-s)^{\lambda} q(s)\rho(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s) \Big\} ds = \infty,$$

then Eq.(E) is oscillatory, where $p^* = 1-\bar{p}$ and

$$G(t,s) = \frac{\rho(s) r(\sigma(s))(t-s)^{\lambda-\alpha-1}}{\big(\sigma'(s)\big)^{\alpha}} \Big(\lambda + \frac{\rho'(s)}{\rho(s)}(t-s)\Big)^{\alpha+1}.$$

Proof. The proof follows from Theorem 3.2.1 such that

$$\begin{aligned} G(t,s) &= \frac{\rho(s) r(\sigma(s))}{\big(\sigma'(s)\big)^{\alpha}(t-s)^{\frac{\lambda(\alpha-1)}{2}}} \Big(\lambda(t-s)^{\frac{\lambda-2}{2}} + \frac{\rho'(s)}{\rho(s)}(t-s)^{\frac{\lambda}{2}}\Big)^{\alpha+1} \\ &= \frac{\rho(s) r(\sigma(s))}{\big(\sigma'(s)\big)^{\alpha}} (t-s)^{\frac{(\lambda-2)(\alpha+1)}{2} - \frac{\lambda(\alpha-1)}{2}} \Big(\lambda + \frac{\rho'(s)}{\rho(s)}(t-s)\Big)^{\alpha+1} \\ &= \frac{\rho(s) r(\sigma(s))(t-s)^{\lambda-\alpha-1}}{\big(\sigma'(s)\big)^{\alpha}} \Big(\lambda + \frac{\rho'(s)}{\rho(s)}(t-s)\Big)^{\alpha+1}. \end{aligned}$$

□

Corollary 3.2.5 *Let $(H_1)-(H_5)$ and (S_1) be satisfied. Suppose that there exists a constant $\bar{p}$ such that $0 \le p(t) \le \bar{p} < 1$. If there exists a positive function $\rho \in C^1\big([t_0,\infty);\mathbb{R}^+\big)$ and $c \in C\big([t_0,\infty);\mathbb{R}^+\big)$ such that for some $\lambda > 1$,*

$$\limsup_{t\to\infty} \frac{1}{\big(C(t)\big)^{\lambda}} \int_{t_0}^{t} \Big\{ \big(C(t)-C(s)\big)^{\lambda} q(s)\rho(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s) \Big\} ds = \infty,$$

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then Eq.(E) is oscillatory, where $p^* = 1 - \bar{p}$, $C(t) = \int_{t_0}^{t} c(s)ds$ *and*

$$G(t,s) = \frac{\rho(s)r(\sigma(s))\big(C(t)-C(s)\big)^{\lambda-\alpha-1}}{\big(\sigma'(s)\big)^{\alpha}}\Big(\lambda c(s) + \frac{\rho'(s)}{\rho(s)}\big(C(t)-C(s)\big)\Big)^{\alpha+1}.$$

Proof. Let us put

$$H(t,s) = \big(C(t)-C(s)\big)^{\lambda}, \quad \text{for} \quad (t,s) \in D.$$

Then with the choice

$$h(t,s) = \lambda c(s)\big(C(t)-C(s)\big)^{(\lambda-2)/2}, \quad \text{for} \quad (t,s) \in D,$$

the conclusion follows directly from Theorem 3.2.1. □

Theorem 3.2.2 *Let* $(H_1) - (H_5)$ *and* (S_1) *be satisfied. Suppose that there exists a constant* $\bar{p}$ *such that* $0 \le p(t) \le \bar{p} < 1$ *and there exists function* $H \in \mathcal{P}$ *such that*

$$0 < \inf_{s \ge t_0}\Big[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\Big] \le \infty \tag{3.27}$$

and there exists a positive function $\rho \in C^1\big([t_0,\infty);\mathbb{R}^+\big)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s)ds < \infty. \tag{3.28}$$

If there exists a function $\phi \in C\big([t_0,\infty)\big)$ *such that for every* $T \ge t_0$

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \Big\{H(t,s)\rho(s)q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s)\Big\}ds \ge \phi(T), \tag{3.29}$$

$$\int_{t_0}^{\infty} \frac{\sigma'(s)}{\big(\rho(s)r(\sigma(s))\big)^{\frac{1}{\alpha}}} \big(\phi_+(s)\big)^{\frac{\alpha+1}{\alpha}} ds = \infty, \tag{3.30}$$

then Eq.(E) is oscillatory, where $p^* = 1 - \bar{p}$, $\phi_+(s) = \max\{\phi(s), 0\}$ *and*

$$G(t,s) = \frac{\rho(s)r(\sigma(s))}{\big(\sigma'(s)\big)^{\alpha}\big(\sqrt{H(t,s)}\big)^{\alpha-1}}\Big(h(t,s) + \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\Big)^{\alpha+1}.$$

Proof. Suppose that there exists a nonoscillatory solution $u(t)$ of Eq.(E). Without loss of generality, we may assume that $u(t) > 0, u(\tau(t)) > 0$, and $u(\sigma(t)) > 0$ on

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$[T, \infty)$ for some sufficiently large $T \geq t_0$. Define w as in (3.24). As in the proof of Theorem 3.2.1, we can obtain (3.25) and (3.26). Then, for $t > T \geq t_0$, we have

$$\lim_{t\to\infty} \sup \frac{1}{H(t,T)} \int_T^t \Big\{ H(t,s)\rho(s)q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s)\Big\} ds \leq w(T)$$

Therefore, by (3.29), we have for $T \geq t_0$

$$\phi(T) \leq w(T) \tag{3.31}$$

and

$$\lim_{t\to\infty} \sup \frac{1}{H(t,T)} \int_T^t H(t,s)\rho(s)q(s)ds \geq \phi(T). \tag{3.32}$$

We define functions

$$A(t) = \frac{1}{H(t,T)} \int_T^t \Big[h(t,s)\sqrt{H(t,s)} + H(t,s)\frac{\rho'(s)}{\rho(s)}\Big] w(s)ds$$

and

$$B(t) = \frac{1}{H(t,T)} \int_T^t H(t,s) \frac{\gamma p^* \sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}} \big(w(s)\big)^{\frac{\alpha+1}{\alpha}} ds.$$

Then, by (3.25) and (3.32), we obtain

$$\begin{aligned}\lim_{t\to\infty} \inf \big[B(t) - A(t)\big] &\leq w(T) - \lim_{t\to\infty} \sup \frac{1}{H(t,T)} \int_T^t H(t,s)\rho(s)q(s)ds \\ &\leq w(T) - \phi(T) \\ &< \infty.\end{aligned}$$

Now we claim that

$$\int_T^\infty \frac{\sigma'(s)}{\big(\rho(s)r(\sigma(s))\big)^{\frac{1}{\alpha}}} \big(w(s)\big)^{\frac{\alpha+1}{\alpha}} ds < \infty. \tag{3.33}$$

Suppose to the contrary that (3.33) fails, i.e., there exists a $T_1 \geq T$ such that

$$\int_T^\infty \frac{\sigma'(s)}{\big(\rho(s)r(\sigma(s))\big)^{\frac{1}{\alpha}}} \big(w(s)\big)^{\frac{\alpha+1}{\alpha}} ds \geq \frac{M^{1/\alpha}\mu}{\gamma p^* \xi} \quad \text{for all} \quad t \geq T_1 \tag{3.34}$$

where μ is an arbitrary positive number and ξ is a positive constant such that

$$\inf_{s \geq t_0} \Big[\lim_{t\to\infty} \inf \frac{H(t,s)}{H(t,t_0)}\Big] > \xi > 0. \tag{3.35}$$

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Using integration by parts and (3.34), we get for all $t \geq T_1$

$$
\begin{aligned}
B(t) &= \frac{\gamma p^*}{M^{1/\alpha} H(t,T)} \int_T^t H(t,s) \frac{\sigma'(s)}{\big(\rho(s) r(\sigma(s))\big)^{1/\alpha}} \big(w(s)\big)^{\frac{\alpha+1}{\alpha}} ds \\
&= \frac{\gamma p^*}{M^{1/\alpha} H(t,T)} \int_T^t H(t,s) d\Big(\int_T^s \frac{\sigma'(v)}{\big(\rho(v) r(\sigma(v))\big)^{1/\alpha}} \big(w(v)\big)^{\frac{\alpha+1}{\alpha}} dv \Big) \\
&= \frac{\gamma p^*}{M^{1/\alpha} H(t,T)} \Big\{ H(t,s) \int_T^s \frac{\sigma'(v)}{\big(\rho(v) r(\sigma(v))\big)^{1/\alpha}} \big(w(v)\big)^{\frac{\alpha+1}{\alpha}} dv \Big\}_{s=T}^{t} \\
&\quad - \frac{\gamma p^*}{M^{1/\alpha} H(t,T)} \int_T^t \Big(\frac{\partial}{\partial s} H(t,s) \Big) \Big(\int_T^s \frac{\sigma'(v)}{\big(\rho(v) r(\sigma(v))\big)^{1/\alpha}} \big(w(v)\big)^{\frac{\alpha+1}{\alpha}} dv \Big) ds \\
&= \frac{\gamma p^*}{M^{1/\alpha} H(t,T)} \int_T^t \Big(- \frac{\partial}{\partial s} H(t,s) \Big) \Big(\int_T^s \frac{\sigma'(v)}{\big(\rho(v) r(\sigma(v))\big)^{1/\alpha}} \big(w(v)\big)^{\frac{\alpha+1}{\alpha}} dv \Big) ds \\
&\geq \frac{\gamma p^*}{M^{1/\alpha} H(t,T)} \int_{T_1}^t \Big(- \frac{\partial}{\partial s} H(t,s) \Big) \Big(\int_T^s \frac{\sigma'(v)}{\big(\rho(v) r(\sigma(v))\big)^{1/\alpha}} \big(w(v)\big)^{\frac{\alpha+1}{\alpha}} dv \Big) ds \\
&\geq \frac{\gamma p^*}{M^{1/\alpha} H(t,T)} \int_{T_1}^t \Big(- \frac{\partial}{\partial s} H(t,s) \Big) \Big(\frac{M^{1/\alpha} \mu}{\gamma p^* \xi} \Big) ds \\
&= \frac{-\mu}{\xi H(t,T)} \int_{T_1}^t \frac{\partial}{\partial s} H(t,s) ds \\
&= \frac{-\mu}{\xi H(t,T)} H(t,s) \Big|_{s=T_1}^{t} \\
&= \frac{\mu}{\xi} \cdot \frac{H(t,T_1)}{H(t,T)} \\
&\geq \frac{\mu}{\xi} \cdot \frac{H(t,T_1)}{H(t,t_0)}.
\end{aligned}
$$

