Chapter 2

Basic Concepts and Preliminaries

2.1 Metric Spaces and Banach Spaces

Definition 2.1.1 ([17]) A *metric space* is a pair (X, d), where X is a set and d is a metric on X (or distance function on X),that is, a real valued function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(1) $d(x, y) \geq 0$

(2) $d(x, y) = 0$ if and only if $x = y$

(3) $d(x, y) = d(y, x)$ (symmetry)

(4) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

Definition 2.1.2 ([17]) A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be *convergent* if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0$$

x is called the *limit* of $\{x_n\}$ and we write

$$\lim_{n\to\infty} x_n = x$$

or, simple, $x_n \to x$

we say that $\{x_n\}$ *converges to* x. If $\{x_n\}$ is not convergent, it is said to be *divergent.*

Lemma 2.1.3 ([39]) Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n = 1, 2, ...,$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(1) $\lim_{n\to\infty} a_n$ exists .
(2) $\lim_{n\to\infty} a_n = 0$ whenever $\liminf_{n\to\infty} a_n = 0$.

Definition 2.1.4 ([17])A sequence (x_n) in a metric space $X = (X, d)$ is said to be *Cauchy* if for every $\epsilon > 0$ there is an $N(\epsilon) \in N$ such that $d(x_m, x_n) < \epsilon$ for every $m, n \geq N(\epsilon)$.

Definition 2.1.5 ([17]) A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges.

Theorem 2.1.6 ([17]) Every convergent sequence in a metric space is a Cauchy sequence.

Theorem 2.1.7 ([20]) Let $\{x_n\}$ be a sequence in $\mathbb{R}$. If every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a convergent subsequence, then $\{x_n\}$ is convergent.

Definition 2.1.8 ([20]) Let X be a metric space and A be any nonempty subset of X. For each x in X, the *distance* $d(x, A)$ from x to A is $\inf\{d(x, y)|\ y \in A\}$.

Definition 2.1.9 ([20]) Let X be a linear space (or vector space). A *norm* on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:

(1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\alpha x\| = |\alpha|\|x\|$,

(3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

The ordered pair $(X, \|\cdot\|)$ is called a *normed space* or *normed vector space* or *normed linear space.*

Definition 2.1.10 ([20]) Let X be normed space. The *metric induced by the norm* of X is the metric d on X defined by the formula $d(x, y) = \|x - y\|$ for all $x, y \in X$. The *norm topology* of X is the topology obtained from this metric.

Definition 2.1.11 ([20]) A Banach norm or complete norm is a norm that induces a complete metric. A normed space is a *Banach space* or *B-space* or *complete normed space* if its norm is a Banach norm.

Definition 2.1.12 ([17]) An *inner product space* is a vector space X with an inner product defined on X. A *Hilbert space* is a complete inner product space. Here, an inner product on X is a mapping of $X \times X$ into the scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$; that is, with every pair of vector x and y there is associated a scalar which is written and is called the inner product of x and y, such that for all vectors x, y, z and scalar $\alpha \in \mathbb{F}$ we have:

(1) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$

(2) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

(3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(4) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

An inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x, x\rangle}$.

Theorem 2.1.13 ([36])(The Schwarz inequality)
If x and y are any two vector in an inner product space X, then $|\langle x, y\rangle| \leq \|x\|\|y\|$.

2.2 Reflexive Spaces and Geometric Properties of Banach Space

Definition 2.2.1 ([17]) Let X be normed space, for each $x \in X$ there corresponds a unique bounded linear functional $g_x \in X^{**}$ given by $g_x(f) = f(x)$, $f \in X^*$. A mapping $C : X \to X^{**}$ defined by $x \mapsto g_x$, is called the *canonical mapping.*

Definition 2.2.2 ([17]) A normed space X is said to be *reflexive* if the canonical mapping $C : X \to X^{**}$ is surjective.

Definition 2.2.3 ([17]) Let x be an element and $\{x_n\}$ a sequence in a normed space X. Then $\{x_n\}$ *converges strongly* to x written by $x_n \to x$, if $\lim_{n\to\infty} \|x_n - x\| = 0$.

Definition 2.2.4 ([17]) Let x be an element and $\{x_n\}$ a sequence in a normed space X. Then $\{x_n\}$ *converges weakly* to x written by $x_n \rightharpoonup x$, if $f(x_n) \to f(x)$ wherever $f \in X^*$.

Theorem 2.2.5 ([36]) A normed space X is reflexive if and only if each of its bounded sequence has a weakly convergent subsequence.

Definition 2.2.6 ([36]) A nonempty subset C of a Banach space X is called *weakly sequentially compact* if every sequence $\{x_n\}$ in C has a subsequence converging to a point of X in the weak topology.

Theorem 2.2.7 ([36]) Let X be a reflexive Banach space. Then a nonempty subset C of X is weakly sequentially compact if and only if C is bounded.

Definition 2.2.8 ([17]) A subset C of a vector space X is said to be *convex* if $x, y \in C$ implies $M = \{z \in X | \ z = tx + (1-t)y, \ 0 \leq t \leq 1\} \subseteq C$.

Definition 2.2.9 ([36]) A Banach space X is *uniformly convex* if for any two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$, imply $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Theorem 2.2.10 ([36]) If a Banach space X is uniformly convex, then X is reflexive.

