CHAPTER 3

MAIN RESULTS

In this chapter, we present some new condition for robust stability of discrete-time linear parameter dependent cellular neural networks with time delay. Based on Lyapunov stability theory and linear matrix inequality (LMI) techniques, stability conditions for time delayed neural networks are derived by using the S-procedure. There are 2 parts in this chapter. In section 3.1, we derive new sufficient conditions for robust stability of system without matrices uncertainties. There results are extended to establish the robust stability criterion for discrete-time cellular neural networks with time delay systems with polytopic type uncertainties in section 3.2. Some numerical simulations are given to illustrate the effectiveness of our theoretical results.

Consider the following state equation

$$\begin{aligned} u(k+1) = & -[A(\xi)+\Delta A]u(k)+[W(\xi)+\Delta W]g(u(k))+[W_1(\xi) \\ & +\Delta W_1]g(u(k-\tau))+b \end{aligned} \tag{3.1}$$

where $u(k) = [u_1(k), \ldots, u_n(k)]^T \in \mathbb{R}^n$ is the neuron state vector, $g(u(\cdot)) = [g_1(u_1(\cdot)), \ldots, g_n(u_n(\cdot))]^T$ is the activation function, $b = [b_1, \ldots, b_n]^T$ is constant input vector, $A(\xi)$ is positive diagonal matrix, $W(\xi)$ and $W_1(\xi)$ are the interconnection matrices of polytopic type where

$$\begin{bmatrix} A(\xi) & W(\xi) & W_1(\xi) \end{bmatrix} \in \Omega,$$

$$\Omega = \left\{ \begin{bmatrix} A(\xi)\ W(\xi)\ W_1(\xi) \end{bmatrix} = \sum_{i=1}^{N} \xi_i \begin{bmatrix} A_i & W_i & W_{1i} \end{bmatrix},\ \ \sum_{i=1}^{N} \xi_i = 1,\ \xi_i \geq 0 \right\}, \tag{3.2}$$

where A_i, W_i and W_{1i} are known constant matrices and ΔA, ΔW and ΔW_1 are uncertainty matrices which are of the form

$$\Delta A = H_0 F_0 E_0,\ \Delta W = HFE \ \text{ and } \Delta W_1 = H_1 F_1 E_1 \tag{3.3}$$

where $H_0,\ H,\ H_1,\ E_0,\ E$ and E_1 are known constant matrices $F_0,\ F$ and F_1 are unknown matrices which satisfy

$$F_0^T F_0 \leq I,\ F^T F \leq I \text{ and } F_1^T F_1 \leq I \tag{3.4}$$

where I is the identity matrix of appropriate dimension. Assume that the activation functions in (3.1) satisfies the following condition

$$0 \leq \frac{g_j(x) - g_j(y)}{x - y} \leq l_j \ \ \forall x,\ y \in \mathbb{R},\ x \neq y,\ j = 1, 2, \ldots, n$$

where $l_j > 0$, for $j = 1, 2, \ldots, n$ are constants. We shift the equilibrium u^* to the zero solution by the transform $x(k) = u(k) - u^*$. Then we obtain the new system

$$x(k+1) = -[A(\xi) + \Delta A]x(k) + [W(\xi) + \Delta W]f(x(k)) + [W_1(\xi) + \Delta W_1]f(x(k-\tau)) \tag{3.5}$$

where $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]$ is the state vector of the transformed system, with $f_j(x_j(k)) = g_j(x_j(k) + u_j^*) - g_j(u_j^*),\ j = 1, 2, \ldots, n$ and f_j satisfies the condition

$$f_j(x_j)(f_j(x_j) - l_j x_j) \leq 0,\ j = 1, 2, \ldots, n. \tag{3.6}$$

Inequality (3.6) is equivalent to the following condition

$$|f_j(x_j)| \leq l_j\, |x_j|\,,\ j = 1, 2, \ldots, n$$

and

$$f_j^2(x_j) \leq l_j x_j f_j(x_j),\ j = 1, 2, \ldots, n, \tag{3.7}$$

$A(\xi)$ is positive diagonal matrix, $W(\xi)$ and $W_1(\xi)$ are the interconnection matrices are the interconnection matrices of polytopic type where

$$\begin{bmatrix} A(\xi) & W(\xi) & W_1(\xi) \end{bmatrix} \in \Omega,$$

$$\Omega = \left\{ \begin{bmatrix} A(\xi)\ W(\xi)\ W_1(\xi) \end{bmatrix} = \sum_{i=1}^{N} \xi_i \begin{bmatrix} A_i & W_i & W_{1i} \end{bmatrix},\ \ \sum_{i=1}^{N} \xi_i = 1,\ \xi_i \geq 0 \right\},$$

where $A_i,\ W_i$ and W_{1i} are known constant matrices and $\Delta A,\ \Delta W,\ \Delta W_1$ are uncertainty matrices which are of the form

$$\Delta A = H_0 F_0 E_0\ ,\ \Delta W = HFE \ \text{ and } \Delta W_1 = H_1 F_1 E_1$$

where $H_0,\ H,\ H_1,\ E_0,\ E$ and E_1 are known constant matrices $F_0,\ F$ and F_1 are unknown matrices which satisfy

$$F_0^T F_0 \leq I,\ F^T F \leq I \ \text{ and } \ F_1^T F_1 \leq I$$

where I is the identity matrix of appropriate dimension.

3.1 Robust Stability of Discrete-Time Cellular Neural Networks with Time Delay Systems

In this we deal with the problem for robust stability of the zero solution of discrete-time linear parameter dependent cellular neural networks with time delay without polytopic type uncertainties. Let $A = A(\xi),\ W = W(\xi)$ and $W_1 = W_1(\xi)$ be constant matrices

Theorem 3.1.1. *The zero solution of system* (3.5) *is robustly stable if there exist* $P = P^T > 0,\ G = G^T \geq 0,\ Q = Q^T > 0$ *and* $L = \ diag\{l_1, l_2, \ldots, l_n\} > 0$ *with* $\tau \geq 0$ *such that the following LMI holds*

$$M = \begin{bmatrix} (1,1) & 0 & 0 \\ 0 & (2,2) & 0 \\ 0 & 0 & (3,3) \end{bmatrix} < 0 \qquad (3.8)$$

where

$$\begin{aligned}(1,1) = {} & A^T PA - P + \epsilon A^T PWW^T PA + \epsilon A^T PH_0E_0E_0^T H_0^T PA \\ & + \epsilon A^T PHEE^T H^T PA + \epsilon E_0^T H_0^T PWW^T PH_0E_0 \\ & + \epsilon E_0^T H_0^T PHEE^T H^T PH_0E_0 + \epsilon A^T PH_1E_1E_1^T H_1^T PA \\ & + \epsilon A^T PW_1W_1^T PA + \epsilon E_0^T H_0^T PW_1W_1^T PH_0E_0 + \tau G + Q\end{aligned}$$

$$+\epsilon LW^TPW_1W_1^TPWL+\epsilon E_0^TH_0^TPH_1E_1E_1^TH_1^TPH_0E_0$$
$$+\epsilon LW^TPHEE^TH^TPWL+\epsilon LW^TPH_1E_1E_1^TH_1^TPWL$$
$$+E_0^TH_0^TPH_0E_0+\epsilon LE^TH^TPW_1W_1^TPHEL+LW^TPWL$$
$$+LE^TH^TPHEL+\epsilon^{-1}I+6\epsilon^{-1}L^2,$$
$$(2,2)=\epsilon LE_1^TH_1^TPHEE^TH^TPH_1E_1L+\epsilon LW_1^TH_1E_1E_1^TH_1^TPW_1L$$
$$+LE_1^TH_1^TPH_1E_1L+LW_1^TPW_1L+8\epsilon^{-1}L^2-Q,$$
$$(3,3)=-\tau G.$$

Proof. Consider the Lyapunov function candidate

$$V(x)=x^T(k)Px(k)+\sum_{i=k-\tau+1}^{k}(\tau-k+i)x^T(i)Gx(i)+\sum_{i=k-\tau+1}^{k}x^T(i)Qx(i).$$

The Lyapunov difference along any trajectory of solution of (3.5) is given by

$$\begin{aligned}\Delta V(x(k))&=V(x(k+1))-V(x(k))\\
&=\Big[-(A+\Delta A)x(k)+(W+\Delta W)f(x(k))+(W_1+\Delta W_1)f(x(t-\tau))\Big]^T\\
&\quad\times P\Big[-(A+\Delta A)x(k)+(W+\Delta W)f(x(k))+(W_1+\Delta W_1)f(x(t-\tau))\Big]\\
&\quad-x^T(k)Px(k)+\tau x^T(k)Gx(k)-\sum_{i=k-\tau+1}^{k}x^T(i)Gx(i)\\
&\quad+x^T(k)Qx(k)-x^T(k-\tau)Qx(k-\tau).\end{aligned}$$

By condition (3.3), we get

$$\begin{aligned}\Delta V(x(k))=&\Big[-(A+H_0F_0E_0)x(k)+(W+HFE)f(x(k))+(W_1+H_1F_1E_1)\\
&\times f(x(t-\tau))\Big]^TP\Big[-(A+H_0F_0E_0)x(k)+(W+HFE)f(x(k))\\
&+(W_1+H_1F_1E_1)f(x(t-\tau))\Big]-x^T(k)Px(k)+\tau x^T(k)Gx(k)\\
&-\sum_{i=k-\tau+1}^{k}x^T(i)Gx(i)+x^T(k)Qx(k)-x^T(k-\tau)Qx(k-\tau)\end{aligned}$$

$$
\begin{aligned}
= \ & x^T(k)(A+H_0F_0E_0)^TP(A+H_0F_0E_0)x(k) - x^T(k)(A+H_0F_0E_0)^T \\
& \times P(W+HFE)f(x(k)) - x^T(k)(A+H_0F_0E_0)^TP(W_1+H_1F_1E_1) \\
& \times f(x(k-\tau)) - f^T(x(k))(W+HFE)^TP(A+H_0F_0E_0)x(k) \\
& + f^T(x(k))(W+HFE)^TP(W+HFE)f(x(k)) + f^T(x(k)) \\
& \times (W+HFE)^TP(W_1+H_1F_1E_1)f(x(k-\tau)) - f^T(x(k-\tau)) \\
& \times (W_1+H_1F_1E_1)^TP(A+H_0F_0E_0)x(k) + f^T(x(k-\tau))(W_1 \\
& + H_1F_1E_1)^TP(W+HFE)f(x(k)) + f^T(x(k-\tau))(W_1+H_1F_1E_1)^T \\
& \times P(W_1+H_1F_1E_1)f(x(k-\tau)) - x^T(k)Px(k) + \tau x^T(k)Gx(k) \\
& - \sum_{i=k-\tau+1}^{k} x^T(i)Gx(i) + x^T(k)Qx(k) - x^T(k-\tau)Qx(k-\tau).
\end{aligned}
$$