By (3.35), there exists $T_2 \geq T_1$ such that

$$\frac{H(t,T_1)}{H(t,t_0)} \geq \xi \quad \text{for all} \quad t \geq T_2,$$

which implies that $B(t) \geq \mu$. Since μ is arbitrary,

$$\lim_{t\to\infty} B(t) = \infty. \tag{3.36}$$

Next, we consider a sequence $\{t_n\}_{n=1}^{\infty}$ in $[t_0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \infty$$

and satisfying

$$\lim_{n\to\infty} \big[B(t_n) - A(t_n)\big] = \lim_{t\to\infty} \inf \big[B(t) - A(t)\big] < \infty.$$

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Hence, there exists a constant K such that

$$B(t_n) - A(t_n) \leq K \tag{3.37}$$

for all sufficiently large $n \in \mathbb{N}$. It follows from (3.36) that

$$\lim_{n\to\infty} B(t_n) = \infty. \tag{3.38}$$

and (3.37) implies that

$$\lim_{n\to\infty} A(t_n) = \infty. \tag{3.39}$$

Furthermore, by (3.37) and (3.38), we derive

$$1 - \frac{A(t_n)}{B(t_n)} \leq \frac{K}{B(t_n)} < \epsilon$$

for large enough value of $n \in \mathbb{N}$, where $\epsilon \in (0,1)$ is a constant. Thus

$$\frac{A(t_n)}{B(t_n)} > 1 - \epsilon > 0$$

for $n \in \mathbb{N}$ large enough, which together with (3.39) implies that

$$\lim_{n\to\infty} \frac{\big(A(t_n)\big)^{\alpha+1}}{\big(B(t_n)\big)^{\alpha}} = \infty. \tag{3.40}$$

On the other hand, by Hölder's inequality, we have for $n \in \mathbb{N}$

$$\begin{aligned}
A(t_n) &= \frac{1}{H(t_n,T)} \int_T^{t_n} \left[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)}\right] w(s)\,ds \\
&= \int_T^{t_n} \left\{ \left(\frac{\big(\gamma p^* \sigma'(s) H(t_n,s)\big)^{\alpha}}{\big(H(t_n,T)\big)^{\alpha} M\rho(s) r(\sigma(s))} \right)^{\frac{1}{\alpha+1}} w(s) \right\} \times \\
&\quad \left\{ \left(\frac{\big(\gamma p^* \sigma'(s)\big)^{-\alpha} M\rho(s) r(\sigma(s))}{H(t_n,T)\big(H(t_n,s)\big)^{\alpha}} \right)^{\frac{1}{\alpha+1}} \left[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)}\right] \right\} ds \\
&\leq \left(\int_T^{t_n} \left\{ \left(\frac{\big(\gamma p^* \sigma'(s) H(t_n,s)\big)^{\alpha}}{\big(H(t_n,T)\big)^{\alpha} M\rho(s) r(\sigma(s))} \right)^{\frac{1}{\alpha+1}} w(s) \right\}^{\frac{\alpha+1}{\alpha}} ds \right)^{\frac{\alpha}{\alpha+1}} \times \\
&\quad \left(\int_T^{t_n} \left\{ \left(\frac{\big(\gamma p^* \sigma'(s)\big)^{-\alpha} M\rho(s) r(\sigma(s))}{H(t_n,T)\big(H(t_n,s)\big)^{\alpha}} \right)^{\frac{1}{\alpha+1}} \times \right. \right. \\
&\quad \left. \left. \left[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)}\right] \right\}^{\alpha+1} ds \right)^{\frac{1}{\alpha+1}} \\
&= \left(\frac{1}{H(t_n,T)} \int_T^{t_n} H(t_n,s) \frac{\gamma p^* \sigma'(s)}{\big(M\rho(s) r(\sigma(s))\big)^{1/\alpha}} \big(w(s)\big)^{\frac{\alpha+1}{\alpha}} ds \right)^{\frac{\alpha}{\alpha+1}} \times \\
&\quad \left(\frac{1}{H(t_n,T)} \int_T^{t_n} \frac{M\rho(s) r(\sigma(s)) \left[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)}\right]^{\alpha+1}}{\big(\gamma p^* \sigma'(s) H(t_n,s)\big)^{\alpha}} ds \right)^{\frac{1}{\alpha+1}},
\end{aligned}$$

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and accordingly,

$$\frac{\big(A(t_n)\big)^{\alpha+1}}{\big(B(t_n)\big)^{\alpha}} \le \frac{M}{H(t_n,T)} \int_T^{t_n} \frac{\rho(s)r(\sigma(s))\Big[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)}\Big]^{\alpha+1}}{\big(\gamma p^*\sigma'(s)H(t_n,s)\big)^{\alpha}} ds$$

$$= \frac{M}{H(t_n,T)} \int_T^{t_n} \frac{\rho(s)r(\sigma(s))\Big[h(t_n,s) + \sqrt{H(t_n,s)}\frac{\rho'(s)}{\rho(s)}\Big]^{\alpha+1}}{\big(\gamma p^*\sigma'(s)\big)^{\alpha}\big(\sqrt{H(t_n,s)}\big)^{\alpha-1}} ds$$

So, from (3.40), we have

$$\lim_{n\to\infty} \frac{1}{H(t_n,T)} \int_T^{t_n} \frac{\rho(s)r(\sigma(s))\Big[h(t_n,s) + \sqrt{H(t_n,s)}\frac{\rho'(s)}{\rho(s)}\Big]^{\alpha+1}}{\big(\sigma'(s)\big)^{\alpha}\big(\sqrt{H(t_n,s)}\big)^{\alpha-1}} ds = \infty,$$

which contradicts (3.28). Therefore, (3.33) holds. Now, from (3.31) we obtain

$$\int_T^{\infty} \frac{\sigma'(s)}{\big(\rho(s)r(\sigma(s))\big)^{\frac{1}{\alpha}}} \big(\phi_+(s)\big)^{\frac{\alpha+1}{\alpha}} ds \le \int_T^{\infty} \frac{\sigma'(s)}{\big(\rho(s)r(\sigma(s))\big)^{\frac{1}{\alpha}}} \big(w(s)\big)^{\frac{\alpha+1}{\alpha}} ds < \infty,$$

which contradicts (3.30). This completes the proof of Theorem 3.2.2. □

The following result is a direct consequence of Theorem 3.2.2 and use the same choice of the functions H and h as in Corollary 3.2.4.

Corollary 3.2.6 *Let $(H_1)-(H_5)$ and (S_1) be satisfied. Suppose that there exists a constant $\bar{p}$ such that $0 \le p(t) \le \bar{p} < 1$ and there exist positive function $\rho \in C^1\big([t_0,\infty);\mathbb{R}^+\big)$ and function $\phi \in C\big([t_0,\infty)\big)$ such that (3.30) along with*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda}} \int_{t_0}^{t} M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s)ds < \infty$$

holds and

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda}} \int_{T}^{t} (t-s)^{\lambda}q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s)\Big\}ds \ge \phi(T)$$

for all $T \ge t_0$ and for some $\lambda > \alpha$. Then Eq.(E) is oscillatory, where $p^* = 1-\bar{p}$ and

$$G(t,s) = \frac{\rho(s)r(\sigma(s))(t-s)^{\lambda-\alpha-1}}{\big(\sigma'(s)\big)^{\alpha}} \Big(\lambda + \frac{\rho'(s)}{\rho(s)}(t-s)\Big)^{\alpha+1}.$$

Proof. The only thing to be checked is condition (3.27). With the above choice of the functions H and h, this is fulfilled automatically since

$$\lim_{t\to\infty} \frac{H(t,s)}{H(t,t_0} = \lim_{t\to\infty} \frac{(t-s)^{\lambda}}{(t-t_0)^{\lambda}} = 1$$

for any $s \ge t_0$. □

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Theorem 3.2.3 *Let $(H_1)-(H_5)$ and (S_1) be satisfied. Suppose that there exists a constant $\bar{p}$ such that $0 \le p(t) \le \bar{p} < 1$ and there exists a function $H \in \mathcal{P}$ such that (3.27) holds and there exists a positive function $\rho \in C^1\big([t_0,\infty);\mathbb{R}^+\big)$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s)ds < \infty. \tag{3.41}$$

If there exists a function $\phi \in C\big([t_0,\infty)\big)$ such that for every $T \ge t_0$

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \Big\{H(t,s)\rho(s)q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s)\Big\}ds \ge \phi(T) \tag{3.42}$$

and (3.30) holds, then Eq.(E) is oscillatory, where $p^* = 1-\bar{p}$ and

$$G(t,s) = \frac{\rho(s)r(\sigma(s))}{(\sigma'(s))^{\alpha}\big(\sqrt{H(t,s)}\big)^{\alpha-1}}\Big(h(t,s)+\frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\Big)^{\alpha+1}.$$

Proof. Suppose that there exists a nonoscillatory solution $u(t)$ of Eq.(E). Without loss of generality, we may assume that $u(t) > 0, u(\tau(t)) > 0$, and $u(\sigma(t)) > 0$ on $[T,\infty)$ for some sufficiently large $T \ge t_0$. Define w as in (3.24). As in the proof of Theorem 3.2.1, we can obtain (3.25), (3.26) and

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \Big\{H(t,s)\rho(s)q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s)\Big\}ds \le w(T).$$

Therefore, by (3.42), we have

$$\phi(T) \le w(T) \qquad \text{for } \ T \ge t_0. \tag{3.43}$$

Using (3.41) and (3.25), we conclude

$$\limsup_{t\to\infty}\big[B(t)-A(t)\big] \le w(T) - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)\rho(s)q(s)ds < \infty,$$

where $A(t)$ and $B(t)$ are defined as in Theorem 3.2.2.

