Chapter 3

Fixed Point Iterations for Nonexpansive Mappings

3.1 Weak and Strong Convergence to a Fixed Point of Nonexpansive Mapping

In this section, a new class of three-step iterative scheme is introduced and studied. The scheme is defined as follows:

Let X be a normed space, C a nonempty convex subset of X, $P : X \to C$ a nonexpansive retraction of X onto C, and $T : C \to X$ a given mapping. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= P((1-a_n-b_n)x_n + a_nTx_n + b_nu_n), \\ y_n &= P((1-c_n-d_n)z_n + c_nTx_n + d_nv_n), \\ x_{n+1} &= P((1-\alpha_n-\beta_n)y_n + \alpha_nTx_n + \beta_nw_n), \quad n \geq 1, \end{aligned} \tag{3.1}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

In this section, we prove weak and strong convergence theorems for the three-step iterative scheme (3.1) for a nonexpansive nonself-mapping in a uniformly convex Banach space. In order to prove our main results, the following lemma is needed.

Lemma 3.1.1 *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T : C \to X$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}$ and $\{d_n\}$ are real sequences in $[0, 1]$ such that c_n+d_n and $\alpha_n + \beta_n$ are in $[0, 1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$,*

$\sum_{n=1}^{\infty} \beta_n < \infty$. *For a given* $x_1 \in C$, *let* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be the sequences defined as in (3.1).*

(i) If p *is a fixed point of* T, *then* $\lim_{n\to\infty} \|x_n - p\|$ *exists.*
(ii) If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *then* $\lim_{n\to\infty} \|Tx_n - y_n\| = 0$.
(iii) If $0 < \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$ *and* $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n+d_n) < 1$, *then* $\lim_{n\to\infty} \|Tx_n - z_n\| = 0$.
(iv) If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$, *and* $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$ *then* $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.

Proof.(i) Let $p \in F(T)$, and

$$\begin{aligned} M_1 &= \sup\{\|u_n - p\| : n \geq 1\}, \\ M_2 &= \sup\{\|v_n - p\| : n \geq 1\}, \\ M_3 &= \sup\{\|w_n - p\| : n \geq 1\}, \\ M &= \max\{M_i : i = 1,2,3\}. \end{aligned}$$

Using (3.1), we have

$$\begin{aligned} \|z_n - p\| &= \|P((1 - a_n - b_n)x_n + a_nTx_n + b_nu_n) - P(p)\| \\ &\leq \|((1 - a_n - b_n)x_n + a_nTx_n + b_nu_n) - p\| \\ &\leq (1 - a_n - b_n)\|x_n - p\| + a_n\|Tx_n - T(p)\| + b_n\|u_n - p\| \\ &\leq (1 - a_n - b_n)\|x_n - p\| + a_n\|x_n - p\| + b_n\|u_n - p\| \\ &\leq \|x_n - p\| + Mb_n, \end{aligned}$$

$$\begin{aligned} \|y_n - p\| &= \|P((1 - c_n - d_n)z_n + c_nTx_n + d_nv_n) - P(p)\| \\ &\leq (1 - c_n - d_n)\|z_n - p\| + c_n\|x_n - p\| + Md_n \\ &\leq (1 - c_n - d_n)(\|x_n - p\| + Mb_n) + c_n\|x_n - p\| + Md_n \\ &\leq \|x_n - p\| + Mb_n + Md_n, \end{aligned}$$

and so

$$\begin{aligned} \|x_{n+1} - p\| &= \|P((1 - \alpha_n - \beta_n)y_n + \alpha_nTx_n + \beta_nw_n) - P(p)\| \\ &\leq (1 - \alpha_n - \beta_n)\|y_n - p\| + \alpha_n\|x_n - p\| + \beta_n\|w_n - p\| \\ &\leq (1 - \alpha_n - \beta_n)(\|x_n - p\| + Mb_n + Md_n) + \alpha_n\|x_n - p\| + M\beta_n \\ &\leq \|x_n - p\| + M(b_n + d_n + \beta_n). \end{aligned}$$

Hence the assertion (i) follows from Lemma 2.1.3.

(ii) By (i), we know that $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(T)$. It follows that $\{x_n - p\}, \{Tx_n - p\}$ and $\{y_n - p\}$ are bounded. Also, $\{u_n - p\}, \{v_n - p\}$ and $\{w_n - p\}$ are bounded by the assumption. Now we set

$$\begin{aligned}
r_1 &= \sup\{\|x_n - p\| : n \geq 1\},\\
r_2 &= \sup\{\|Tx_n - p\| : n \geq 1\},\\
r_3 &= \sup\{\|y_n - p\| : n \geq 1\},\\
r_4 &= \sup\{\|z_n - p\| : n \geq 1\},\\
r_5 &= \sup\{\|u_n - p\| : n \geq 1\},\\
r_6 &= \sup\{\|v_n - p\| : n \geq 1\},\\
r_7 &= \sup\{\|w_n - p\| : n \geq 1\},\\
r &= \max\{r_i : i = 1, 2, 3, 4, 5, 6, 7\}. \qquad (3.2)
\end{aligned}$$