By Lemma 2.3.5, condition (3.4) and condition (3.7), we have

$$
\begin{aligned}
& x^T(k)A^TP(H_0E_0F_0) + x^T(k)(H_0E_0F_0)^TPAx(k) \\
& \quad \leq \epsilon x^T(k)A^TP(H_0F_0E_0)(H_0F_0E_0)^TPAx(k) + \epsilon^{-1}x^T(k)x(k) \\
& \quad \leq \epsilon x^T(k)A^TPH_0E_0E_0^TH_0^TPAx(k) + \epsilon^{-1}x^T(k)x(k), && (3.9) \\
& -x^T(k)A^TPWf(x(k)) - f^T(x(k))W^TPAx(k) \\
& \quad \leq \epsilon x^T(k)A^TPWW^TPAx(k) + \epsilon^{-1}f^T(x(k))f(x(k)) \\
& \quad \leq \epsilon x^T(k)A^TPWW^TPAx(k) + \epsilon^{-1}x^T(k)L^2x(k), && (3.10) \\
& -x^T(k)A^TP(HFE)f(x(k)) - f^T(x(k))(HFE)^TPAx(k) \\
& \quad \leq \epsilon x^T(k)A^TP(HFE)(HFE)^TPAx(k) + \epsilon^{-1}f^T(x(k))f(x(k)) \\
& \quad \leq \epsilon x^T(k)A^TPHEE^TH^TPAx(k) + \epsilon^{-1}f^T(x(k))f(x(k)) \\
& \quad \leq \epsilon x^T(k)A^TPHEE^TH^TPAx(k) + \epsilon^{-1}x^T(k)L^2x(k), && (3.11) \\
& -x^T(k)(H_0F_0E_0)^TPWf(x(k)) - f^T(x(k))W^TP(H_0F_0E_0)x(k) \\
& \quad \leq \epsilon x^T(k)(H_0F_0E_0)^TPWW^TP(H_0F_0E_0)x(k) + \epsilon^{-1}f^T(x(k))f(x(k)) \\
& \quad \leq \epsilon x^T(k)E_0^TH_0^TPWW^TPH_0E_0x(k) + \epsilon^{-1}f^T(x(k))f(x(k)) \\
& \quad \leq \epsilon x^T(k)E_0^TH_0^TPWW^TPH_0E_0x(k) + \epsilon^{-1}x^T(k)L^2x(k), && (3.12)
\end{aligned}
$$

$$
\begin{aligned}
&-x^T(k)(H_0F_0E_0)^TP(HFE)f(x(k)) - f^T(x(k))(HFE)^TP(H_0F_0E_0)x(k) \\
&\quad \le \epsilon x^T(k)(H_0F_0E_0)^TP(HFE)(HFE)^TP(H_0F_0E_0)x(k) + \epsilon^{-1}f^T(x(k))f(x(k)) \\
&\quad \le \epsilon x^T(k)E_0^TH_0^TPHEE^TH^TPH_0E_0)x(k) + \epsilon^{-1}f^T(x(k))f(x(k)) \\
&\quad \le \epsilon x^T(k)E_0^TH_0^TPHEE^TH^TPH_0E_0)x(k) + \epsilon^{-1}x^T(k)L^2x(k), \qquad (3.13) \\
&-x^T(k)A^TPW_1f(x(k-\tau)) - f^T(x(k-\tau))W_1^TPAx(k) \\
&\quad \le \epsilon x^T(k)A^TPW_1W_1^TPAx(k) + \epsilon^{-1}f^T(x(k-\tau))f(x(k-\tau)) \\
&\quad \le \epsilon x^T(k)A^TPW_1W_1^TPAx(k) + \epsilon^{-1}x^T(k-\tau)L^2x(k-\tau), \qquad (3.14) \\
&-x^T(k)A^TP(H_1F_1E_1)f(x(k-\tau)) - f^T(x(k-\tau))(H_1F_1E_1)^TPAx(k) \\
&\quad \le \epsilon x^T(k)A^TP(H_1F_1E_1)(H_1F_1E_1)^TPAx(k) + \epsilon^{-1}f^T(x(k-\tau))f(x(k-\tau)) \\
&\quad \le \epsilon x^T(k)A^TPH_1E_1E_1^TH_1^TPAx(k) + \epsilon^{-1}f^T(x(k-\tau))f(x(k-\tau)) \\
&\quad \le \epsilon x^T(k)A^TPH_1E_1E_1^TH_1^TPAx(k) + \epsilon^{-1}x^T(k-\tau)L^2x(k-\tau), \qquad (3.15) \\
&-x^T(k)(H_0F_0E_0)^TPW_1f(x(k-\tau)) - f^T(x(k-\tau))W_1^TP(H_0F_0E_0)x(k) \\
&\quad \le \epsilon x^T(k)(H_0F_0E_0)^TPW_1W_1^TP(H_0F_0E_0)x(k) + \epsilon^{-1}f^T(x(k-\tau))f(x(k-\tau)) \\
&\quad \le \epsilon x^T(k)E_0^TH_0^TPW_1W_1^TPH_0E_0x(k) + \epsilon^{-1}f^T(x(k-\tau))f(x(k-\tau)) \\
&\quad \le \epsilon x^T(k)E_0^TH_0^TPW_1W_1^TPH_0E_0x(k) + \epsilon^{-1}x^T(k-\tau)L^2x(k-\tau), \qquad (3.16) \\
&-x^T(k)(H_0F_0E_0)^TP(H_1F_1E_1)f(x(k-\tau)) - f^T(x(k-\tau))(H_1F_1E_1)^T \\
&\qquad \times P(H_0F_0E_0)x(k) \\
&\quad \le \epsilon x^T(k)(H_0F_0E_0)^TP(H_1F_1E_1)(H_1F_1E_1)^TP(H_0F_0E_0)x(k) \\
&\qquad + \epsilon^{-1}f^T(x(k-\tau))f(x(k-\tau)) \\
&\quad \le \epsilon x^T(k)E_0^TH_0^TPH_1E_1E_1^TH_1^TPH_0E_0x(k) + \epsilon^{-1}f^T(x(k-\tau))f(x(k-\tau)) \\
&\quad \le \epsilon x^T(k)E_0^TH_0^TPH_1E_1E_1^TH_1^TPH_0E_0x(k) + \epsilon^{-1}x^T(k-\tau)L^2x(k-\tau), \qquad (3.17) \\
&f^T(x(k))W^TP(HFE)f(x(k)) + f^T(x(k))(HFE)^TPWf(x(k)) \\
&\quad \le \epsilon f^T(x(k))W^TP(HFE)(HFE)^TPWf(x(k)) + \epsilon^{-1}f^T(x(k))f(x(k)) \\
&\quad \le \epsilon f^T(x(k))W^TPHEE^TH^TPWf(x(k)) + \epsilon^{-1}f^T(x(k))f(x(k)) \\
&\quad \le \epsilon x^T(k)LW^TPHEE^TH^TPWLx(k) + \epsilon^{-1}x^T(k)L^2x(k), \qquad (3.18) \\
&f^T(x(k))W^TPW_1f(x(k-\tau)) + f^T(x(k-\tau))W_1^TPWf(x(k)) \\
&\quad \le \epsilon f^T(x(k))W^TPW_1W_1^TPWf(x(k)) + \epsilon^{-1}f^T(x(k-\tau))f(x(k-\tau)) \\
&\quad \le \epsilon x^T(k)LW^TPW_1W_1^TPWLx(k) + \epsilon^{-1}x^T(k-\tau)L^2x(k-\tau) \qquad (3.19)
\end{aligned}
$$

$$
\begin{aligned}
& f^T(x(k))W^TP(H_1F_1E_1)f(x(k-\tau)) - f^T(x(k-\tau))(H_1F_1E_1)^TPWf(x(k)) \\
& \quad \le \ \epsilon f^T(x(k))W^TP(H_1F_1E_1)(H_1F_1E_1)^TPWf(x(k)) \\
& \qquad + \epsilon^{-1}f^T(x(k-\tau))f(x(k-\tau)) \\
& \quad \le \ \epsilon f^T(x(k))W^TPH_1E_1E_1^TH_1^TPWf(x(k)) + \epsilon^{-1}f^T(x(k-\tau))f(x(k-\tau)) \\
& \quad \le \ \epsilon x^T(k)LW^TPH_1E_1E_1^TH_1^TPWLx(k)) + \epsilon^{-1}x^T(k-\tau)L^2x(k-\tau), \qquad (3.20)
\end{aligned}
$$

$$
\begin{aligned}
& f^T(x(k))(HFE)^TPW_1f(x(k-\tau)) + f^T(x(k-\tau))W_1^TP(HFE)f(x(k)) \\
& \quad \le \ \epsilon f^T(x(k))(HFE)^TPW_1W_1^TP(HFE)f(x(k)) + \epsilon^{-1}f^T(x(k-\tau))f(x(k-\tau)) \\
& \quad \le \ \epsilon f^T(x(k))E^TH^TPW_1W_1^TPHEf(x(k)) + \epsilon^{-1}f^T(x(k-\tau))f(x(k-\tau)) \\
& \quad \le \ \epsilon x^T(k)LE^TH^TPW_1W_1^TPHELx(k) + \epsilon^{-1}x^T(k-\tau)L^2x(k-\tau), \qquad (3.21)
\end{aligned}
$$

$$
\begin{aligned}
& f^T(x(k-\tau))(H_1F_1E_1)^TP(HFE)f(x(k)) + f^T(x(k))(HFE)^TP(H_1F_1E_1)f(x(k-\tau)) \\
& \quad \le \ \epsilon f^T(x(k-\tau))(H_1F_1E_1)^TP(HFE)(HFE)^TP(H_1F_1E_1)f(x(k-\tau)) \\
& \qquad + \epsilon^{-1}f^T(x(k))f(x(k)) \\
& \quad \le \ \epsilon f^T(x(k)E_1^T)H_1^TPHEE^TH^TPH_1E_1f(x(k)) + \epsilon^{-1}f^T(x(k))f(x(k)) \\
& \quad \le \ \epsilon x^T(k)LE_1^T)H_1^TPHEE^TH^TPH_1E_1Lx(k) + \epsilon^{-1}x^T(k)L^2x(k), \qquad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
& f^T(x(k-\tau))W_1^TP(H_1F_1E_1)f(x(k-\tau)) + f^T(x(k-\tau))(H_1F_1E_1)^TPW_1f(x(k-\tau)) \\
& \quad \le \ \epsilon f^T(x(k-\tau))W_1^TP(H_1F_1E_1)(H_1F_1E_1)^TPW_1f(x(k-\tau)) \\
& \qquad + \epsilon^{-1}f^T(x(k-\tau))f(x(k-\tau)) \\
& \quad \le \ \epsilon f^T(x(k-\tau))W_1^TPH_1E_1E_1^TH_1^TPW_1f(x(k-\tau)) \\
& \qquad + \epsilon^{-1}f^T(x(k-\tau))f(x(k-\tau)) \\
& \quad \le \ \epsilon x^T(k-\tau))LW_1^TPH_1E_1E_1^TH_1^TPW_1Lx(k-\tau) + \epsilon^{-1}x^T(k-\tau))L^2x(k-\tau), \\
& \qquad \qquad (3.23)
\end{aligned}
$$

$$
x^T(k)(H_0F_0E_0)^TP(H_0F_0E_0)x(k) \le \ x^T(k)E_0^TH_0^TPH_0E_0, \qquad (3.24)
$$

$$
\begin{aligned}
& f^T(x(k-\tau))(H_1F_1E_1)^TP(H_1F_1E_1)f(x(k-\tau)) \\
& \quad \le \ f^T(x(k-\tau))E_1^TH_1^TPH_1E_1f(x(k-\tau)) \\
& \quad \le \ x^T(k-\tau)LE_1^TH_1^TPH_1E_1Lx(k-\tau), \qquad (3.25)
\end{aligned}
$$

$$
f^T(x(k-\tau))W_1^TPW_1f(x(k-\tau)) \le \ x^T(k-\tau)LW_1^TPW_1Lx(k-\tau), \qquad (3.26)
$$