It follows from (3.42) that

$$\begin{aligned}\phi(T) \le &\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)\rho(s)q(s)ds \\ &- \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s)ds.\end{aligned}$$

Hence, (3.41) implies that

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s)ds < \infty. \tag{3.44}$$

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Next, we claim that

$$\int_T^\infty \frac{\sigma'(s)}{\big(\rho(s)r(\sigma(s))\big)^{\frac{1}{\alpha}}} \big(w(s)\big)^{\frac{\alpha+1}{\alpha}} ds < \infty. \tag{3.45}$$

Suppose to the contrary that there exists a number $T_1 \geq T$ such that

$$\int_T^\infty \frac{\sigma'(s)}{\big(\rho(s)r(\sigma(s))\big)^{\frac{1}{\alpha}}} \big(w(s)\big)^{\frac{\alpha+1}{\alpha}} ds \geq \frac{M^{1/\alpha}\mu}{\gamma p^*\xi} \quad \text{for all} \quad t \geq T_1$$

where μ is an arbitrary positive number and ξ is a positive constant with satisfy (3.35). Using integration by parts, we get for all $t \geq T_1$

$$\begin{aligned}
B(t) &= \frac{\gamma p^*}{M^{1/\alpha}H(t,T)} \int_T^t H(t,s) d\Big(\int_T^s \frac{\sigma'(v)}{\big(\rho(v)r(\sigma(v))\big)^{1/\alpha}} \big(w(v)\big)^{\frac{\alpha+1}{\alpha}} dv\Big) \\
&= \frac{\gamma p^*}{M^{1/\alpha}H(t,T)} \int_T^t \Big(-\frac{\partial}{\partial s}H(t,s)\Big)\Big(\int_T^s \frac{\sigma'(v)}{\big(\rho(v)r(\sigma(v))\big)^{1/\alpha}} \big(w(v)\big)^{\frac{\alpha+1}{\alpha}} dv\Big) ds \\
&\geq \frac{\gamma p^*}{M^{1/\alpha}H(t,T)} \int_{T_1}^t \Big(-\frac{\partial}{\partial s}H(t,s)\Big)\Big(\frac{M^{1/\alpha}\mu}{\gamma p^*\xi}\Big) ds \\
&= \frac{-\mu}{\xi H(t,T)} \int_{T_1}^t \frac{\partial}{\partial s} H(t,s) ds \\
&= \frac{\mu}{\xi} . \frac{H(t,T_1)}{H(t,T)} \\
&\geq \frac{\mu}{\xi} . \frac{H(t,T_1)}{H(t,t_0)}.
\end{aligned}$$

By (3.35), there exists $T_2 \geq T_1$ such that

$$\frac{H(t,T_1)}{H(t,t_0)} \geq \xi \quad \text{for all} \quad t \geq T_2,$$

which implies that $B(t) \geq \mu$. Since μ is arbitrary,

$$\lim_{t\to\infty} B(t) = \infty. \tag{3.46}$$

Considering a sequence $\Big\{t_n\Big\}_{n=1}^\infty$ in $[t_0, \infty)$ with $\lim_{n\to\infty} t_n = \infty$ and satisfying

$$\lim_{n\to\infty} \big[B(t_n) - A(t_n)\big] = \limsup_{t\to\infty} \big[B(t) - A(t)\big] < \infty.$$

Hence, there exists a constant K such that

$$B(t_n) - A(t_n) \leq K \tag{3.47}$$

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for all sufficiently large $n \in \mathbb{N}$. It follows from (3.46) that

$$\lim_{n\to\infty} B(t_n) = \infty. \tag{3.48}$$

and (3.47) implies that

$$\lim_{n\to\infty} A(t_n) = \infty. \tag{3.49}$$

Furthermore, by (3.47) and (3.48), we derive

$$1 - \frac{A(t_n)}{B(t_n)} \le \frac{K}{B(t_n)} < \epsilon$$

for large enough value of $n \in \mathbb{N}$, where $\epsilon \in (0,1)$ is a constant. Thus

$$\frac{A(t_n)}{B(t_n)} > 1 - \epsilon > 0$$

for $n \in \mathbb{N}$ large enough, which together with (3.49) implies that

$$\lim_{n\to\infty} \frac{\big(A(t_n)\big)^{\alpha+1}}{\big(B(t_n)\big)^{\alpha}} = \infty. \tag{3.50}$$

On the other hand, by Hölder's inequality, we have for $n \in \mathbb{N}$

$$\begin{aligned}
&A(t_n)\\
&= \frac{1}{H(t_n,T)} \int_T^{t_n} \Big[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)}\Big] w(s)ds\\
&= \int_T^{t_n} \Big\{\Big[\frac{\big(\gamma p^*\sigma'(s)\big)^{\alpha}\big(H(t_n,s)\big)^{\alpha}}{M\rho(s)r(\sigma(s)\big(H(t_n,T)\big)^{\alpha}}\Big]^{\frac{1}{\alpha+1}} w(s)\Big\}\times\\
&\quad \Big\{\Big[\frac{\big(\gamma p^*\sigma'(s)\big)^{-\alpha}M\rho(s)r(\sigma(s)}{H(t_n,T)\big(H(t_n,s)\big)^{\alpha}}\Big]^{\frac{1}{\alpha+1}}\Big[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)}\Big]\Big\}ds\\
&\le \Big(\frac{1}{H(t_n,T)}\int_T^{t_n} H(t_n,s)\frac{\gamma p^*\sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}}\big(w(s)\big)^{\frac{\alpha+1}{\alpha}}ds\Big)^{\frac{\alpha}{\alpha+1}}\times\\
&\quad \Big(\frac{1}{H(t_n,T)}\int_T^{t_n} \frac{M\rho(s)r(\sigma(s))\Big[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)}\Big]^{\alpha+1}}{\big(\gamma p^*\sigma'(s)H(t_n,s)\big)^{\alpha}}ds\Big)^{\frac{1}{\alpha+1}},
\end{aligned}$$

and accordingly,

$$\begin{aligned}
\frac{\big(A(t_n)\big)^{\alpha+1}}{\big(B(t_n)\big)^{\alpha}} &\le \frac{M}{H(t_n,T)}\int_T^{t_n} \frac{\rho(s)r(\sigma(s))\Big[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)}\Big]^{\alpha+1}}{\big(\gamma p^*\sigma'(s)H(t_n,s)\big)^{\alpha}}ds\\
&= \frac{M}{H(t_n,T)}\int_T^{t_n} \frac{\rho(s)r(\sigma(s))\Big[h(t_n,s) + \sqrt{H(t_n,s)}\frac{\rho'(s)}{\rho(s)}\Big]^{\alpha+1}}{\big(\gamma p^*\sigma'(s)\big)^{\alpha}\big(\sqrt{H(t_n,s)}\big)^{\alpha-1}}ds
\end{aligned}$$

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So, from (3.50), we have

$$\lim_{n\to\infty}\frac{1}{H(t_n,T)}\int_T^{t_n}\frac{\rho(s)r(\sigma(s))\Big[h(t_n,s)+\sqrt{H(t_n,s)}\frac{\rho'(s)}{\rho(s)}\Big]^{\alpha+1}}{(\sigma'(s))^{\alpha}\big(\sqrt{H(t_n,s)}\big)^{\alpha-1}}ds=\infty,$$

which contradicts (3.44). Therefore, (3.45) holds. Now, from (3.43) we obtain

$$\int_T^{\infty}\frac{\sigma'(s)}{\big(\rho(s)r(\sigma(s))\big)^{\frac{1}{\alpha}}}\,\big(\phi_+(s)\big)^{\frac{\alpha+1}{\alpha}}ds\le\int_T^{\infty}\frac{\sigma'(s)}{\big(\rho(s)r(\sigma(s))\big)^{\frac{1}{\alpha}}}\,\big(w(s)\big)^{\frac{\alpha+1}{\alpha}}ds<\infty,$$

which contradicts (3.30). This completes the proof of Theorem 3.2.3. □

Theorem 3.2.4 *Let $(H_1)-(H_5)$ and (S_2) be satisfied. If there exist function $H\in\mathcal{P}$ and positive function $\rho\in C^1\big([t_0,\infty);\mathbb{R}^+\big)$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Big\{\beta H(t,s)\rho(s)Q(s)-M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}G(t,s)\Big\}ds=\infty,\quad(3.51)$$

then Eq.(E) is oscillatory, where $Q(t)=q(t)\big(1-p(\sigma(t))\big)^{\alpha}$ and

$$G(t,s)=\frac{\rho(s)r(\sigma(s))}{(\sigma'(s))^{\alpha}\big(\sqrt{H(t,s)}\big)^{\alpha-1}}\Big(h(t,s)+\frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\Big)^{\alpha+1}.$$