Definition 2.2.11 ([36]) Let X be a linear space and let C be a convex subset of X. A function $F : C \to (-\infty, \infty]$ is *convex* on C if for any $x, y \in C$ and $t \in [0,1]$, then $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$.

Lemma 2.2.12 ([42]) Let $p > 1$, $r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g(\|x-y\|),$$

for all x, y in $B_r = \{x \in X : \|x\| \leq r\}$, $\lambda \in [0,1]$, where

$$w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda).$$

Lemma 2.2.13 ([9]) Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty), g(0) = 0$ such that

$$\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g(\|x-y\|),$$

for all $x, y, z \in B_r$, and all $\lambda, \beta, \gamma \in [0,1]$ with $\lambda + \beta + \gamma = 1$.

Lemma 2.2.14 ([21], Lemma 1.4) Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty), g(0) = 0$ such that

$$\|\alpha x + \beta y + \mu z + \lambda w\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \mu\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g(\|x-y\|),$$

for all $x, y, z, w \in B_r$, and all $\alpha, \beta, \mu, \lambda \in [0,1]$ with $\alpha + \beta + \mu + \lambda = 1$.

Definition 2.2.15 ([25]) A Banach space X is said to satisfy *Opial's condition* if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

A Banach space X is said to have the *Kadec-Klee property* if for every sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$. The mapping $T : C \to X$ with $F(T) \neq \emptyset$ is said to satisfy *condition*(A) [32] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in C$

$$\|x - Tx\| \geq f(d(x, F(T))).$$

A family $\{T_i : i \in J\}$ of N self-mappings of C with $F := \cap_{i=1}^{N} F(T_i) \neq \varnothing$ is said to satisfy *condition* (B) on C [8] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\max_{1 \leqslant l \leqslant N} \{\|x - T_l x\|\} \geqslant f(d(x, F))$$

for all $x \in C$.

Lemma 2.2.16 ([35]) Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X . Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exists. If $\{x_{n_k}\}$ and $\{x_{n_j}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.

Lemma 2.2.17 ([21], Lemma 2.1) If $\{b_n\}, \{c_n\}$ and $\{\mu_n\}$ are sequences in $[0, 1]$ such that $\limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$ and $\{k_n\}$ is a sequence of real number with $k_n \geq 1$ for all $n \geq 1$ and $\lim_{n\to\infty} k_n = 1$, then there exist a positive integer N_1 and $\gamma \in (0, 1)$ such that $c_n k_n < \gamma$ for all $n \geq N_1$.

2.3 Fixed Points of Nonexpansive Mappings

Definition 2.3.1 ([44]) Let C be subset of a Banach space X. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of all fixed points of T is denoted by $F(T)$, i.e. $F(T) = \{x \in C | x = Tx\}$.

Definition 2.3.2 ([44]) Let C be subset of a Banach space X. A self-mapping $f : C \to C$ is called *contraction* on C if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$ for all $x, y \in C$. We use Π_C to denote the collection of all contraction on C.

Theorem 2.3.3 ([36])(The Banach contraction principle)
Let X be complete metric space and let f be a contraction of X. Then f has a unique fixed point.

Definition 2.3.4 ([4]) A mapping $T : C \to X$ is called *demiclosed* with respect to y if for each sequence $\{x_n\}$ in C and each $x \in X$, $x_n \to x$ weakly and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Lemma 2.3.5 ([2]) Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X and $T : C \to X$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed point of T.

Lemma 2.3.6 ([14]) Let X be a real reflexive Banach space such that its dual X^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in X and $x^*, y^* \in \omega_w(x_n)$; here $\omega_w(x_n)$ denote the set of all weak subsequential limits of $\{x_n\}$. Suppose $\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\|$ exists for all $t \in [0, 1]$. Then $x^* = y^*$.

We denote by Γ the set of strictly increasing, continuous convex function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$. Let C be a convex subset of the Banach space X. A mapping $T : C \to C$ is said to be type (γ) if $\gamma \in \Gamma$ and $0 \leqslant \alpha \leqslant 1$,

$$\gamma(\|\alpha Tx + (1-\alpha)Ty - T(\alpha x + (1-\alpha)y)\|) \leqslant \|x - y\| - \|Tx - Ty\|$$

for all x, y in C.

Lemma 2.3.7 ([5], [24]) Let X be a uniformly convex Banach space and C a convex subset of X. Then there exists $\gamma \in \Gamma$ such that for each mapping $S : C \to C$ with *Lipschitz constant* L,

$$\|\alpha Sx + (1-\alpha)Sy - S(\alpha x + (1-\alpha)y)\| \leqslant L\gamma^{-1}(\|x - y\| - \frac{1}{L}\|Sx - Sy\|)$$

for all $x, y \in C$ and $0 < \alpha < 1$.

2.4 Fixed Points of Asymptotically Nonexpansive Mappings

Lemma 2.4.1 ([9]) Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X and $T : C \to C$ be an asymptotically nonexpansive mapping. Then $I - T$ is demi-closed at zero, i.e., for each sequence $\{x_n\}$ in C, if $\{x_n\}$ converges weakly to $q \in C$ and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)q = 0$.

Lemma 2.4.2 ([7], Theorem 3.4) Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and let $T : C \to X$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed point of T.