$$f^T(x(k))W^TPWf(x(k)) \leq \; x^T(k)LW^TPWLx(k). \tag{3.27}$$

By Using (3.9) - (3.27) and Lemma 2.3.5, we obtain

$$\begin{aligned}\Delta V(x(k)) \leq x^T(k)\Big[& A^TPA - P + \epsilon A^TPWW^TPA + \epsilon A^TPH_0E_0E_0^TH_0^TPA \\ & + \epsilon A^TPHEE^TH^TPA + \epsilon E_0^TH_0^TPWW^TPH_0E_0 + \tau G + Q \\ & + \epsilon A^TPH_1E_1E_1^TH_1^TPA + \epsilon LW_1^TH_1E_1E_1^TH_1^TPW_1L \\ & + \epsilon E_0^TH_0^TPHEE^TH^TPH_0E_0 + \epsilon LW^TPHEE^TH^TPWL \\ & + \epsilon A^TPW_1W_1^TPA + \epsilon E_0^TH_0^TPW_1W_1^TPH_0E_0 + E_0^TH_0^TPH_0E_0 \\ & + \epsilon E_0^TH_0^TPH_1E_1E_1^TH_1^TPH_0E_0 + \epsilon LW^TPW_1W_1^TPWL \\ & + \epsilon LW^TPH_1E_1E_1^TH_1^TPWL + \epsilon LE^TH^TPW_1W_1^TPHEL \\ & + LE^TH^TPHEL + LW^TPWL + \epsilon^{-1}I + 6\epsilon^{-1}L^2\Big]x(k) \\ & + x^T(k-\tau)\Big[\epsilon LE_1^TH_1^TPHEE^TH^TPH_1E_1L \\ & + LE_1^TH_1^TPH_1E_1L + LW_1^TPW_1L + 8\epsilon^{-1}L^2 - Q,\Big]x(k-\tau) \\ & - \left(\frac{1}{\tau}\sum_{i=k-\tau+1}^{k} x(i)\right)^T(\tau G)\left(\frac{1}{\tau}\sum_{i=k-\tau+1}^{k} x(i)\right).\end{aligned}$$

Let $y = \left[x^T(k) \;\; x^T(k-\tau) \;\; \left(\frac{1}{\tau}\sum_{i=k-\tau+1}^{k} x(i)\right)^T\right]^T$. We have

$$\Delta V(x) \leq y^TMy < 0.$$

Hence system (3.5) is asymptotically stable and by Definition 2.3.4, the system (3.5) is robustly stable. The proof is complete. □

Example 3.1.1. Consider the CNNs (3.5) without parameter matrices where

$$A = \begin{bmatrix} a(1) & 0 \\ 0 & a(2) \end{bmatrix}, \quad W = \begin{bmatrix} w(1) & 0.5 \\ 0.5 & w(2) \end{bmatrix}, \quad W_1 = \begin{bmatrix} w_1(1) & 0.8 \\ 0.8 & w_1(2) \end{bmatrix},$$

$$H_0 = \begin{bmatrix} -0.003 & 0.015 \\ -0.018 & 0.025 \end{bmatrix}, \quad H = \begin{bmatrix} -0.035 & 0.042 \\ 0.008 & -0.004 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0.002 & -0.036 \\ -0.006 & 0.007 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 0.9 & 0 \\ 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_0 = H_0, \quad E = H, \quad E_1 = H_1, \quad \epsilon = 0.2, \quad \tau = 1,$$

$$a(1),\ a(2) \in [0.09\ ,\ 0.15], \quad w(1),\ w(2),\ w_1(1),\ w_1(2) \in [-0.9\ ,-0.1]$$

$$f_1(x) = 0.4\tanh(x), \quad f_2(x) = 0.2\tanh(x).$$

By using the Matlab LMI toolbox, we can solved for matrices P, Q, G and L which satisfies the criterion of Theorem 3.1.1, thus the zero solution of system (3.5) is robust stability system (3.1) is robust stability. A solution of (3.5) is given by

$$P = \begin{bmatrix} 30.458 & 10.852 \\ 10.852 & 29.584 \end{bmatrix}, \quad G = \begin{bmatrix} 2.256 & 0.825 \\ 0.825 & 1.358 \end{bmatrix},$$

$$Q = \begin{bmatrix} 10.253 & 7.894 \\ 7.894 & 6.451 \end{bmatrix}, \quad L = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.007 \end{bmatrix}.$$

The simulation is illustrated in Fig. 3.1.

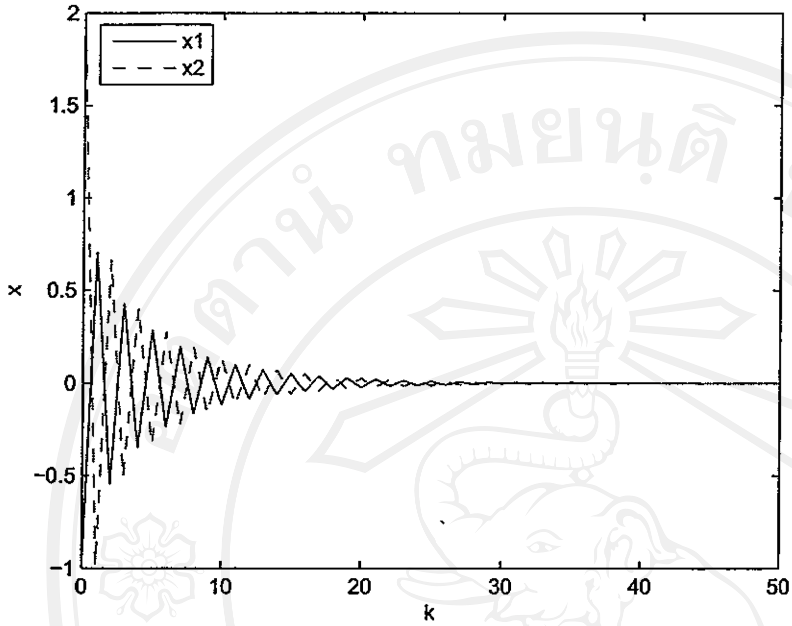

Figure 3.1: The solution trajectory of system (3.5) in Example 3.1.1.

Theorem 3.1.2. *The zero solution of system* (3.5) *is robustly stable if there exist* $P = P^T > 0$, $Q = Q^T > 0$, $T = diag\{t_1, t_2, \ldots, t_n\} \geq 0$, $S = diag\{s_1, s_2, \ldots, s_n\} \geq 0$ *and scalars* $e_0 > 0$, $e > 0$ *and* $e_1 > 0$ *such that the following LMI holds*

$$M = \begin{bmatrix} \prod_{11} & 0 & \prod_{13} & -A^TPW_1 & A^TPH_0 & -A^TPH & -A^TPH_1 \\ * & \prod_{22} & 0 & LS & 0 & 0 & 0 \\ * & * & \prod_{33} & W^TPW_1 & -W^TPH_0 & W^TPH & W^TPH_1 \\ * & * & * & \prod_{44} & -W_1^TPH_0 & W_1^TPH & W_1^TPH_1 \\ * & * & * & * & \prod_{55} & -H_0^TPH & -H_0^TPH_1 \\ * & * & * & * & * & \prod_{66} & H^TPH_1 \\ * & * & * & * & * & * & \prod_{77} \end{bmatrix} < 0 \tag{3.28}$$

where

$\prod_{11} = A^TPA - P + Q + e_0E_0^TE_0$,

$\prod_{13} = -A^TPW - LT$,

$\prod_{22} = -Q$,

$\prod_{33} = W^T PW - 2T + eE^T E,$

$\prod_{44} = W_1^T PW_1 - 2S + e_1 E_1^T E_1,$

$\prod_{55} = H_0^T PH_0 - e_0 I,$

$\prod_{66} = H^T PH - eI,$

$\prod_{77} = H_1^T PH_1 - e_1 I.$

Proof Consider the Lyapunov function candidate

$$V(x) = x^T(k)Px(k) + \sum_{j=k-\tau}^{k-1} x^T(j)Qx(j).$$

The Lyapunov difference along any trajectory of solution of (3.5) is given by

$$\begin{aligned}
\Delta V(x(k)) &= V(x(k+1)) - V(x(k)) \\
&= \Big[-(A+\Delta A)x(k) + (W+\Delta W)f(x(k)) + (W_1+\Delta W_1)f(x(t-\tau))\Big]^T \\
&\quad \times P\Big[-(A+\Delta A)x(k) + (W+\Delta W)f(x(k)) + (W_1+\Delta W_1)f(x(t-\tau))\Big] \\
&\quad - x^T(k)Px(k) + x^T(k)Qx(k) - x^T(k-\tau)Qx(k-\tau).
\end{aligned}$$

By condition (3.3), we get

$$\begin{aligned}
\Delta V(x(k)) = &\Big[-(A+H_0F_0E_0)x(k) + (W+HFE)f(x(k)) + (W_1+H_1F_1E_1) \\
&\times f(x(t-\tau))\Big]^T P\Big[-(A+H_0F_0E_0)x(k) + (W+HFE)f(x(k)) \\
&+ (W_1+H_1F_1E_1)f(x(t-\tau))\Big] - x^T(k)Px(k) + x^T(k)Qx(k) \\
&- x^T(k-\tau)Qx(k-\tau) \\
= &\; x^T(k)(A+H_0F_0E_0)^T P(A+H_0F_0E_0)x(k) - x^T(k)(A+H_0F_0E_0)^T \\
&\times P(W+HFE)f(x(k)) - x^T(k)(A+H_0F_0E_0)^T P(W_1+H_1F_1E_1) \\
&\times f(x(k-\tau)) - f^T(x(k))(W+HFE)^T P(A+H_0F_0E_0)x(k) \\
&+ f^T(x(k))(W+HFE)^T P(W+HFE)f(x(k)) + f^T(x(k)) \\
&\times (W+HFE)^T P(W_1+H_1F_1E_1)f(x(k-\tau)) - f^T(x(k-\tau)) \\
&\times (W_1+H_1F_1E_1)^T P(A+H_0F_0E_0)x(k) + f^T(x(k-\tau))(W_1 \\
&+ H_1F_1E_1)^T P(W+HFE)f(x(k)) + f^T(x(k-\tau))(W_1+H_1F_1E_1)^T \\
&\times P(W_1+H_1F_1E_1)f(x(k-\tau)) - x^T(k)Px(k) + x^T(k)Qx(k) \\
&- x^T(k-\tau)Qx(k-\tau).
\end{aligned}$$

By condition (3.6), we get

$$f_j(x_j(k))(f_j(x_j(k)) - l_j x_j(k)) \leq 0, \; j = 1, 2, \ldots, n \tag{3.29}$$

and

$$f_j(x_j(k-\tau))(f_j(x_j(k-\tau)) - l_j x_j(k-\tau)) \leq 0, \; j = 1, 2, \ldots, n. \tag{3.30}$$

Then, by applying S-procedure, system (3.5) is asymptotically stable if there exist $T = \; diag\{t_1, t_2, \ldots, t_n\} \geq 0$ and $S = \; diag\{s_1, s_2, \ldots, s_n\} \geq 0$ such that

$$\begin{aligned} \Delta V(x(k)) - 2\sum_{j=1}^{n} t_j f_j[x_j(k))(f_j(x_j(k)) - l_j x_j(k)] & \\ - 2\sum_{j=1}^{n} s_j f_j(x_j(k-\tau))[f_j(x_j(k) - l_j x_j(k-\tau)] & \leq y^T M_1 y \end{aligned} \tag{3.31}$$

where
$y = [x^T(k) \;\; x^T(k-\tau) \;\; f^T(x(k)) \;\; f^T(x(k-\tau)) \;\; (F_0 E_0 x(k))^T \;\; (FEf(x(k)))^T$
$(F_1 E_1 f(x(k-\tau)))^T]^T$,

$$M_1 = \begin{bmatrix} \prod_{11} & 0 & \prod_{13} & -A^T P W_1 & A^T P H_0 & -A^T P H & -A^T P H_1 \\ * & \prod_{22} & 0 & LS & 0 & 0 & 0 \\ * & * & \prod_{33} & W^T P W_1 & -W^T P H_0 & W^T P H & W^T P H_1 \\ * & * & * & \prod_{44} & -W_1^T P H_0 & W_1^T P H & W_1^T P H_1 \\ * & * & * & * & \prod_{55} & -H_0^T P H & -H_0^T P H_1 \\ * & * & * & * & * & \prod_{66} & H^T P H_1 \\ * & * & * & * & * & * & \prod_{77} \end{bmatrix}$$

where
$\prod_{11} = A^T P A - P + Q$,
$\prod_{13} = -A^T P W - LT$,
$\prod_{22} = -Q$,
$\prod_{33} = W^T P W - 2T$,
$\prod_{44} = W_1^T P W_1 - 2S$,
$\prod_{55} = H_0^T P H_0$,
$\prod_{66} = H^T P H$,

$\prod_{77} = H_1^T P H_1$.