Proof. On the contrary, we may assume that $u(t)$ is an eventually positive solution of Eq.(E). From the proof of Lemma 3.1.2 and Theorem 3.1.2, we conclude that $u(t)\ge z(t)\big(1-p(t)\big)$, $r(t)\psi(u(t))|z'(t)|^{\alpha-1}z'(t)$ is decreasing function and $z'(t)>0$ on $[T,\infty)$ for some sufficiently large $T\ge t_0$. Hence, from Eq.(E) and $\sigma(t)\le t$, we have for $t\ge T$

$$z'(\sigma(t))\ge z'(t)\Big(\frac{r(t)\psi(u(t))}{Mr(\sigma(t))}\Big)^{\frac{1}{\alpha}}$$

and

$$\big(r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}\big)'+\beta Q(t)z^{\alpha}(\sigma(t))\le 0$$

where $Q(t)=q(t)\big(1-p(\sigma(t))\big)^{\alpha}$. We define a function w by

$$w(t)=\rho(t)\frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}}{\big(z(\sigma(t))\big)^{\alpha}},\quad t\ge T.\qquad(3.52)$$

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Then $w(t) > 0$ and

$$
\begin{aligned}
w'(t) &= \frac{d}{dt}\Big(\rho(t)\frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}}{\big(z(\sigma(t))\big)^{\alpha}}\Big) \\
&= \rho'(t)\frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}}{\big(z(\sigma(t))\big)^{\alpha}} + \rho(t)\frac{\Big(r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}\Big)'}{\big(z(\sigma(t))\big)^{\alpha}} \\
&\quad - \rho(t)\frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}\alpha z'(\sigma(t))\sigma'(t)}{\big(z(\sigma(t))\big)^{\alpha+1}} \\
&\le \frac{\rho'(t)}{\rho(t)}w(t) - \beta\rho(t)Q(t) - \alpha\sigma'(t)\rho(t)\frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha+1}}{\big(z(\sigma(t))\big)^{\alpha}}\Big(\frac{r(t)\psi(u(t))}{Mr(\sigma(t))}\Big)^{\frac{1}{\alpha}} \\
&= \frac{\rho'(t)}{\rho(t)}w(t) - \beta\rho(t)Q(t) - \frac{\alpha\sigma'(t)}{\big(M\rho(t)r(\sigma(t))\big)^{1/\alpha}}\Big(\rho(t)\frac{r(t)\psi(u(t))\big(z'(t)\big)^{\alpha}}{\big(z(\sigma(t))\big)^{\alpha}}\Big)^{\frac{\alpha+1}{\alpha}} \\
&= \frac{\rho'(t)}{\rho(t)}w(t) - \beta\rho(t)Q(t) - \frac{\alpha\sigma'(t)}{\big(M\rho(t)r(\sigma(t))\big)^{1/\alpha}}\big(w(t)\big)^{\frac{\alpha+1}{\alpha}}.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\int_T^t \beta H(t,s)\rho(s)Q(s)ds \le -\int_T^t H(t,s)w'(s)ds + \int_T^t H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds \\
- \int_T^t H(t,s)\frac{\alpha\sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}}\big(w(s)\big)^{\frac{\alpha+1}{\alpha}}ds.
\end{aligned}
$$

Since

$$
\int_T^t H(t,s)w'(s)ds = -H(t,T)w(T) - \int_T^t w(s)\frac{\partial}{\partial s}H(t,s)ds
$$

and in view of (3.22), the previous inequality becomes

$$
\begin{aligned}
&\int_T^t \beta H(t,s)\rho(s)Q(s)ds \\
&\qquad \le H(t,T)w(T) + \int_T^t w(s)\frac{\partial}{\partial s}H(t,s)ds + \int_T^t H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds \\
&\qquad\quad - \int_T^t H(t,s)\frac{\alpha\sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}}\big(w(s)\big)^{\frac{\alpha+1}{\alpha}}ds \\
&\qquad = H(t,T)w(T) - \int_T^t w(s)h(t,s)\sqrt{H(t,s)}ds + \int_T^t H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds \\
&\qquad\quad - \int_T^t H(t,s)\frac{\alpha\sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}}\big(w(s)\big)^{\frac{\alpha+1}{\alpha}}ds \\
&\qquad \le H(t,T)w(T) + \int_T^t \Big[h(t,s)\sqrt{H(t,s)} + H(t,s)\frac{\rho'(s)}{\rho(s)}\Big]w(s)ds \\
&\qquad\quad - \int_T^t H(t,s)\frac{\alpha\sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}}\big(w(s)\big)^{\frac{\alpha+1}{\alpha}}ds \qquad (3.53)
\end{aligned}
$$

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In Lemma 3.1.1, we let

$$\lambda = \frac{\alpha+1}{\alpha} > 1, \qquad X = \Big(\frac{\big(\alpha\sigma'(s)H(t,s)\big)^{\alpha}}{M\rho(s)r(\sigma(s))}\Big)^{\frac{1}{\alpha+1}} w(t)$$

and

$$Y = \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha}\Big(h(t,s)\sqrt{H(t,s)} + H(t,s)\frac{\rho'(s)}{\rho(s)}\Big)^{\alpha}\Big(\frac{M\rho(s)r(\sigma(s))}{\big(\alpha\sigma'(s)H(t,s)\big)^{\alpha}}\Big)^{\frac{\alpha}{\alpha+1}}.$$

From Lemma 3.1.1, we then obtain for $t > s \geq T$

$$\begin{aligned}
&\Big[h(t,s)\sqrt{H(t,s)} + H(t,s)\frac{\rho'(s)}{\rho(s)}\Big]w(s) - H(t,s)\frac{\alpha\sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}}\big(w(s)\big)^{\frac{\alpha+1}{\alpha}}\\
&\leq \frac{1}{\alpha}\Big[\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha}\Big(h(t,s)\sqrt{H(t,s)} + H(t,s)\frac{\rho'(s)}{\rho(s)}\Big)^{\alpha}\Big(\frac{M\rho(s)r(\sigma(s))}{\big(\alpha\sigma'(s)H(t,s)\big)^{\alpha}}\Big)^{\frac{\alpha}{\alpha+1}}\Big]^{\frac{\alpha+1}{\alpha}}\\
&= \frac{1}{\alpha}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\Big(h(t,s)\sqrt{H(t,s)} + H(t,s)\frac{\rho'(s)}{\rho(s)}\Big)^{\alpha+1}\frac{M\rho(s)r(\sigma(s))}{\big(\alpha\sigma'(s)H(t,s)\big)^{\alpha}}\\
&= M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\frac{\rho(s)r(\sigma(s))}{\big(\sigma'(s)\big)^{\alpha}\big(\sqrt{H(t,s)}\big)^{\alpha-1}}\Big(h(t,s) + \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\Big)^{\alpha+1}\\
&= M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}G(t,s).
\end{aligned}$$

Hence, (3.53) implies for $t > s \geq T$

$$\begin{aligned}
\int_T^t \Big\{\beta H(t,s)\rho(s)Q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}G(t,s)\Big\}ds &\leq H(t,T)w(T) \qquad (3.54)\\
&\leq H(t,t_0)w(T).
\end{aligned}$$

Thereby, including (ii), we conclude that for $t > t_0$

$$\begin{aligned}
&\int_{t_0}^t \Big\{\beta H(t,s)\rho(s)Q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}G(t,s)\Big\}ds\\
&\qquad = \int_{t_0}^T \Big\{\beta H(t,s)\rho(s)Q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}G(t,s)\Big\}ds\\
&\qquad\quad + \int_T^t \Big\{\beta H(t,s)\rho(s)Q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}G(t,s)\Big\}ds\\
&\qquad \leq \int_{t_0}^T \beta H(t,s)\rho(s)Q(s)ds + H(t,t_0)w(T)\\
&\qquad \leq H(t,t_0)\Big[\int_{t_0}^T \beta\rho(s)Q(s)ds + w(T)\Big].
\end{aligned}$$

Accordingly, we obtain

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^t \Big\{\beta H(t,s)\rho(s)Q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}G(t,s)\Big\}ds\\
&\qquad\qquad \leq \int_{t_0}^T \beta\rho(s)Q(s)ds + w(T) < \infty
\end{aligned}$$

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for $t \geq t_0$. Taking the limit superior as $t \to \infty$ in the above inequality, we obtain a contradiction to (3.51), which completes the proof of Theorem 3.2.4. □

As immediate consequences of Theorem 3.2.4, we obtain the following Corollaries.