By using Lemma 2.2.13 we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|P((1 - a_n - b_n)x_n + a_nTx_n + b_nu_n) - P(p)\|^2\\
&\leq \|(1 - a_n - b_n)(x_n - p) + a_n(Tx_n - p) + b_n(u_n - p)\|^2\\
&\leq (1 - a_n - b_n)\|x_n - p\|^2 + a_n\|Tx_n - p\|^2 + b_n\|u_n - p\|^2\\
&\quad - a_n(1 - a_n - b_n)g(\|Tx_n - x_n\|)\\
&\leq (1 - a_n - b_n)\|x_n - p\|^2 + a_n\|x_n - p\|^2 + b_n\|u_n - p\|^2\\
&\leq \|x_n - p\|^2 + r^2b_n,
\end{aligned}$$

$$\begin{aligned}
\|y_n - p\|^2 &= \|P((1 - c_n - d_n)z_n + c_nTx_n + d_nv_n) - P(p)\|^2\\
&\leq \|(1 - c_n - d_n)(z_n - p) + c_n(Tx_n - p) + d_n(v_n - p)\|^2\\
&\leq (1 - c_n - d_n)\|z_n - p\|^2 + c_n\|Tx_n - p\|^2 + d_n\|v_n - p\|^2\\
&\quad - c_n(1 - c_n - d_n)g(\|Tx_n - z_n\|)\\
&\leq (1 - c_n - d_n)\|z_n - p\|^2 + c_n\|x_n - p\|^2 + d_n\|v_n - p\|^2\\
&\quad - c_n(1 - c_n - d_n)g(\|Tx_n - z_n\|)\\
&\leq (1 - c_n - d_n)(\|x_n - p\|^2 + r^2b_n) + c_n\|x_n - p\|^2 + r^2d_n\\
&\quad - c_n(1 - c_n - d_n)g(\|Tx_n - z_n\|)\\
&\leq (1 - d_n)\|x_n - p\|^2 + r^2b_n + r^2d_n\\
&\quad - c_n(1 - c_n - d_n)g(\|Tx_n - z_n\|)\\
&\leq \|x_n - p\|^2 + r^2b_n + r^2d_n,
\end{aligned}$$

and so

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|P((1 - \alpha_n - \beta_n)y_n + \alpha_nTx_n + \beta_nw_n) - P(p)\|^2\\
&\leq \|(1 - \alpha_n - \beta_n)y_n + \alpha_nTx_n + \beta_nw_n - p\|^2\\
&= \|(1 - \alpha_n - \beta_n)(y_n - p) + \alpha_n(Tx_n - p) + \beta_n(w_n - p)\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n-\beta_n)\|y_n-p\|^2+\alpha_n\|Tx_n-p\|^2+\beta_n\|w_n-p\|^2 \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|Tx_n-y_n\|) \\
&\leq (1-\alpha_n-\beta_n)\|y_n-p\|^2+\alpha_n\|x_n-p\|^2+\beta_n\|w_n-p\|^2 \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|Tx_n-y_n\|) \\
&\leq (1-\alpha_n-\beta_n)(\|x_n-p\|^2+r^2b_n+r^2d_n)+\alpha_n\|x_n-p\|^2+r^2\beta_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|Tx_n-y_n\|) \\
&\leq (1-\beta_n)\|x_n-p\|^2+r^2b_n+r^2d_n+r^2\beta_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|Tx_n-y_n\|) \\
&\leq \|x_n-p\|^2+r^2b_n+r^2d_n+r^2\beta_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|Tx_n-y_n\|) \\
&\leq \|x_n-p\|^2+r^2(b_n+d_n+\beta_n)-\alpha_n(1-\alpha_n-\beta_n)g(\|Tx_n-y_n\|),
\end{aligned}
$$

which leads to the following:

$$
\begin{aligned}
\alpha_n(1-\alpha_n-\beta_n)g(\|Tx_n-y_n\|) \leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&+r^2(b_n+d_n+\beta_n), \qquad (3.3)
\end{aligned}
$$