From (3.4), we have for $e_0 > 0,\ e > 0,\ e_1 > 0$

$$
\begin{aligned}
&e_0[F_0E_0x(k)]^T[F_0E_0x(k)] \le e_0x^T(k)E_0^TE_0x(k),\\
&e[FEf(x(k))]^T[FEf(x(k))] \le ef^T(x(k))E^TEf(x(k)),\\
&e_1[F_1E_1f(x(k-\tau))]^T[F_1E_1f(x(k-\tau))] \le e_1f^T(x(k-\tau))E_1^TE_1f(x(k-\tau)).
\end{aligned} \tag{3.32}
$$

Substituting (3.32) into (3.31), we get LMI (3.28). Therefore,

$$\Delta V(x(k)) \le y^TMy < 0.$$

From the Lyapunov stability theorem, the system is asymptotically stable and by Definition 2.3.4 we get the system (3.5) is robustly stable. □

Example 3.1.2. Consider the CNNs (3.5) without parameter matrices where

$$A = \begin{bmatrix} a(1) & 0 \\ 0 & a(2) \end{bmatrix},\ W = \begin{bmatrix} 0.1 & -w(1) \\ w(2) & -0.2 \end{bmatrix},\ W_1 = \begin{bmatrix} 0.1 & -w_1(1) \\ w_1(2) & -0.2 \end{bmatrix},$$

$$H_0 = \begin{bmatrix} -0.003 & 0.015 \\ -0.018 & -0.025 \end{bmatrix},\ H = \begin{bmatrix} -0.035 & 0.042 \\ 0.008 & -0.004 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0.002 & -0.036 \\ -0.006 & 0.007 \end{bmatrix},\ L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ F_0 = \begin{bmatrix} 0.9 & 0 \\ 0 & -1 \end{bmatrix},$$

$$F = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix},\ F_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, E_0 = H_0,\ E = H,\ E_1 = H_1,$$

$f_1(x) = \tanh(x),\ f_2(x) = \tanh(x)$.

$a(1),\ a(2) \in [0.09\ ,\ 0.3],\ \ w(1),\ w(2) \in [0.09\ ,\ 0.5],\ \ w_1(1),\ w_1(2) \in [0.09\ ,\ 0.3]$

By using the Matlab LMI toolbox, we can solved for matrices P, Q, S and T and constant e_0, e and e_1 which satisfies the criterion of Theorem 3.1.2, thus the zero solution of system (3.5) is robust stability system (3.1) is robust stability.

A solution of (3.5) is given by

$$P = \begin{bmatrix} 233.4198 & -79.0100 \\ -79.0100 & 187.6041 \end{bmatrix}, \quad Q = \begin{bmatrix} 76.6728 & -32.5821 \\ -32.5821 & 73.3814 \end{bmatrix},$$
$$S = \begin{bmatrix} 54.8796 & 0 \\ 0 & 54.8796 \end{bmatrix}, \quad T = \begin{bmatrix} 80.4508 & 0 \\ 0 & 80.4508 \end{bmatrix},$$
$$e_0 = 121.0134, \; e = 122.1542, \; e_1 = 120.8576.$$

The simulation is illustrated in Fig. 3.2.

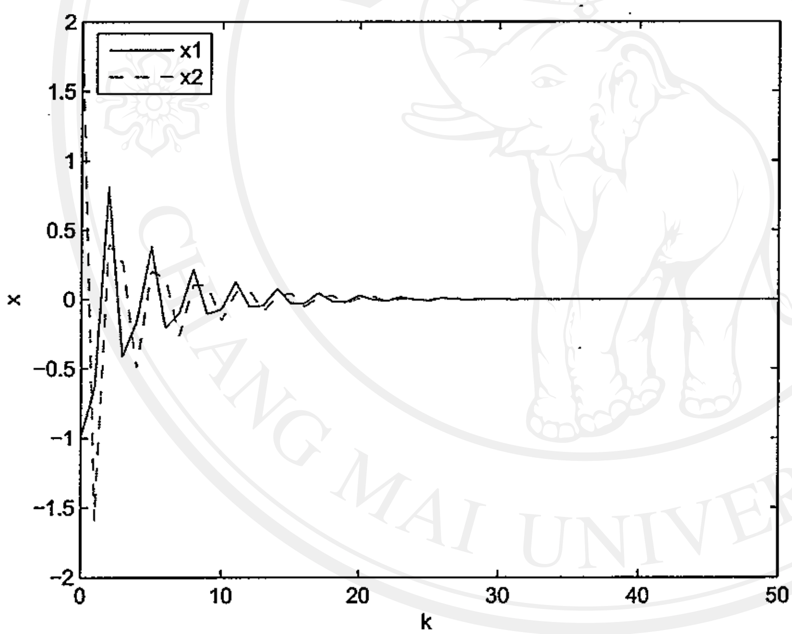

Figure 3.2: The solutions trajectory of system (3.5) in Example 3.1.2.

Theorem 3.1.3. *The zero solution of system* (3.5) *is robustly stable if there exist* $P = P^T > 0,\ Q = Q^T > 0,\ T = \ diag\{t_1, t_2, \ldots, t_n\} \geq 0,\ S = \ diag\{s_1, s_2, \ldots, s_n\}$ ≥ 0 *and scalars* $\epsilon > 0,\ e_0 > 0,\ e > 0$ *and* $e_1 > 0$ *such that the following LMI holds*

$$M = \begin{bmatrix} \prod_{11} & 0 & \prod_{13} & -A^TPW_1 & A^TPH_0 & -A^TPH & -A^TPH_1 \\ * & \prod_{22} & 0 & LS & 0 & 0 & 0 \\ * & * & \prod_{33} & W^TPW_1 & -W^TPH_0 & W^TPH & W^TPH_1 \\ * & * & * & \prod_{44} & -W_1^TPH_0 & W_1^TPH & W_1^TPH_1 \\ * & * & * & * & \prod_{55} & -H_0^TPH & -H_0^TPH_1 \\ * & * & * & * & * & \prod_{66} & H^TPH_1 \\ * & * & * & * & * & * & \prod_{77} \end{bmatrix} < 0 \tag{3.33}$$

where

$\prod_{11} = A^T(P^{-1} - \epsilon^{-1}H_0H_0^T)A + \epsilon E_0^TE_0 - P + Q + e_0E_0^TE_0,$

$\prod_{13} = -A^TPW - LT,$

$\prod_{22} = -Q,$

$\prod_{33} = W^T(P^{-1} - \epsilon^{-1}HH^T)W + \epsilon E^TE - 2T + eE^TE,$

$\prod_{44} = W_1^T(P^{-1} - \epsilon^{-1}H_1H_1^T)W_1^T + \epsilon E_1^TE_1 - 2S + e_1E_1^TE_1,$

$\prod_{55} = -e_0I\ ,\ \prod_{66} = -eI\ ,\ \prod_{77} = -e_1I.$

Proof Consider the Lyapunov function candidate

$$V(x) = x^T(k)Px(k) + \sum_{j=k-\tau}^{k-1} x^T(j)Qx(j).$$

The Lyapunov difference of system along any trajectory of solution (3.5) is given by

$$\begin{aligned}\Delta V(x(k)) =& V(x(k+1)) - V(x(k)) \\ =& \Big[- (A + \Delta A)x(k) + (W + \Delta W)f(x(k)) + (W_1 + \Delta W_1)f(x(t-\tau))\Big]^T \\ & \times P\Big[- (A + \Delta A)x(k) + (W + \Delta W)f(x(k)) + (W_1 + \Delta W_1)f(x(t-\tau))\Big] \\ & - x^T(k)Px(k) + x^T(k)Qx(k) - x^T(k-\tau)Qx(k-\tau).\end{aligned}$$

By condition (3.3), we get

$$\begin{aligned}\Delta V(x(k)) = &\Big[-(A+H_0F_0E_0)x(k)+(W+HFE)f(x(k))+(W_1+H_1F_1E_1)\\ &\times f(x(t-\tau))\Big]^T P\Big[-(A+H_0F_0E_0)x(k)+(W+HFE)f(x(k))\\ &+(W_1+H_1F_1E_1)f(x(t-\tau))\Big]-x^T(k)Px(k)+x^T(k)Qx(k)\\ &-x^T(k-\tau)Qx(k-\tau).\end{aligned}$$

By Lemma 2.3.3, condition (3.29) and (3.30) we get

$$\begin{aligned}\Delta V(x(k)) =&x^T(k)\Big(A^T(P^{-1}-\epsilon^{-1}H_0H_0^T)A+\epsilon E_0^TE_0\Big)x(k)-x^T(k)(A+H_0F_0E_0)^T\\ &\times P(W+HFE)f(x(k))-x^T(k)(A+H_0F_0E_0)^TP(W_1+H_1F_1E_1)\\ &\times f(x(k-\tau))-f^T(x(k))(W+HFE)^TP(A+H_0F_0E_0)x(k)\\ &+f^T(x(k))\Big(W^T(P^{-1}-\epsilon^{-1}HH^T)W+\epsilon E^TE\Big)f(x(k))+f^T(x(k))\\ &\times(W+HFE)^TP(W_1+H_1F_1E_1)f(x(k-\tau))-f^T(x(k-\tau))(W_1\\ &+H_1F_1E_1)^TP(A+H_0F_0E_0)x(k)+f^T(x(k-\tau))(W_1+H_1F_1E_1)^TP\\ &\times(W+HFE)f(x(k))+f^T(x(k-\tau))\Big(W_1^T(P^{-1}-\epsilon^{-1}H_1H_1^T)W_1^T\\ &+\epsilon E_1^TE_1\Big)f(x(k-\tau))-x^T(k)Px(k)+x^T(k)Qx(k)\\ &-x^T(k-\tau)Qx(k-\tau).\end{aligned}$$