Corollary 3.2.7 *Let $(H_1)-(H_5)$ and (S_2) be satisfied. If there exist function $H \in \mathcal{P}$ and positive function $\rho \in C^1([t_0, \infty); \mathbb{R}^+)$ such that*

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} \beta H(t,s)\rho(s)Q(s)ds = \infty,$$

and

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1} G(t,s)ds < \infty,$$

then Eq.(E) is oscillatory, where $Q(t) = q(t)\big(1 - p(\sigma(t))\big)^{\alpha}$ and

$$G(t,s) = \frac{\rho(s)r(\sigma(s))}{(\sigma'(s))^{\alpha}\big(\sqrt{H(t,s)}\big)^{\alpha-1}}\Big(h(t,s) + \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\Big)^{\alpha+1}.$$

Theorem 3.2.5 *Let $(H_1)-(H_5)$ and (S_2) be satisfied. Suppose that there exists a function $H \in \mathcal{P}$ such that (3.27) holds and there exists a positive function $\rho \in C^1([t_0, \infty); \mathbb{R}^+)$ such that*

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1} G(t,s)ds < \infty. \tag{3.55}$$

If there exists a function $\phi \in C([t_0, \infty))$ such that for every $T \geq t_0$

$$\lim_{t\to\infty} \sup \frac{1}{H(t,T)} \int_{T}^{t} \Big\{\beta H(t,s)\rho(s)Q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1} G(t,s)\Big\}ds \geq \phi(T) \tag{3.56}$$

and (3.30) holds, then Eq.(E) is oscillatory, where $Q(t) = q(t)\big(1 - p(\sigma(t))\big)^{\alpha}$ and

$$G(t,s) = \frac{\rho(s)r(\sigma(s))}{(\sigma'(s))^{\alpha}\big(\sqrt{H(t,s)}\big)^{\alpha-1}}\Big(h(t,s) + \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\Big)^{\alpha+1}.$$

Proof. Suppose that there exists a nonoscillatory solution $u(t)$ of Eq.(E). Without loss of generality, we may assume that $u(t) > 0, u(\tau(t)) > 0$, and $u(\sigma(t)) > 0$ on

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$[T,\infty)$ for some sufficiently large $T \geq t_0$. Define w as in (3.52). As in the proof of Theorem 3.2.4, we can obtain (3.53) and (3.54). Then, for $t > T \geq t_0$, we have

$$\lim_{t\to\infty} \sup \frac{1}{H(t,T)} \int_T^t \Big\{\beta H(t,s)\rho(s)Q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1} G(t,s)\Big\} ds \leq w(T)$$

Therefore, by (3.56), we have for $T \geq t_0$

$$\phi(T) \leq w(T) \tag{3.57}$$

and

$$\lim_{t\to\infty} \sup \frac{1}{H(t,T)} \int_T^t \beta H(t,s)\rho(s)Q(s)ds \geq \phi(T). \tag{3.58}$$

We define functions

$$\overline{A}(t) = \frac{1}{H(t,T)} \int_T^t \Big[h(t,s)\sqrt{H(t,s)} + H(t,s)\frac{\rho'(s)}{\rho(s)}\Big] w(s)ds$$

and

$$\overline{B}(t) = \frac{1}{H(t,T)} \int_T^t H(t,s)\frac{\alpha\sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}} \big(w(s)\big)^{\frac{\alpha+1}{\alpha}} ds.$$

Then, by (3.53) and (3.58), we obtain

$$\begin{aligned}\lim_{t\to\infty} \inf \big[\overline{B}(t) - \overline{A}(t)\big] &\leq w(T) - \lim_{t\to\infty} \sup \frac{1}{H(t,T)} \int_T^t \beta H(t,s)\rho(s)Q(s)ds \\ &\leq w(T) - \phi(T) \\ &< \infty.\end{aligned}$$

Now we claim that

$$\int_T^\infty \frac{\sigma'(s)}{\big(\rho(s)r(\sigma(s))\big)^{\frac{1}{\alpha}}} \big(w(s)\big)^{\frac{\alpha+1}{\alpha}} ds < \infty. \tag{3.59}$$

Suppose to the contrary that there exists a $T_1 \geq T$ such that

$$\int_T^\infty \frac{\sigma'(s)}{\big(\rho(s)r(\sigma(s))\big)^{\frac{1}{\alpha}}} \big(w(s)\big)^{\frac{\alpha+1}{\alpha}} ds \geq \frac{M^{1/\alpha}\mu}{\alpha\xi} \quad \text{for all} \quad t \geq T_1 \tag{3.60}$$

where μ is an arbitrary positive number and ξ is a positive constant such that

$$\inf_{s\geq t_0} \Big[\lim_{t\to\infty} \inf \frac{H(t,s)}{H(t,t_0)}\Big] > \xi > 0. \tag{3.61}$$

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Using integration by parts, we get for all $t \geq T_1$

$$\begin{aligned}
\overline{B}(t) &= \frac{\alpha}{M^{1/\alpha}H(t,T)}\int_T^t H(t,s)d\Big(\int_T^s \frac{\sigma'(v)}{\big(\rho(v)r(\sigma(v))\big)^{1/\alpha}}\big(w(v)\big)^{\frac{\alpha+1}{\alpha}}dv\Big)\\
&= \frac{\alpha}{M^{1/\alpha}H(t,T)}\int_T^t \Big(-\frac{\partial}{\partial s}H(t,s)\Big)\Big(\int_T^s \frac{\sigma'(v)}{\big(\rho(v)r(\sigma(v))\big)^{1/\alpha}}\big(w(v)\big)^{\frac{\alpha+1}{\alpha}}dv\Big)ds\\
&\geq \frac{\alpha}{M^{1/\alpha}H(t,T)}\int_{T_1}^t \Big(-\frac{\partial}{\partial s}H(t,s)\Big)\Big(\frac{M^{1/\alpha}\mu}{\alpha\xi}\Big)ds\\
&= \frac{-\mu}{\xi H(t,T)}\int_{T_1}^t \frac{\partial}{\partial s}H(t,s)ds\\
&= \frac{\mu}{\xi}.\frac{H(t,T_1)}{H(t,T)}\\
&\geq \frac{\mu}{\xi}.\frac{H(t,T_1)}{H(t,t_0)}.
\end{aligned}$$

By (3.61), there exists $T_2 \geq T_1$ such that

$$\frac{H(t,T_1)}{H(t,t_0)} \geq \xi \qquad \text{for all} \quad t \geq T_2,$$

which implies that $\overline{B}(t) \geq \mu$. Since μ is arbitrary,

$$\lim_{t\to\infty} \overline{B}(t) = \infty. \tag{3.62}$$

Considering a sequence $\Big\{t_n\Big\}_{n=1}^{\infty}$ in $[t_0, \infty)$ with $\lim_{n\to\infty} t_n = \infty$ and satisfying

$$\lim_{n\to\infty}\big[\overline{B}(t_n) - \overline{A}(t_n)\big] = \lim_{t\to\infty}\inf\big[\overline{B}(t_n) - \overline{A}(t_n)\big] < \infty.$$

Hence, there exists a constant K such that

$$\overline{B}(t_n) - \overline{A}(t_n) \leq K \tag{3.63}$$

for all sufficiently large $n \in \mathbb{N}$. It follows from (3.62) that

$$\lim_{n\to\infty} \overline{B}(t_n) = \infty. \tag{3.64}$$

and (3.63) implies that

$$\lim_{n\to\infty} \overline{A}(t_n) = \infty. \tag{3.65}$$

Furthermore, by (3.63) and (3.64), we derive

$$1 - \frac{\overline{A}(t_n)}{\overline{B}(t_n)} \leq \frac{K}{\overline{B}(t_n)} < \epsilon$$

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for large enough value of $n \in \mathbb{N}$, where $\epsilon \in (0,1)$ is a constant. Thus

$$\frac{\overline{A}(t_n)}{\overline{B}(t_n)} > 1 - \epsilon > 0$$

for $n \in \mathbb{N}$ large enough, which together with (3.65) implies that

$$\lim_{n\to\infty} \frac{\left(\overline{A}(t_n)\right)^{\alpha+1}}{\left(\overline{B}(t_n)\right)^{\alpha}} = \infty. \tag{3.66}$$

On the other hand, by Hölder's inequality, we have for $n \in \mathbb{N}$

$$\begin{aligned}
&\overline{A}(t_n)\\
&= \frac{1}{H(t_n,T)} \int_T^{t_n} \left[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)}\right] w(s)\,ds\\
&= \int_T^{t_n} \Big\{ \Big[\frac{(\alpha\sigma'(s))^{\alpha}(H(t_n,s))^{\alpha}}{M\rho(s)r(\sigma(s)(H(t_n,T))^{\alpha}} \Big]^{\frac{1}{\alpha+1}} w(s) \Big\} \times\\
&\quad \Big\{ \Big[\frac{(\alpha\sigma'(s))^{-\alpha} M\rho(s)r(\sigma(s)}{H(t_n,T)(H(t_n,s))^{\alpha}} \Big]^{\frac{1}{\alpha+1}} \Big[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)} \Big] \Big\} ds\\
&\le \Big(\frac{1}{H(t_n,T)} \int_T^{t_n} H(t_n,s) \frac{\alpha\sigma'(s)}{(M\rho(s)r(\sigma(s)))^{1/\alpha}} (w(s))^{\frac{\alpha+1}{\alpha}} ds \Big)^{\frac{\alpha}{\alpha+1}} \times\\
&\quad \Big(\frac{1}{H(t_n,T)} \int_T^{t_n} \frac{M\rho(s)r(\sigma(s)) \Big[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)} \Big]^{\alpha+1}}{(\alpha\sigma'(s)H(t_n,s))^{\alpha}} ds \Big)^{\frac{1}{\alpha+1}},
\end{aligned}$$

and accordingly,

$$\begin{aligned}
\frac{\left(\overline{A}(t_n)\right)^{\alpha+1}}{\left(\overline{B}(t_n)\right)^{\alpha}} &\le \frac{M}{H(t_n,T)} \int_T^{t_n} \frac{\rho(s)r(\sigma(s)) \Big[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)} \Big]^{\alpha+1}}{(\alpha\sigma'(s)H(t_n,s))^{\alpha}} ds\\
&= \frac{M}{H(t_n,T)} \int_T^{t_n} \frac{\rho(s)r(\sigma(s)) \Big[h(t_n,s) + \sqrt{H(t_n,s)}\frac{\rho'(s)}{\rho(s)} \Big]^{\alpha+1}}{(\alpha\sigma'(s))^{\alpha}\left(\sqrt{H(t_n,s)}\right)^{\alpha-1}} ds
\end{aligned}$$