Then, by applying S-procedure, system is asymptotically stable if there exist $T = diag\{t_1, t_2, \ldots, t_n\} \geq 0$ and $S = diag\{s_1, s_2, \ldots, s_n\} \geq 0$ such that

$$\begin{aligned}&\Delta V(x(k))-2\sum_{j=1}^{n}t_jf_j[x_j(k))(f_j(x_j(k))-l_jx_j(k)]\\ &\quad-2\sum_{j=1}^{n}s_jf_j(x_j(k-\tau))[f_j(x_j(k)-l_jx_j(k-\tau)]\leq y^TM_1y \qquad (3.34)\end{aligned}$$

where

$y=[x^T(k)\ \ x^T(k-\tau)\ \ f^T(x(k))\ \ f^T(x(k-\tau))\ \ (F_0E_0x(k))^T\ \ (FEf(x(k)))^T$
$(F_1E_1f(x(k-\tau)))^T]^T$,

$$M_1 = \begin{bmatrix} \prod_{11} & 0 & \prod_{13} & -A^TPW_1 & A^TPH_0 & -A^TPH & -A^TPH_1 \\ * & \prod_{22} & 0 & LS & 0 & 0 & 0 \\ * & * & \prod_{33} & W^TPW_1 & -W^TPH_0 & W^TPH & W^TPH_1 \\ * & * & * & \prod_{44} & -W_1^TPH_0 & W_1^TPH & W_1^TPH_1 \\ * & * & * & * & \prod_{55} & -H_0^TPH & -H_0^TPH_1 \\ * & * & * & * & * & \prod_{66} & H^TPH_1 \\ * & * & * & * & * & * & \prod_{77} \end{bmatrix}$$

where

$\prod_{11} = A^T(P^{-1} - \epsilon^{-1}H_0H_0^T)A + \epsilon E_0^TE_0 - P + Q$, $\prod_{13} = -A^TPW - LT$,

$\prod_{22} = -Q$,

$\prod_{33} = W^T(P^{-1} - \epsilon^{-1}HH^T)W + \epsilon E^TE - 2T$,

$\prod_{44} = W_1^T(P^{-1} - \epsilon^{-1}H_1H_1^T)W_1^T + \epsilon E_1^TE_1 - 2S$,

$\prod_{55} = 0$, $\prod_{66} = 0$, $\prod_{77} = 0$.

Substituting (3.32) into (3.34), we get LMI (3.33). Therefore,

$$\Delta V(x(k)) \leq y^TMy < 0.$$

From the Lyapunov stability theorem, the system is asymptotically stable and by Definition 2.3.4 we get the system (3.5) is robustly stable. □

Example 3.1.3. Consider the CNNs (3.5) without parameter matrices where.

$$A = \begin{bmatrix} a(1) & 0 \\ 0 & a(2) \end{bmatrix}, \quad W = \begin{bmatrix} 0.1 & -w(1) \\ w(2) & -0.2 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.1 & -w_1(1) \\ w_1(2) & -0.2 \end{bmatrix},$$

$$H_0 = \begin{bmatrix} 0.003 & -0.005 \\ -0.001 & -0.006 \end{bmatrix}, \quad H = \begin{bmatrix} -0.035 & 0.042 \\ 0.008 & -0.004 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0.002 & -0.036 \\ -0.006 & 0.007 \end{bmatrix}, \quad L = \begin{bmatrix} 1.1 & 0 \\ 0 & 1.1 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 0.9 & 0 \\ 0 & -1 \end{bmatrix},$$

$$F = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, \quad F_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, E_0 = H_0, \quad E = H, \quad E_1 = H_1,$$

$f_1(x) = \tanh(x)$, $f_2(x) = \tanh(x)$, $\epsilon = 2$,

$a(1)$, $a(2) \in [0.09 , 0.3]$, $w(1), w(2) \in [0.09 , 0.5]$, $w_1(1)$, $w_1(2) \in [0.09 , 0.3]$

By using the Matlab LMI toolbox, we can solved for matrices P, Q, S and T and constant e_0, e and e_1 which satisfies the criterion of Theorem 3.1.3, thus the zero solution of system (3.5) is robust stability system (3.1) is robust stability. A solution of (3.5) is given by

$$P = \begin{bmatrix} 8.1679 & -2.6588 \\ -2.6588 & 6.6406 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.0314 & -0.4968 \\ -0.4968 & 1.1640 \end{bmatrix},$$

$$S = \begin{bmatrix} 0.7505 & 0 \\ 0 & 0.7505 \end{bmatrix}, \quad T = \begin{bmatrix} 3.9311 & 0 \\ 0 & 3.9311 \end{bmatrix},$$

$$e_0 = 4.8915, \ e = 5.3373, \ e_1 = 4.9354.$$

The simulation is illustrated in Fig. 3.3.

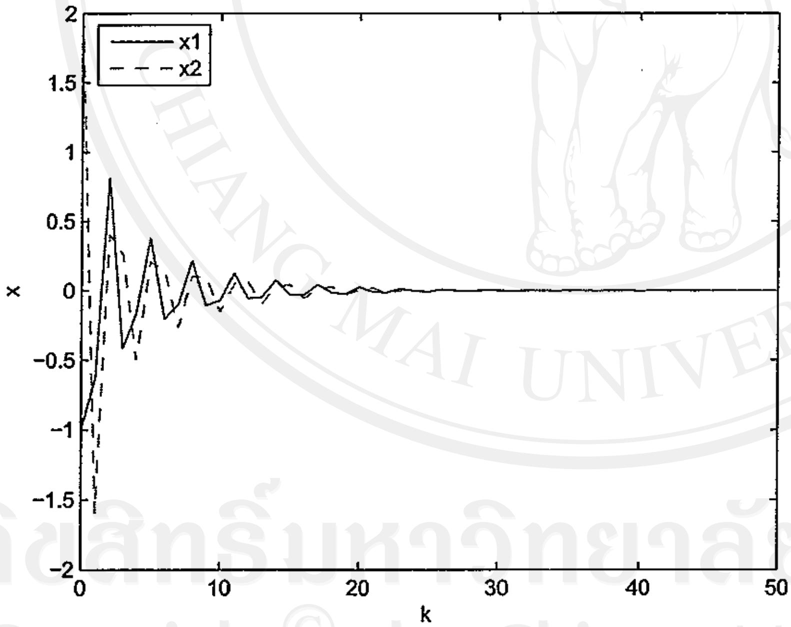

Figure 3.3: The solution trajectory of system (3.5) in Example 3.1.3.

3.2 Robust Stability of Discrete-Time Cellular Neural Networks with Time Delay Systems with Polytopic Type Uncertainties

In this section we extend result in 3.1 to obtain robust stability criteria for discrete-time cellular neural networks with time delay system with polytopic type uncertainties (3.2), Theorem 3.1.2 and 3.1.3 will be extended to provide an LMI-base robust stability conditions for system (3.5) with polytopic type uncertainties.

Theorem 3.2.1. *The zero solution of system* (3.5) *with polytopic type uncertainties* (3.2) *is robustly stable if there exist* $P_i = P_i^T > 0$, $Q_i = Q_i^T > 0$, $T_i = diag\{t_{1i}, t_{2i}, \dots, t_{ni}\} \geq 0$, $S_i = diag\{s_{1i}, s_{2i}, \dots, s_{ni}\} \geq 0$ *and scalars* $e_{0i} > 0$, $e_i > 0$ *and* $e_{1i} > 0$, $i = 1, 2, \dots, N$ *satisfy this condition*

$$(i)\ M_{i,i,i} + N_i < -I,\ i = 1, 2, \dots, N$$

$$(ii)\ M_{i,i,j} + M_{j,i,i} + M_{i,j,i} + 2N_i + N_j < \frac{1}{(N-1)^2} I,$$
$$i = 1, 2, \dots, N,\ i \neq j,\ j = 1, 2, \dots, N$$

$$(iii)\ M_{i,j,l} + M_{i,l,j} + M_{j,i,l} + M_{j,l,i} + M_{l,i,j} + M_{l,j,i}$$
$$+ 2N_i + 2N_j + 2N_l < \frac{6}{(N-1)^2} I,$$
$$i = 1, 2, \dots, N-2,\ j = i+1, 2, \dots, N-1,\ l = 1, 2, \dots, N, \tag{3.35}$$

where

$$M_{i,j,l} =$$
$$\begin{bmatrix} A_i^T P_j A_l & 0 & -A_i^T P_j W_l & -A_i^T P_j W_{1l} & A_i^T P_j H_0 & -A_i^T P_j H & -A_i^T P_j H_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -W_l^T P_j A_i & 0 & W_i^T P_j W_l & W_i^T P_j W_{1l} & -W_i^T P_j H_0 & W_i^T P_j H & W_i^T P_j H_1 \\ -W_{1l}^T P_j A_i & 0 & W_{1l}^T P_j W_i & W_{1i}^T P_j W_{1l} & -W_{1i}^T P_j H_0 & W_{1i}^T P_j H & W_{1i}^T P_j H_1 \\ H_0^T P_j A_i & 0 & -H_0^T P_j W_i & -H_0^T P_j W_{1i} & 0 & 0 & 0 \\ -H^T P_j A_i & 0 & H^T P_j W_i & H^T P_j W_{1i} & 0 & 0 & 0 \\ -H_1^T P_j A_i & 0 & H_1^T P_j W_i & H_1^T P_j W_{1i} & 0 & 0 & 0 \end{bmatrix},$$

$N_i =$

$$\begin{bmatrix} \prod_{11}(i) & 0 & -LT_i & 0 & 0 & 0 & 0 \\ 0 & -Q_i & 0 & LS_i & 0 & 0 & 0 \\ -LT_i & 0 & \prod_{33}(i) & 0 & 0 & 0 & 0 \\ 0 & LS_i & 0 & \prod_{44}(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \prod_{55}(i) & -H_0^T P_i H & -H_0^T P_i H_1 \\ 0 & 0 & 0 & 0 & -H^T P_i H_0 & \prod_{66}(i) & H^T P_i H_1 \\ 0 & 0 & 0 & 0 & -H_1^T P_i H_0 & H_1^T P_i H & \prod_{77}(i) \end{bmatrix},$$

and $\prod_{11}(i) = e_{0i}E_0^T E_0 - P_i + Q_i$, $\prod_{33}(i) = -2T_i + e_i E^T E$,

$\prod_{44}(i) = -2S_i + e_{1i}E_1^T E_1$, $\prod_{55}(i) = H_0^T P_i H_0 - e_{0i}I$, $\prod_{66}(i) = H^T P_i H - e_i I$,

$\prod_{77}(i) = H_1^T P_i H_1 - e_{1i}I$.

Proof Consider the Lyapunov function candidate

$$V(x(k),\ \xi) = \sum_{i=1}^{N} x^T(k)\xi_i P_i x(k) + \sum_{i=1}^{N} \sum_{l=k-\tau}^{k-1} x^T(l)\xi_i Q_i x(l).$$

The Lyapunov difference along any trajectory of solution of (3.5) is given by

$$\begin{aligned} \Delta V(x(k),\ \xi) = \Big[&- (A(\xi) + \Delta A)x(k) + (W(\xi) + \Delta W)f(x(k)) + (W_1(\xi) + \Delta W_1) \\ &\times f(x(t-\tau))\Big]^T P(\xi)\Big[- (A(\xi) + \Delta A)x(k) + (W(\xi) + \Delta W) \\ &\times f(x(k)) + (W_1(\xi) + \Delta W_1)f(x(t-\tau))\Big] - x^T(k)P(\xi)x(k) \\ &+ x^T(k)Q(\xi)x(k) - x^T(k-\tau)Q(\xi)x(k-\tau). \end{aligned}$$

By condition (3.3), we get

$$\begin{aligned} \Delta V(x(k),\ \xi) =& \Big[- (A(\xi) + H_0F_0E_0)x(k) + (W(\xi) + HFE)f(x(k)) \\ &+ (W_1(\xi) + H_1F_1E_1)f(x(t-\tau))\Big]^T P(\xi)\Big[- (A(\xi) + H_0F_0E_0)x(k) \\ &+ (W(\xi) + HFE)f(x(k)) + (W_1(\xi) + H_1F_1E_1)f(x(t-\tau))\Big] \\ &- x^T(k)P(\xi)x(k) + x^T(k))Q(\xi)x(k) - x^T(k-\tau)Q(\xi)x(k-\tau), \end{aligned}$$

and by condition (3.29) and (3.30) therefore applying S-procedure, system is asymptotically stable if there exist $T(\xi) = diag\{t_{1i}, t_{2i}, \ldots, t_{ni}\} \geq 0$ and $S(\xi) = diag\{s_{1i}, s_{2i}, \ldots, s_{ni}\} \geq 0$ such that

$$
\begin{aligned}
\Delta V(x(k),\ \xi) &- 2\sum_{i=1}^{N}\sum_{j=1}^{n}\xi_i t_{ji} f_j[x_j(k))(f_j(x_j(k)) - l_j x_j(k)] \\
&- 2\sum_{i=1}^{N}\sum_{j=1}^{n}\xi_i s_{ji} f_j(x_j(k-\tau))[f_j(x_j(k-\tau) - l_j x_j(k-\tau)] \\
= &\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k)\Big(A_i^T P_j A_l - P_i + Q_i\Big)x(k)\Big] \\
&+ \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k)\Big(-A_i^T P_j W_l - LT_i\Big)f(x(k))\Big] \\
&+ \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k)\Big(-A_i^T P_j W_{1l}\Big)f(x(k-\tau))\Big] \\
&+ \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k)\Big(A_i^T P_j H_0\Big)(F_0 E_0(x(k))\Big] \\
&+ \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k)\Big(-A_i^T P_j H\Big)(FEf(x(k))\Big] \\
&+ \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k)\Big(-A_i^T P_j H_1\Big)(F_1 E_1 f(x(k-\tau))\Big] \\
&+ \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k-\tau)\Big(Q_i\Big)x(k-\tau)\Big] \\
&+ \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k-\tau)\Big(LS_i\Big)x(k-\tau)\Big] \\
&+ \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k))\Big(-W_l^T P_j A_i - LT_i\Big)x(k)\Big] \\
&+ \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k))\Big(W_i^T P_j W_l - 2T_i\Big)f(x(k))\Big] \\
&+ \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k))\Big(W_i^T P_j W_{1l}\Big)x(k)\Big] \\
&+ \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k))\Big(-W_i^T P_j H_0\Big)(F_0 E_0 x(k))\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k))\Big(W_i^TP_jH\Big)(FEf(x(k)))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k))\Big(W_i^TP_jH_1\Big)(F_1E_1f(x(k-\tau)))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k-\tau))\Big(-W_{1j}^TP_jA_i\Big)x(k)\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k-\tau))\Big(LS_i\Big)x(k-\tau)\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k-\tau))\Big(W_{1l}^TP_jW_i\Big)f(x(k))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k-\tau))\Big(W_{1i}^TP_jW_{1l}-2S_i\Big)f(x(k-\tau))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k-\tau))\Big(-W_{1i}^TP_jH_0\Big)(F_0E_0x(k))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k-\tau))\Big(W_{1i}^TP_jH\Big)(FEfx(k))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k-\tau))\Big(W_{1i}^TP_jH_1\Big)(F_1E_1f(x(k-\tau)))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_0E_0x(k))\Big(H_0^TP_jA_i\Big)x(k)\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_0E_0x(k))\Big(-H_0^TP_jW_i\Big)f(x(k))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_0E_0x(k))\Big(-H_0^TP_jW_{1i}\Big)f(x(k-\tau))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_0E_0x(k))\Big(H_0^TP_iH_0\Big)(F_0E_0x(k))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_0E_0x(k))\Big(-H_0^TP_iH\Big)(FEf(x(k)))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_0E_0x(k))\Big(-H_0^TP_iH_1\Big)(F_1E_1f(x(k-\tau)))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(FEf(x(k)))\Big(-H^TP_jA_i\Big)x(k)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(FEf(x(k)))\Big(H^TP_jW\Big)f(x(k))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(FEf(x(k)))\Big(H^TP_jW_{1i}\Big)f(x(k-\tau))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(FEf(x(k)))\Big(-H^TP_iH_0\Big)(F_0E_0x(k))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(FEf(x(k)))\Big(H^TP_iH\Big)(FEf(x(k)))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(FEf(x(k)))\Big(H^TP_iH_1\Big)(F_1E_1f(x(k-\tau)))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_1E_1f(x(k-\tau)))\Big(-H_1^TP_jA_i\Big)x(k)\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_1E_1f(x(k-\tau)))\Big(H_1^TP_jW_i\Big)f(x(k))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_1E_1f(x(k-\tau)))\Big(H_1^TP_jW_{1i}\Big)f(x(k-\tau))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_1E_1f(x(k-\tau)))\Big(-H_1^TP_iH_0\Big)(F_0E_0x(k))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_1E_1f(x(k-\tau)))\Big(H_1^TP_iH\Big)(FEf(x(k)))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_1E_1f(x(k-\tau)))\Big(H_1^TP_iH_1\Big)(F_1E_1f(x(k-\tau)))\Big]. \qquad (3.36)
\end{aligned}
$$

Substituting (3.32) into (3.36). Therefore

$$
\begin{aligned}
\Delta V(x(k),\ \xi)\le\ &\sum_{i=1}^{N}\xi_i^3y^T\Big[M_{i,i,i}+N_i\Big]y\\
&+\sum_{i=1}^{N}\xi_i\sum_{j\neq i;\ j=1}^{N}\xi_jy^T\Big[M_{i,j,i}+M_{i,i,j}+M_{j,i,i}+2N_i+N_j\Big]y\\
&+\sum_{i=1}^{N-2}\xi_i\sum_{j=i+1}^{N-1}\xi_j\sum_{l=j+1}^{N}\xi_ly^T\begin{bmatrix}M_{i,j,l}+M_{i,l,j}+M_{j,i,l}+M_{j,l,i}\\+M_{l,i,j}+M_{l,j,i}+2N_i+2N_j+2N_l\end{bmatrix}y,
\end{aligned}
$$

where $y=[x^T(k)\ \ x^T(k-\tau)\ \ f^T(x(k))\ \ f^T(x(k-\tau))\ \ (F_0E_0x(k))^T\ \ (FEf(x(k)))^T$ $(F_1E_1f(x(k-\tau)))^T]^T$.

By condition (3.35) ,we have

$$\Delta V(x(k),\ \xi) < -y^T\Big[\sum_{i=1}\xi_i^3 - \frac{1}{(N-1)^2}\sum_{i=1}^{N}\sum_{j\neq i;\ j=1}^{N}\xi_i^2\xi_j I - \frac{6}{(N-1)^2} + \sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{l=j+1}^{N}\xi_i\xi_j\xi_l I\Big]y.$$

We define Θ and Λ as

$$\Theta \equiv \sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i(\xi_i-\xi_j)^2 = (N-1)\sum_{i=1}^{N}\xi^3 - \sum_{i=1}^{N-2}\sum_{j\neq i;\ j=1}^{N}\xi_i^2\xi_j \geq 0,$$

$$\Lambda \equiv \sum_{i=1}^{N}\sum_{j=i+1}^{N-1}\sum_{l=j+1}^{N}\xi_i(\xi_j-\xi_l)^2 = (N-2)\sum_{i=1}^{N}\sum_{j\neq i;\ j=1}^{N}\xi_i^2\xi_j - 6\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{l=j+1}^{N}\xi_i\xi_j\xi_l \geq 0$$

and compute $(N-1)\Theta + \Lambda$, we obtain

$$(N-1)^2\sum_{i=1}^{N}\xi_i^3 - \sum_{i=1}^{N}\sum_{j\neq i;\ j=1}^{N}\xi_i^2\xi_j - 6\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{l=j+1}^{N}\xi_i\xi_j\xi_l \geq 0.$$

Thus $\Delta V(x(k),\ \xi) < 0$ From the Lyapunov stability theorem, the system is asymptotically stable and by Definition 2.3.4 we get the system (3.5) is robustly stable.□

Example 3.2.1. Consider the CNNs (3.5) with polytopic type uncertainties with N = 2 where

$$A_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.1 & -0.5 \\ 0.09 & -0.2 \end{bmatrix}, \; W_2 = \begin{bmatrix} 0.1 & -0.5 \\ 0.4 & -0.2 \end{bmatrix},$$

$$W_{11} = \begin{bmatrix} 0.1 & -0.3 \\ 0.1 & -0.2 \end{bmatrix}, \; W_{12} = \begin{bmatrix} 0.1 & -0.2 \\ 0.1 & -0.2 \end{bmatrix},$$

$$H_0 = \begin{bmatrix} -0.003 & 0.015 \\ -0.018 & -0.025 \end{bmatrix}, \; H = \begin{bmatrix} -0.035 & 0.042 \\ 0.008 & -0.004 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0.002 & -0.036 \\ -0.006 & 0.007 \end{bmatrix}, \; L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$F_0 = \begin{bmatrix} 0.9 & 0 \\ 0 & -1 \end{bmatrix}, \; F = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, \; F_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$f_1(x) = \tanh(x), \; f_2(x) = \tanh(x), \; E_0 = H_0, \; E = H, \; E_1 = H_1.$$

By using the Matlab LMI toolbox, we can solved for matrices P_i , Q_i, S_i and T_i, $i = 1, 2, 3$ which satisfy the criterion of Theorem 3.2.1, thus the zero solution of system (3.5) is robust stability system (3.1) is robust stability. A set of solutions of (3.5) are given the following

$$P_1 = \begin{bmatrix} 77.3679 & -34.0999 \\ -34.0999 & 124.2649 \end{bmatrix}, \; Q_1 = \begin{bmatrix} 29.7607 & -12.6706 \\ -12.6706 & 57.2807 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 20.9749 & 0 \\ 0 & 20.9749 \end{bmatrix}, \; T_1 = \begin{bmatrix} 26.5428 & 0 \\ 0 & 26.5428 \end{bmatrix},$$

$$e_{01} = 49.3011, \; e_1 = 49.5071, \; e_{11} = 49.3053,$$

The simulation is illustrated in Fig. 3.4.

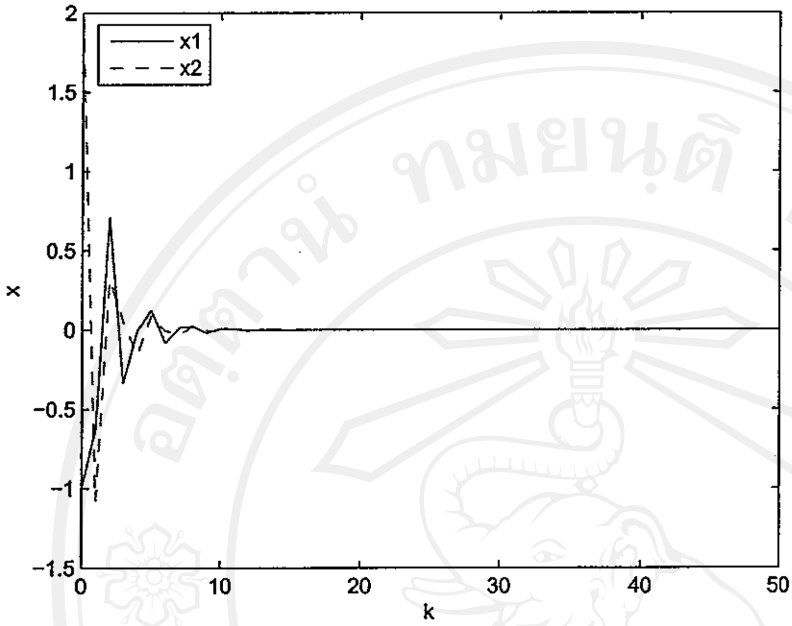

Figure 3.4: The solutions trajectory of system (3.5) in Example 3.2.1.

$$P_2 = \begin{bmatrix} 127.2143 & -48.2870 \\ -48.2870 & 140.5125 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 51.5535 & -19.3739 \\ -19.3739 & 57.2807 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 26.5189 & 0 \\ 0 & 26.5189 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 40.4664 & 0 \\ 0 & 40.4664 \end{bmatrix},$$

$e_{02} = 64.5882,\ e_2 = 64.8376,\ e_{12} = 64.5850,$

The simulation is illustrated in Fig. 3.5.