So, from (3.66), we have

$$\lim_{n\to\infty} \frac{1}{H(t_n,T)} \int_T^{t_n} \frac{\rho(s)r(\sigma(s)) \Big[h(t_n,s) + \sqrt{H(t_n,s)}\frac{\rho'(s)}{\rho(s)} \Big]^{\alpha+1}}{(\sigma'(s))^{\alpha}\left(\sqrt{H(t_n,s)}\right)^{\alpha-1}} ds = \infty,$$

which contradicts (3.55). Therefore, (3.59) holds. Now, from (3.57) we obtain

$$\int_T^{\infty} \frac{\sigma'(s)}{(\rho(s)r(\sigma(s)))^{\frac{1}{\alpha}}} \left(\phi_+(s)\right)^{\frac{\alpha+1}{\alpha}} ds \le \int_T^{\infty} \frac{\sigma'(s)}{(\rho(s)r(\sigma(s)))^{\frac{1}{\alpha}}} \left(w(s)\right)^{\frac{\alpha+1}{\alpha}} ds < \infty,$$

which contradicts (3.30). This completes the proof of Theorem 3.2.5. □

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Theorem 3.2.6 *Let* $(H_1) - (H_5)$ *and* (S_2) *be satisfied. Suppose that there exists a function* $H \in \mathcal{P}$ *such that* (3.27) *holds and there exists a positive function* $\rho \in C^1([t_0, \infty); \mathbb{R}^+)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \beta H(t,s)\rho(s)Q(s)ds < \infty. \tag{3.67}$$

If there exists a function $\phi \in C([t_0, \infty))$ *such that for every* $T \geq t_0$

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \Big\{\beta H(t,s)\rho(s)Q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1} G(t,s)\Big\}ds \geq \phi(T) \tag{3.68}$$

and (3.30) *holds, then Eq.*(E) *is oscillatory, where* $Q(t) = q(t)\big(1 - p(\sigma(t))\big)^{\alpha}$ *and*

$$G(t,s) = \frac{\rho(s)r(\sigma(s))}{(\sigma'(s))^{\alpha}\big(\sqrt{H(t,s)}\big)^{\alpha-1}} \Big(h(t,s) + \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\Big)^{\alpha+1}.$$

Proof. Suppose that there exists a nonoscillatory solution $u(t)$ of Eq.(E). Without loss of generality, we may assume that $u(t) > 0, u(\tau(t)) > 0$, and $u(\sigma(t)) > 0$ on $[T, \infty)$ for some sufficiently large $T \geq t_0$. Define w as in (3.52). As in the proof of Theorem 3.2.4, we can obtain (3.53), (3.54) and

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \Big\{\beta H(t,s)\rho(s)Q(s) - M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1} G(t,s)\Big\}ds \leq w(T).$$

Therefore, by (3.68), we have

$$\phi(T) \leq w(T) \qquad \text{for } \; T \geq t_0. \tag{3.69}$$

Using (3.67) and (3.53), we conclude

$$\limsup_{t\to\infty} \big[\overline{B}(t) - \overline{A}(t)\big] \leq w(T) - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \beta H(t,s)\rho(s)Q(s)ds < \infty,$$

where $\overline{A}(t)$ and $\overline{B}(t)$ are defined as in Theorem 3.2.5.
It follows from (3.68) that

$$\phi(T) \leq \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \beta H(t,s)\rho(s)Q(s)ds - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1} G(t,s)ds.$$

Hence, (3.67) implies that

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1} G(t,s)ds < \infty. \tag{3.70}$$

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Next, we claim that

$$\int_T^\infty \frac{\sigma'(s)}{\big(\rho(s)r(\sigma(s))\big)^{\frac{1}{\alpha}}}\,\big(w(s)\big)^{\frac{\alpha+1}{\alpha}}ds < \infty. \tag{3.71}$$

Suppose to the contrary that there exists a number $T_1 \geq T$ such that

$$\int_T^\infty \frac{\sigma'(s)}{\big(\rho(s)r(\sigma(s))\big)^{\frac{1}{\alpha}}}\,\big(w(s)\big)^{\frac{\alpha+1}{\alpha}}ds \geq \frac{M^{1/\alpha}\mu}{\alpha\xi} \quad \text{for all} \quad t \geq T_1$$

where μ is an arbitrary positive number and ξ is a positive constant with satisfy (3.61). Using integration by parts, we get for all $t \geq T_1$

$$\begin{aligned}
\overline{B}(t) &= \frac{\alpha}{M^{1/\alpha}H(t,T)}\int_T^t H(t,s)d\Big(\int_T^s \frac{\sigma'(v)}{\big(\rho(v)r(\sigma(v))\big)^{1/\alpha}}\big(w(v)\big)^{\frac{\alpha+1}{\alpha}}dv\Big)\\
&= \frac{\alpha}{M^{1/\alpha}H(t,T)}\int_T^t \Big(-\frac{\partial}{\partial s}H(t,s)\Big)\Big(\int_T^s \frac{\sigma'(v)}{\big(\rho(v)r(\sigma(v))\big)^{1/\alpha}}\big(w(v)\big)^{\frac{\alpha+1}{\alpha}}dv\Big)ds\\
&\geq \frac{\alpha}{M^{1/\alpha}H(t,T)}\int_{T_1}^t \Big(-\frac{\partial}{\partial s}H(t,s)\Big)\Big(\frac{M^{1/\alpha}\mu}{\alpha\xi}\Big)ds\\
&= \frac{-\mu}{\xi H(t,T)}\int_{T_1}^t \frac{\partial}{\partial s}H(t,s)ds\\
&= \frac{\mu}{\xi}.\frac{H(t,T_1)}{H(t,T)}\\
&\geq \frac{\mu}{\xi}.\frac{H(t,T_1)}{H(t,t_0)}.
\end{aligned}$$

By (3.61), there exists $T_2 \geq T_1$ such that

$$\frac{H(t,T_1)}{H(t,t_0)} \geq \xi \quad \text{for all} \quad t \geq T_2,$$

which implies that $\overline{B}(t) \geq \mu$. Since μ is arbitrary,

$$\lim_{t\to\infty} \overline{B}(t) = \infty. \tag{3.72}$$

Considering a sequence $\Big\{t_n\Big\}_{n=1}^{\infty}$ in $[t_0,\infty)$ with $\lim_{n\to\infty} t_n = \infty$ and satisfying

$$\lim_{n\to\infty}\big[\overline{B}(t_n) - \overline{A}(t_n)\big] = \liminf_{t\to\infty}\big[\overline{B}(t_n) - \overline{A}(t_n)\big] < \infty.$$

Hence, there exists a constant K such that

$$\overline{B}(t_n) - \overline{A}(t_n) \leq K \tag{3.73}$$

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for all sufficiently large $n \in \mathbb{N}$. It follows from (3.72) that

$$\lim_{n\to\infty} \overline{B}(t_n) = \infty. \tag{3.74}$$

and (3.73) implies that

$$\lim_{n\to\infty} \overline{A}(t_n) = \infty. \tag{3.75}$$

Furthermore, by (3.73) and (3.74), we derive

$$1 - \frac{\overline{A}(t_n)}{\overline{B}(t_n)} \le \frac{K}{\overline{B}(t_n)} < \epsilon$$

for large enough value of $n \in \mathbb{N}$, where $\epsilon \in (0,1)$ is a constant. Thus

$$\frac{\overline{A}(t_n)}{\overline{B}(t_n)} > 1 - \epsilon > 0$$

for $n \in \mathbb{N}$ large enough, which together with (3.75) implies that

$$\lim_{n\to\infty} \frac{\big(\overline{A}(t_n)\big)^{\alpha+1}}{\big(\overline{B}(t_n)\big)^{\alpha}} = \infty. \tag{3.76}$$