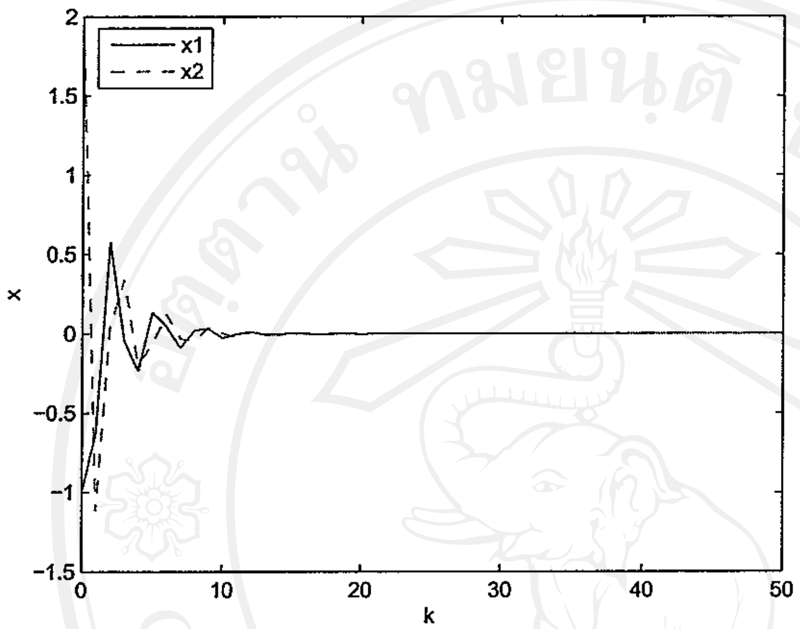

Figure 3.5: The solutions trajectory of system (3.5) in Example 3.2.1.

Theorem 3.2.2. *The zero solution of system* (3.5) *with polytopic type uncertainties* (3.2) *is robustly stable if there exist* $P_i = P_i^T > 0$, $Q_i = Q_i^T > 0$, $T_i = diag\{t_{1i}, t_{2i}, \ldots, t_{ni}\} \geq 0$, $S_i = diag\{s_{1i}, s_{2i}, \ldots, s_{ni}\} \geq 0$ *and scalars* $\epsilon > 0$, $e_{0i} > 0$, $e_i > 0$ *and* $e_{1i} > 0$, $i = 1, 2, \ldots, N$ *satisfy this condition where*

$$(i)\ M_{i,i,i} + N_i < -I,\ i = 1, 2, \ldots, N$$

$$(ii)\ M_{i,i,j} + M_{j,i,i} + M_{i,j,i} + 2N_i + N_j < \frac{1}{(N-1)^2} I,$$
$$i = 1, 2, \ldots, N,\ i \neq j,\ j = 1, 2, \ldots, N$$

$$(iii)\ M_{i,j,l} + M_{i,l,j} + M_{j,i,l} + M_{j,l,i} + M_{l,i,j} + M_{l,j,i}$$
$$+ 2N_i + 2N_j + 2N_l < \frac{6}{(N-1)^2} I,$$
$$i = 1, 2, \ldots, N-2,\ j = i+1, 2, \ldots, N-1,\ l = 1, 2, \ldots, N, \tag{3.37}$$

where

$$M_{i,j,l} =$$
$$\begin{bmatrix} \theta_{11}(i) & 0 & -A_i^T P_j W_l & -A_i^T P_j W_{1l} & A_i^T P_j H_0 & -A_i^T P_j H & -A_i^T P_j H_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -W_l^T P_j A_i & 0 & \theta_{33}(i) & W_i^T P_j W_{1l} & -W_i^T P_j H_0 & W_i^T P_j H & W_i^T P_j H_1 \\ -W_{1l}^T P_j A_i & 0 & W_{1l}^T P_j W_i & \theta_{44}(i) & -W_{1i}^T P_j H_0 & W_{1i}^T P_j H & W_{1i}^T P_j H_1 \\ H_0^T P_j A_i & 0 & -H_0^T P_j W_i & -H_0^T P_j W_{1i} & 0 & 0 & 0 \\ -H^T P_j A_i & 0 & H^T P_j W_i & H^T P_j W_{1i} & 0 & 0 & 0 \\ -H_1^T P_j A_i & 0 & H_1^T P_j W_i & H_1^T P_j W_{1i} & 0 & 0 & 0 \end{bmatrix},$$

$\theta_{11}(i) = A_i^T (P_j^{-1} - \epsilon^{-1} H_0 H_0^T) A_l$, $\theta_{33}(i) = W_i^T (P_j^{-1} - \epsilon^{-1} H H^T) W_l$

$\theta_{44}(i) = W_{1i}^T (P_j^{-1} - \epsilon^{-1} H_1 H_1^T) W_{1l}^T$

$N_i =$

$$
\begin{bmatrix}
\prod_{11}(i) & 0 & -LT_i & 0 & 0 & 0 & 0 \\
0 & -Q_i & 0 & LS_i & 0 & 0 & 0 \\
-LT_i & 0 & \prod_{33}(i) & 0 & 0 & 0 & 0 \\
0 & LS_i & 0 & \prod_{44}(i) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \prod_{55}(i) & -H_0^T P_i H & -H_0^T P_i H_1 \\
0 & 0 & 0 & 0 & -H^T P_i H_0 & \prod_{66}(i) & H^T P_i H_1 \\
0 & 0 & 0 & 0 & -H_1^T P_i H_0 & H_1^T P_i H & \prod_{77}(i)
\end{bmatrix},
$$

and $\prod_{11}(i) = \epsilon E_0^T E_0 + e_{0i} E_0^T E_0 - P_i + Q_i$, $\prod_{33}(i) = \epsilon E^T E - 2T_i + e_i E^T E$, $\prod_{44}(i) = \epsilon E_1^T E_1 - 2S_i + e_{1i} E_1^T E_1$, $\prod_{55}(i) = -e_{0i} I$, $\prod_{66}(i) = -e_i I$, $\prod_{77}(i) = -e_{1i} I$.

Proof Consider the Lyapunov function candidate

$$V(x(k),\ \xi) = \sum_{i=1}^{N} x^T(k)\xi_i P_i x(k) + \sum_{i=1}^{N} \sum_{l=k-\tau}^{k-1} x^T(l)\xi_i Q_i x(l).$$

The Lyapunov difference along any trajectory of solution of (3.5) is given by

$$
\begin{aligned}
\Delta V(x(k),\ \xi) = \Big[&-(A(\xi)+\Delta A)x(k) + (W(\xi)+\Delta W)f(x(k)) + (W_1(\xi)+\Delta W_1) \\
&\times f(x(t-\tau))\Big]^T P(\xi)\Big[-(A(\xi)+\Delta A)x(k) + (W(\xi)+\Delta W) \\
&\times f(x(k)) + (W_1(\xi)+\Delta W_1)f(x(t-\tau))\Big] - x^T(k)P(\xi)x(k) \\
&+ x^T(k)Q(\xi)x(k) - x^T(k-\tau)Q(\xi)x(k-\tau)
\end{aligned}
$$

by condition (3.3) we get

$$
\begin{aligned}
\Delta V(x(k),\ \xi) = \Big[&-(A(\xi)+H_0F_0E_0)x(k) + (W(\xi)+HFE)f(x(k)) + (W_1(\xi) \\
&+ H_1F_1E_1)f(x(t-\tau))\Big]^T P(\xi)\Big[-(A(\xi)+H_0F_0E_0)x(k) \\
&+ (W(\xi)+HFE)f(x(k)) + (W_1(\xi)+H_1F_1E_1)f(x(t-\tau))\Big] \\
&- x^T(k)P(\xi)x(k) + x^T(k)Q(\xi)x(k) - x^T(k-\tau)Q(\xi)x(k-\tau)
\end{aligned}
$$

by Lemma 2.3.3, condition (3.29) and (3.30) therefore applying S-procedure, system is asymptotically stable if there exist $T(\xi) = diag\{t_{1i}, t_{2i}, \dots, t_{ni}\} \geq 0$ and $S(\xi) = diag\{s_{1i}, s_{2i}, \dots, s_{ni}\} \geq 0$ such that

$$
\begin{aligned}
&\Delta V(x(k),\ \xi) - 2\sum_{i=1}^{N}\sum_{j=1}^{n}\xi_i t_{ji} f_j[x_j(k))(f_j(x_j(k)) - l_j x_j(k)]\\
&\quad - 2\sum_{i=1}^{N}\sum_{j=1}^{n}\xi_i s_{ji} f_j(x_j(k-\tau))[f_j(x_j(k-\tau) - l_j x_j(k-\tau)]\\
&\quad = \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k)\Big(A_i^T(P_j^{-1} - \epsilon^{-1}H_0H_0^T)A_l + \epsilon E_0^T E_0 - P_i\\
&\quad + Q_i\Big)x(k)\Big] + \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k)\Big(-A_i^T P_j W_l - LT_i\Big)f(x(k))\Big]\\
&\quad + \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k)\Big(-A_i^T P_j W_{1l}\Big)f(x(k-\tau))\Big]\\
&\quad + \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k)\Big(A_i^T P_j H_0\Big)(F_0E_0(x(k))\Big]\\
&\quad + \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k)\Big(-A_i^T P_j H\Big)(FEf(x(k))\Big]\\
&\quad + \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k)\Big(-A_i^T P_j H_1\Big)(F_1E_1f(x(k-\tau))\Big]\\
&\quad + \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k-\tau)\Big(Q_i\Big)x(k-\tau)\Big]\\
&\quad + \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[x^T(k-\tau)\Big(LS_i\Big)x(k-\tau)\Big]\\
&\quad + \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k))\Big(-W_l^T P_j A_i - LT_i\Big)x(k)\Big]\\
&\quad + \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k))\Big(W_i^T(P_j^{-1} - \epsilon^{-1}HH^T)W_l\\
&\quad + \epsilon E^T E - 2T_i\Big)f(x(k))\Big] + \sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k))\Big(W_i^T P_j W_{1l}\Big)x(k)\Big]
\end{aligned}
$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k))\Big(-W_i^TP_jH_0\Big)(F_0E_0x(k))\Big]$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k))\Big(W_i^TP_jH\Big)(FEf(x(k)))\Big]$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k))\Big(W_i^TP_jH_1\Big)(F_1E_1f(x(k-\tau)))\Big]$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k-\tau))\Big(-W_{1l}^TP_jA_i\Big)x(k)\Big]$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k-\tau))\Big(LS_i\Big)x(k-\tau)\Big]$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k-\tau))\Big(W_{1l}^TP_jW_i\Big)f(x(k))\Big]$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k-\tau))\Big(W_{1i}^T(P_j^{-1}-\epsilon^{-1}H_1H_1^T)W_{1l}^T+\epsilon E_1^TE_1$$

$$-2S_i\Big)f(x(k-\tau))\Big]+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k-\tau))\Big(-W_{1i}^TP_jH_0\Big)(F_0E_0x(k))\Big]$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k-\tau))\Big(W_{1i}^TP_jH\Big)(FEfx(k))\Big]$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[f^T(x(k-\tau))\Big(W_{1i}^TP_jH_1\Big)(F_1E_1f(x(k-\tau)))\Big]$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_0E_0x(k))\Big(H_0^TP_jA_i\Big)x(k)\Big]$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_0E_0x(k))\Big(-H_0^TP_jW_i\Big)f(x(k))\Big]$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_0E_0x(k))\Big(-H_0^TP_jW_{1i}\Big)f(x(k-\tau))\Big]$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_0E_0x(k))\Big(-H_0^TP_iH\Big)(FEf(x(k)))\Big]$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_0E_0x(k))\Big(-H_0^TP_iH_1\Big)(F_1E_1f(x(k-\tau)))\Big]$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(FEf(x(k)))\Big(-H^TP_jA_i\Big)x(k)\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(FEf(x(k)))\Big(H^TP_jW_i\Big)f(x(k))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(FEf(x(k)))\Big(H^TP_jW_{1i}\Big)f(x(k-\tau))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(FEf(x(k)))\Big(-H^TP_iH_0\Big)(F_0E_0x(k))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(FEf(x(k)))\Big(H^TP_iH_1\Big)(F_1E_1f(x(k-\tau)))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_1E_1f(x(k-\tau)))\Big(-H_1^TP_jA_i\Big)x(k)\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_1E_1f(x(k-\tau)))\Big(H_1^TP_jW_i\Big)f(x(k))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_1E_1f(x(k-\tau)))\Big(H_1^TP_jW_{1i}\Big)f(x(k-\tau))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_1E_1f(x(k-\tau)))\Big(-H_1^TP_iH_0\Big)(F_0E_0x(k))\Big]\\
&+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\xi_j\sum_{l=1}^{N}\xi_l\Big[(F_1E_1f(x(k-\tau)))\Big(H_1^TP_iH\Big)(FEf(x(k)))\Big]
\end{aligned}
\tag{3.38}
$$

Substituting (3.32) into (3.38). Therefore

$$
\begin{aligned}
\Delta V(x(k),\,\xi)\leq\ &\sum_{i=1}^{N}\xi_i^3y^T\Big[M_{i,i,i}+N_i\Big]y\\
&+\sum_{i=1}^{N}\xi_i\sum_{j\neq i;\ j=1}^{N}\xi_jy^T\Big[M_{i,j,i}+M_{i,i,j}+M_{j,i,i}+2N_i+N_j\Big]y\\
&+\sum_{i=1}^{N-2}\xi_i\sum_{j=i+1}^{N-1}\xi_j\sum_{l=j+1}^{N}\xi_ly^T\begin{bmatrix}M_{i,j,l}+M_{i,l,j}+M_{j,i,l}+M_{j,l,i}\\+M_{l,i,j}+M_{l,j,i}+2N_i+2N_j+2N_l\end{bmatrix}y,
\end{aligned}
$$

where

$y=[x^T(k)\ \ x^T(k-\tau)\ \ f^T(x(k))\ \ f^T(x(k-\tau))\ \ (F_0E_0x(k))^T\ \ (FEf(x(k)))^T$
$(F_1E_1f(x(k-\tau)))^T]^T$,

By condition (3.37) ,we have

$$\Delta V(x(k),\ \xi) < -y^T\Big[\sum_{i=1}\xi_i^3 - \frac{1}{(N-1)^2}\sum_{i=1}^{N}\sum_{j\neq i;\ j=1}^{N}\xi_i^2\xi_j I - \frac{6}{(N-1)^2} + \sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{l=j+1}^{N}\xi_i\xi_j\xi_l I\Big]y.$$

We define Θ and Λ as

$$\Theta \equiv \sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i(\xi_i-\xi_j)^2 = (N-1)\sum_{i=1}^{N}\xi^3 - \sum_{i=1}^{N-2}\sum_{j\neq i;\ j=1}^{N}\xi_i^2\xi_j \geq 0,$$

$$\Lambda \equiv \sum_{i=1}^{N}\sum_{j=i+1}^{N-1}\sum_{l=j+1}^{N}\xi_i(\xi_j-\xi_l)^2 = (N-2)\sum_{i=1}^{N}\sum_{j\neq i;\ j=1}^{N}\xi_i^2\xi_j - 6\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{l=j+1}^{N}\xi_i\xi_j\xi_l \geq 0$$

and compute $(N-1)\Theta + \Lambda$, we obtain

$$(N-1)^2\sum_{i=1}^{N}\xi_i^3 - \sum_{i=1}^{N}\sum_{j\neq i;\ j=1}^{N}\xi_i^2\xi_j - 6\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{l=j+1}^{N}\xi_i\xi_j\xi_l \geq 0.$$

Thus $\Delta V(x(k),\ \xi) < 0$ From the Lyapunov stability theorem, the system is asymptotically stable and by Definition 2.3.4 we get the system (3.5) is robustly stable.□

Example 3.2.2. Consider the CNNs (3.5) with polytopic type uncertainties with N = 2 where

$$A_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.1 & -0.5 \\ 0.4 & -0.2 \end{bmatrix}, \ W_2 = \begin{bmatrix} 0.1 & -0.3 \\ 0.1 & -0.2 \end{bmatrix},$$

$$W_{11} = \begin{bmatrix} 0.1 & -0.2 \\ 0.1 & -0.2 \end{bmatrix}, \ W_{12} = \begin{bmatrix} 0.1 & -0.15 \\ 0.09 & -0.2 \end{bmatrix},$$

$$H_0 = \begin{bmatrix} 0.003 & -0.005 \\ -0.001 & -0.006 \end{bmatrix}, \ H = \begin{bmatrix} -0.035 & 0.042 \\ 0.008 & -0.004 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0.002 & -0.036 \\ -0.006 & 0.007 \end{bmatrix}, \ L = \begin{bmatrix} 1.1 & 0 \\ 0 & 1.1 \end{bmatrix}$$

$$F_0 = \begin{bmatrix} 0.9 & 0 \\ 0 & -1 \end{bmatrix}, \ F = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, \ F_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$f_1(x) = 2\tanh(x)$, $f_2(x) = 1.25\tanh(x)$, $E_0 = H_0$, $E = H$, $E_1 = H_1$, $\epsilon = 2$.

By using the Matlab LMI toolbox, we can solved for matrices P_i ,Q_i, S_i and T_i, $i = 1, 2, 3$ which satisfy which satisfies the criterion of Theorem 3.2.2, thus the zero solution of system (3.5) is robust stability system (3.1) is robust stability. A set of solutions of (3.5) are given the following

$$P_1 = \begin{bmatrix} 102.5121 & -58.5343 \\ -58.5343 & 117.5862 \end{bmatrix}, \ Q_1 = \begin{bmatrix} 13.6395 & -5.2266 \\ -5.2266 & 24.0687 \end{bmatrix},$$

$$S_1 = 10^{-10} \begin{bmatrix} 0.6476 & 0 \\ 0 & 0.6476 \end{bmatrix}, \ T_1 = \begin{bmatrix} 0.7463 & 0 \\ 0 & 0.7463 \end{bmatrix},$$

$e_{01} = 54.6859$, $e_1 = 55.3860$, $e_{11} = 54.9032$,

The simulation is illustrated in Fig. 3.6.

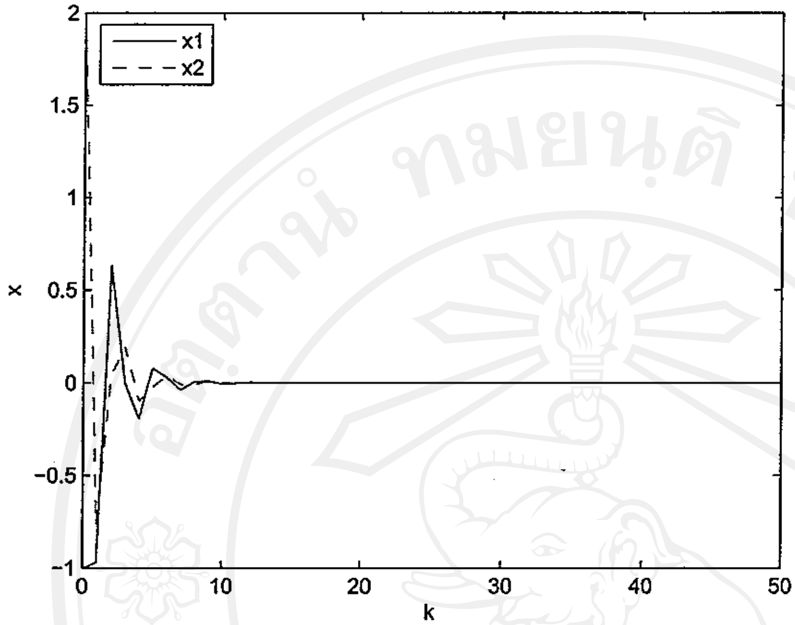

Figure 3.6: The solutions trajectory of system (3.5) in Example 3.2.2.

$$P_2 = \begin{bmatrix} 13.4897 & -9.4789 \\ -9.4789 & 15.7616 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.9926 & -0.8419 \\ -0.8419 & 0.1266 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0.8128 & 0 \\ 0 & 0.8128 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 5.3231 & 0 \\ 0 & 5.3231 \end{bmatrix},$$

$e_{02} = 7.1652, \; e_2 = 7.5813, \; e_{12} = 7.2984,$

The simulation is illustrated in Fig. 3.7.

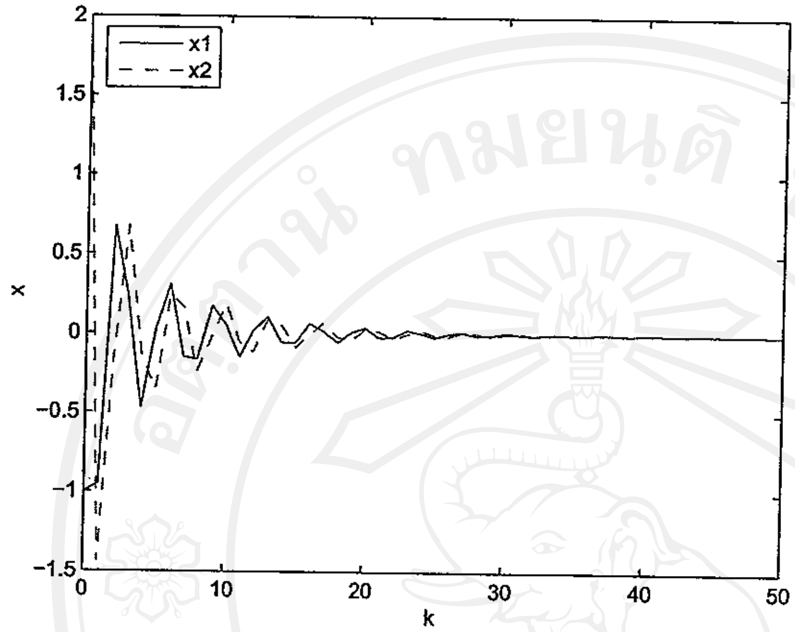

Figure 3.7: The solutions trajectory of system (3.5) in Example 3.2.2.