On the other hand, by Hölder's inequality, we have for $n \in \mathbb{N}$

$$\begin{aligned}
&\overline{A}(t_n)\\
&= \frac{1}{H(t_n,T)} \int_T^{t_n} \Big[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)}\Big] w(s)ds\\
&= \int_T^{t_n} \Big\{\Big[\frac{\big(\alpha\sigma'(s)\big)^{\alpha}\big(H(t_n,s)\big)^{\alpha}}{M\rho(s)r(\sigma(s)\big(H(t_n,T)\big)^{\alpha}}\Big]^{\frac{1}{\alpha+1}} w(s)\Big\}\times\\
&\quad \Big\{\Big[\frac{\big(\alpha\sigma'(s)\big)^{-\alpha} M\rho(s)r(\sigma(s)}{H(t_n,T)\big(H(t_n,s)\big)^{\alpha}}\Big]^{\frac{1}{\alpha+1}} \Big[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)}\Big]\Big\} ds\\
&\le \Big(\frac{1}{H(t_n,T)} \int_T^{t_n} H(t_n,s)\frac{\alpha\sigma'(s)}{\big(M\rho(s)r(\sigma(s))\big)^{1/\alpha}}\big(w(s)\big)^{\frac{\alpha+1}{\alpha}} ds\Big)^{\frac{\alpha}{\alpha+1}}\times\\
&\quad \Big(\frac{1}{H(t_n,T)} \int_T^{t_n} \frac{M\rho(s)r(\sigma(s))\Big[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)}\Big]^{\alpha+1}}{\big(\alpha\sigma'(s)H(t_n,s)\big)^{\alpha}} ds\Big)^{\frac{1}{\alpha+1}},
\end{aligned}$$

and accordingly,

$$\begin{aligned}
\frac{\big(\overline{A}(t_n)\big)^{\alpha+1}}{\big(\overline{B}(t_n)\big)^{\alpha}} &\le \frac{M}{H(t_n,T)} \int_T^{t_n} \frac{\rho(s)r(\sigma(s))\Big[h(t_n,s)\sqrt{H(t_n,s)} + H(t_n,s)\frac{\rho'(s)}{\rho(s)}\Big]^{\alpha+1}}{\big(\alpha\sigma'(s)H(t_n,s)\big)^{\alpha}} ds\\
&= \frac{M}{H(t_n,T)} \int_T^{t_n} \frac{\rho(s)r(\sigma(s))\Big[h(t_n,s) + \sqrt{H(t_n,s)}\frac{\rho'(s)}{\rho(s)}\Big]^{\alpha+1}}{\big(\alpha\sigma'(s)\big)^{\alpha}\big(\sqrt{H(t_n,s)}\big)^{\alpha-1}} ds
\end{aligned}$$

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So, from (3.76), we have

$$\lim_{n\to\infty}\frac{1}{H(t_n,T)}\int_T^{t_n}\frac{\rho(s)r(\sigma(s))\left[h(t_n,s)+\sqrt{H(t_n,s)}\frac{\rho'(s)}{\rho(s)}\right]^{\alpha+1}}{(\sigma'(s))^{\alpha}\left(\sqrt{H(t_n,s)}\right)^{\alpha-1}}ds=\infty,$$

which contradicts (3.70). Therefore, (3.71) holds. Now, from (3.69) we obtain

$$\int_T^{\infty}\frac{\sigma'(s)}{(\rho(s)r(\sigma(s)))^{\frac{1}{\alpha}}}\,(\phi_+(s))^{\frac{\alpha+1}{\alpha}}ds\le\int_T^{\infty}\frac{\sigma'(s)}{(\rho(s)r(\sigma(s)))^{\frac{1}{\alpha}}}\,(w(s))^{\frac{\alpha+1}{\alpha}}ds<\infty,$$

which contradicts (3.30). This completes the proof of Theorem 3.2.6. □

3.3 Applications

For illustration we consider the following example of second order nonlinear neutral delay differential equations.

Example 3.3.1 Consider the nonlinear neutral differential equation

$$\Big[\frac{1}{1+u^2(t)}\big|(u(t)+(\tfrac{1}{2}-e^{-t})u(t-\kappa))'\big|^{\alpha-1}(u(t)+(\tfrac{1}{2}-e^{-t})u(t-\kappa))'\Big]'$$
$$+\frac{a}{t^{\alpha+1}}|u(\lambda t)|^{\alpha-1}u(\lambda t)=0,\quad t\ge 1, \tag{3.77}$$

where $0<\lambda<1,\ \kappa\ge 0,\ a\ge 0,\ 0<\alpha<1$. We have

$$r(t)=1,\psi(u)=\frac{1}{1+u^2}\le 1,p(t)=\frac{1}{2}-e^{-t}\le\frac{1}{2},q(t)=\frac{a}{t^{\alpha+1}},\tau(t)=t-\kappa,\sigma(t)=\lambda t.$$

Since

$$q(t)\frac{r^{1/\alpha}(\sigma(t))R^{\alpha+1}(\sigma(t))}{\sigma'(t)}=\frac{a}{t^{\alpha+1}}\frac{(\lambda t-1)^{\alpha+1}}{\lambda}=a\lambda^{\alpha}\quad\text{as}\quad t\to\infty$$

and

$$M\Big(\frac{\alpha}{\beta p^*}\Big)^{\alpha}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}=\Big(\frac{1}{\frac{1}{2}}\Big)^{\alpha}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}=2^{\alpha}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1},$$

it is easy to see that condition (3.12) reduces to

$$a\lambda^{\alpha}>2^{\alpha}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}. \tag{3.78}$$

By Corollary 3.1.2, Eq.(3.77) is oscillatory if (3.78) holds.

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Example 3.3.2 Consider the nonlinear neutral differential equation

$$\Big[\frac{1}{2+u^2(t)}\big|\big(u(t)+(1-e^{-\mu t})u(t-\kappa)\big)'\big|^{\alpha-1}\big(u(t)+(1-e^{-\mu t})u(t-\kappa)\big)'\Big]' + \frac{ae^{\lambda\alpha\mu t}}{t^{\alpha+1}}|u(\lambda t)|^{\alpha-1}u(\lambda t)=0, \quad t\ge 1, \tag{3.79}$$

where $0<\lambda<1,\ \kappa\ge 0,\ a\ge 0,\ \alpha>0,\ \mu\ge 0$. We have

$$r(t)=1, \psi(u)=\frac{1}{2+u^2}, p(t)=1-e^{-\mu t}, q(t)=\frac{ae^{\lambda\alpha\mu t}}{t^{\alpha+1}}, \tau(t)=t-\kappa, \sigma(t)=\lambda t.$$

Since

$$\begin{aligned}\beta\hat{p}(t)\frac{r^{1/\alpha}(\sigma(t))R^{\alpha+1}(\sigma(t))}{\sigma'(t)} &= \frac{ae^{\lambda\alpha\mu t}\big(1-(1-e^{-\mu t})\big)^{\alpha}}{t^{\alpha+1}}\frac{(\lambda t-1)^{\alpha+1}}{\lambda}\\ &= \frac{ae^{-\alpha\mu t(1-\lambda)}}{\lambda}\Big(\lambda-\frac{1}{t}\Big)^{\alpha+1}\\ &= a\lambda^{\alpha} \quad \text{as} \quad t\to\infty\end{aligned}$$

and

$$M\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}=\frac{1}{2}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1},$$

it is easy to see that condition (3.20) reduces to

$$a\lambda^{\alpha}>\frac{1}{2}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}. \tag{3.80}$$

By Corollary 3.1.4, Eq.(3.79) is oscillatory if (3.80) holds.

Example 3.3.3 Consider the nonlinear neutral differential equation

$$\Big[\frac{1}{1+u^4(t)}\big|\big(u(t)+(1-\frac{1}{t})u(t-|\sin t|)\big)'\big|^{\alpha-1}\big(u(t)+(1-\frac{1}{t})u(t-|\sin t|)\big)'\Big]' + \frac{a}{t}|u(\frac{t}{\lambda})|^{\alpha-1}u(\frac{t}{\lambda})=0, \quad t\ge 1, \tag{3.81}$$

where $\lambda>1,\ a\ge 0,\ \alpha>0$. We have

$$r(t)=1, \psi(u)=\frac{1}{1+u^4}, p(t)=1-\frac{1}{t}, q(t)=\frac{a}{t}, \tau(t)=t-|\sin t|, \sigma(t)=\frac{t}{\lambda}.$$

Since

$$\begin{aligned}\beta\hat{p}(t)\frac{r^{1/\alpha}(\sigma(t))R^{\alpha+1}(\sigma(t))}{\sigma'(t)} &= \frac{a\big(1-(1-\frac{1}{t})\big)^{\alpha}}{t}\frac{(\frac{t}{\lambda}-1)^{\alpha+1}}{1/\lambda}\\ &= a\lambda\Big(\frac{1}{\lambda}+\frac{1}{t}\Big)^{\alpha+1}\\ &= \frac{a}{\lambda^{\alpha}} \quad \text{as} \quad t\to\infty\end{aligned}$$

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and

$$M\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1} = \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1},$$

it is easy to see that condition (3.20) reduces to

$$\frac{a}{\lambda^{\alpha}} > \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}, \tag{3.82}$$

for $n \geq 1$. By Corollary 3.1.4, Eq.(3.81) is oscillatory if (3.82) holds.

Example 3.3.4 Consider the nonlinear neutral differential equation

$$\Big[\frac{t^{-\kappa}}{1+u^2(t)}\big|\big(u(t)+(\frac{1}{2}-e^{-t})u(\vartheta t)\big)'\big|^{\alpha-1}\big(u(t)+(\frac{1}{2}-e^{-t})u(\vartheta t)\big)'\Big]' + t^{\nu-2}\Big(\frac{\nu(2-\cos t)}{t}+\sin t\Big)|u(\mu t)|^{\alpha-1}u(\mu t) = 0, \quad t \geq 1, \tag{3.83}$$

where $0 < \vartheta, \mu \leq 1$, ν is arbitrary positive constant and κ, α are constant such that $\kappa + \alpha > 2$ and $\alpha < 1$. We have $r(t) = t^{-\kappa}$, $\psi(u) = \dfrac{1}{1+u^2(t)} \leq 1$, $q(t) = t^{\nu-2}\Big(\dfrac{\nu(2-\cos t)}{t}+\sin t\Big)$, $p(t) = \dfrac{1}{2} - e^{-t} \leq \dfrac{1}{2}$, $\tau(t) = \vartheta t$, $\sigma(t) = \mu t$. Here, we choose $\rho(t) = t^2$ and $H(t,s) = (t-s)^2$ for $t \geq s \geq t_0$. Since

$$\begin{aligned}\rho(t)q(t) &= t^2\Big[t^{\nu-2}\Big(\frac{\nu(2-\cos t)}{t}+\sin t\Big)\Big]\\ &= t^{\nu}\Big(\frac{\nu(2-\cos t)}{t}+\sin t\Big)\\ &= \frac{d}{dt}\big(t^{\nu}(2-\cos t)\big),\end{aligned}$$

we get

$$\begin{aligned}\int_{t_0}^{t}\rho(s)q(s)ds &= \int_{t_0}^{t} d\big(s^{\nu}(2-\cos s)\big)\\ &= t^{\nu}(2-\cos t) - t_0^{\nu}(2-\cos t_0)\\ &\geq t^{\nu} - k_0,\end{aligned}$$

where $k_0 = t_0^{\nu}(2-\cos t_0)$, and

$$\begin{aligned}G(t,s) &= \frac{\rho(s)r(\sigma(s))}{(\sigma'(s))^{\alpha}\big(\sqrt{H(t,s)}\big)^{\alpha-1}}\Big(h(t,s)+\frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\Big)^{\alpha+1}\\ &= \frac{s^2(\mu s)^{-\kappa}(t-s)^{1-\alpha}}{\mu^{\alpha}}\Big(2+\frac{2s}{s^2}(t-s)\Big)^{\alpha+1}\\ &= \frac{(2t)^{\alpha+1}(t-s)^{1-\alpha}}{\mu^{\alpha+\kappa}s^{\alpha+\kappa-1}}\end{aligned}$$

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Therefore, we have

$$
\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\rho(s)q(s)ds \\
&\quad= \frac{1}{(t-t_0)^2}\int_{t_0}^{t}(t-s)^2\rho(s)q(s)ds \\
&\quad= \frac{1}{(t-t_0)^2}\int_{t_0}^{t}(t-s)^2 d\Big(\int_{t_0}^{s}\rho(r)q(r)dr\Big) \\
&\quad\geq \frac{1}{(t-t_0)^2}\Big[(t-s)^2\int_{t_0}^{s}\rho(r)q(r)dr\Big|_{s=t_0}^{t} + \int_{t_0}^{t}2(t-s)(s^{\nu}-k_0)ds\Big] \\
&\quad= \frac{2}{(t-t_0)^2}\int_{t_0}^{t}(ts^{\nu}-tk_0-s^{\nu+1}+sk_0)ds \\
&\quad= \frac{2}{(t-t_0)^2}\Big[\frac{ts^{\nu+1}}{\nu+1}-tk_0s-\frac{s^{\nu+2}}{\nu+2}+\frac{s^2k_0}{2}\Big]_{s=t_0}^{t} \\
&\quad= \frac{2}{(t-t_0)^2}\Big[t^2\Big(\frac{t^{\nu}}{(\nu+1)(\nu+2)}-\frac{k_0}{2}\Big)+t\Big(k_0t_0-\frac{t_0^{\nu+1}}{\nu+1}\Big)+\Big(\frac{t_0^{\nu+2}}{\nu+2}-\frac{t_0^2k_0}{2}\Big)\Big] \\
&\quad= \frac{1}{(1-\frac{t_0}{t})^2}\Big[\frac{2t^{\nu}}{(\nu+1)(\nu+2)}-k_0+\frac{2}{t}\Big(k_0t_0-\frac{t_0^{\nu+1}}{\nu+1}\Big)+\frac{2}{t^2}\Big(\frac{t_0^{\nu+2}}{\nu+2}-\frac{t_0^2k_0}{2}\Big)\Big] \\
&\quad= \infty \qquad \text{as} \quad t\to\infty.
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^{t} M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\Big(\frac{\alpha}{\gamma p^*}\Big)^{\alpha} G(t,s)ds \\
&\quad= \frac{1}{(t-t_0)^2}\int_{t_0}^{t}2^{\alpha}\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\frac{(2t)^{\alpha+1}(t-s)^{1-\alpha}}{\mu^{\alpha+\kappa}s^{\alpha+\kappa-1}}ds \\
&\quad\leq \frac{k_1\ t^{\alpha+1}(t-t_0)^{1-\alpha}}{(t-t_0)^2}\int_{t_0}^{t}\frac{1}{s^{\alpha+\kappa-1}}ds \\
&\quad= \frac{k_1\ t^{\alpha+1}}{(t-t_0)^{\alpha+1}}\Big[\frac{s^{2-\alpha-\kappa}}{2-\alpha-\kappa}\Big]_{s=t_0}^{t} \\
&\quad= \frac{k_1}{2-\alpha-\kappa}\Big(1-\frac{t_0}{t}\Big)^{-(\alpha+1)}\Big(t^{-(\alpha+\kappa-2)}-t_0^{-(\alpha+\kappa-2)}\Big) \\
&\quad= \frac{k_1\ t_0^{-(\alpha+\kappa-2)}}{\alpha+\kappa-2} \\
&\quad< \infty \qquad \text{as} \quad t\to\infty,
\end{aligned}
$$

where $k_1 = \Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}\frac{2^{2\alpha+1}}{\mu^{\alpha+\kappa}}$. Consequently, by Corollary 3.2.1, Eq.(3.83) is oscillatory.

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Example 3.3.5 Consider the nonlinear neutral differential equation

$$\Big[\frac{t^{-\kappa}}{2+u^2(t)}\big|\big(u(t)+(1-\frac{1}{t})u(\vartheta t)\big)'\big|^{\alpha-1}\big(u(t)+(1-\frac{1}{t})u(\vartheta t)\big)'\Big]'$$
$$+\,t^{\nu+\alpha-2}|u(\mu t)|^{\alpha-1}u(\mu t)=0, \quad t\geq 1, \tag{3.84}$$

where $0<\vartheta,\mu\leq 1$, ν is arbitrary positive constant and κ, α are constant such that $\kappa+\alpha>2$ and $\alpha<1$. We have $r(t)=t^{-\kappa}$, $\psi(u)=\dfrac{1}{2+u^2(t)}\leq\dfrac{1}{2}$, $q(t)=t^{\nu+\alpha-2}$, $p(t)=1-\dfrac{1}{t}$, $\tau(t)=\vartheta t$, $\sigma(t)=\mu t$. Here, we choose $\rho(t)=t^2$ and $H(t,s)=(t-s)^2$ for $t\geq s\geq t_0$. Hence

$$\begin{aligned}
&\frac{1}{H(t,t_0}\int_{t_0}^{t}\beta H(t,s)\rho(s)Q(s)ds\\
&=\frac{1}{(t-t_0)^2}\int_{t_0}^{t}(t-s)^2s^2s^{\nu+\alpha-2}\big(1-(1-\frac{1}{s})\big)^{\alpha}ds\\
&=\frac{1}{(t-t_0)^2}\int_{t_0}^{t}(t-s)^2s^{\nu}ds\\
&=\frac{1}{(t-t_0)^2}\int_{t_0}^{t}\big(t^2s^{\nu}-2ts^{\nu+1}+s^{\nu+2}\big)ds\\
&=\frac{1}{(t-t_0)^2}\Big[\frac{t^2s^{\nu+1}}{\nu+1}-\frac{2ts^{\nu+2}}{\nu+2}+\frac{s^{\nu+3}}{\nu+3}\Big]_{s=t_0}^{t}\\
&=\frac{1}{(t-t_0)^2}\Big[\frac{2t^{\nu+3}}{(\nu+1)(\nu+2)(\nu+3)}-\frac{t^2t_0^{\nu+1}}{\nu+1}+\frac{2tt_0^{\nu+2}}{\nu+2}-\frac{t_0^{\nu+3}}{\nu+3}\Big]\\
&=\frac{1}{(1-\frac{t_0}{t})^2}\Big[\frac{2t^{\nu+1}}{(\nu+1)(\nu+2)(\nu+3)}-\frac{t_0^{\nu+1}}{\nu+1}+\frac{2t_0^{\nu+2}}{t(\nu+2)}-\frac{t_0^{\nu+3}}{t^2(\nu+3)}\Big]\\
&=\infty \qquad \text{as} \quad t\to\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^{t}M\Big(\frac{1}{\alpha+1}\Big)^{\alpha+1}G(t,s)ds\\
&\qquad=\frac{k_2t^{\alpha+1}}{(t-t_0)^2}\int_{t_0}^{t}\frac{(t-s)^{1-\alpha}}{s^{\alpha+\kappa-1}}ds\\
&\qquad\leq\frac{k_2\ t^{\alpha+1}(t-t_0)^{1-\alpha}}{(t-t_0)^2}\int_{t_0}^{t}\frac{1}{s^{\alpha+\kappa-1}}ds\\
&\qquad=\frac{k_2}{2-\alpha-\kappa}\Big(1-\frac{t_0}{t}\Big)^{-(\alpha+1)}\Big(t^{-(\alpha+\kappa-2)}-t_0^{-(\alpha+\kappa-2)}\Big)\\
&\qquad=\frac{k_2\ t_0^{-(\alpha+\kappa-2)}}{\alpha+\kappa-2}<\infty \qquad \text{as} \quad t\to\infty,
\end{aligned}$$

where $k_2=\dfrac{1}{2\mu^{\alpha+\kappa}}\Big(\dfrac{2}{\alpha+1}\Big)^{\alpha+1}$. Consequently, by Corollary 3.2.7, Eq.(3.84) is oscillatory.

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