If $0<\liminf_{n\to\infty}\alpha_n\leq\limsup_{n\to\infty}(\alpha_n+\beta_n)<1$, then there exist a positive integer n_0 and $\eta,\eta'\in(0,1)$ such that $0<\eta<\alpha_n$ and $\alpha_n+\beta_n<\eta'<1$ for all $n\geq n_0$. It follows from (3.3) that

$$
\begin{aligned}
\eta(1-\eta')g(\|Tx_n-y_n\|) \leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&+r^2(b_n+d_n+\beta_n), \qquad (3.4)
\end{aligned}
$$

for all $n\geq n_0$. Applying (3.4) for $m\geq n_0$, we have

$$
\begin{aligned}
\sum_{n=n_0}^{m} g(\|Tx_n-y_n\|) &\leq \frac{1}{\eta(1-\eta')}\Bigg(\sum_{n=n_0}^{m}(\|x_n-p\|^2-\|x_{n+1}-p\|^2) \\
&\qquad +r^2\sum_{n=n_0}^{m}(b_n+d_n+\beta_n)\Bigg) \\
&\leq \frac{1}{\eta(1-\eta')}\Bigg(\|x_{n_0}-p\|^2 \\
&\qquad +r^2\sum_{n=n_0}^{m}(b_n+d_n+\beta_n)\Bigg). \qquad (3.5)
\end{aligned}
$$

Letting $m\to\infty$ in the inequality (3.5), we get that $\sum_{n=n_0}^{\infty} g(\|Tx_n-y_n\|)<\infty$, and therefore $\lim_{n\to\infty} g(\|Tx_n-y_n\|)=0$. Since g is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|Tx_n-y_n\|=0$.

(iii) If $0 < \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$ and $0 < \liminf_{n\to\infty} c_n \le \limsup_{n\to\infty}(c_n+d_n) < 1$, then by the same argument as that given in (ii), it can be shown that $\lim_{n\to\infty}\|Tx_n - z_n\| = 0$.

(iv) If $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$, and $0 < \liminf_{n\to\infty} c_n \le \limsup_{n\to\infty}(c_n + d_n) < 1$ and $\limsup_{n\to\infty} a_n < 1$, by (ii) and (iii) we have

$$\lim_{n\to\infty}\|Tx_n - y_n\| = 0 \text{ and } \lim_{n\to\infty}\|Tx_n - z_n\| = 0. \tag{3.6}$$

From $y_n = P((1 - c_n - d_n)z_n + c_nTx_n + d_nv_n)$, we have

$$\begin{aligned}
\|y_n - x_n\| &= \|P((1 - c_n - d_n)z_n + c_nTx_n + d_nv_n) - P(x_n)\| \\
&\le \|(1 - c_n - d_n)z_n + c_nTx_n + d_nv_n - x_n\| \\
&= \|(z_n - x_n) + c_n(Tx_n - z_n) + d_n(v_n - z_n)\| \\
&\le \|z_n - x_n\| + c_n\|Tx_n - z_n\| + d_n\|v_n - z_n\| \\
&= \|P((1 - a_n - b_n)x_n + a_nTx_n + b_nu_n) - P(x_n)\| \\
&\quad + c_n\|Tx_n - z_n\| + d_n\|v_n - z_n\| \\
&\le \|(1 - a_n - b_n)x_n + a_nTx_n + b_nu_n - x_n\| \\
&\quad + c_n\|Tx_n - z_n\| + d_n\|v_n - z_n\| \\
&= \|a_n(Tx_n - x_n) + b_n(u_n - x_n)\| \\
&\quad + c_n\|Tx_n - z_n\| + d_n\|v_n - z_n\| \\
&\le a_n\|Tx_n - x_n\| + b_n\|u_n - x_n\| \\
&\quad + c_n\|Tx_n - z_n\| + d_n\|v_n - z_n\| \\
&\le a_n\|Tx_n - x_n\| + c_n\|Tx_n - z_n\| + 2rb_n + 2rd_n,
\end{aligned}$$

where r is defined by (3.2). Thus

$$\begin{aligned}
\|Tx_n - x_n\| &\le \|Tx_n - y_n\| + \|y_n - x_n\| \\
&\le \|Tx_n - y_n\| + a_n\|Tx_n - x_n\| + c_n\|Tx_n - z_n\| \\
&\quad + 2rb_n + 2rd_n,
\end{aligned}$$

and so

$$(1 - a_n)\|Tx_n - x_n\| \le \|Tx_n - y_n\| + c_n\|Tx_n - z_n\| + 2rb_n + 2rd_n.$$

Since $\limsup_{n\to\infty} a_n < 1$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} d_n = 0$, it follows from (3.6) that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. □

Theorem 3.1.2 *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T : C \to X$ a completely continuous nonexpansive nonself-mapping with*

$F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}$ and $\{d_n\}$ are sequences of real numbers in $[0,1]$ with $c_n + d_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, and

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$.

For $\{x_n\}, \{y_n\}$ and $\{z_n\}$ being the sequences defined by the three-step iterative scheme (3.1), we have $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

Proof. By Lemma 3.1.1(iv), we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| \;=\; 0. \tag{3.7}$$

Since T is completely continuous and $\{x_n\} \subseteq C$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore from (3.7), $\{x_{n_k}\}$ converges. Let $q = \lim_{k\to\infty} x_{n_k}$. By the continuity of T and (3.7) we have that $Tq = q$, so q is a fixed point of T. By Lemma 3.1.1(i), $\lim_{n\to\infty} \|x_n - q\|$ exists. Then $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$. Thus $\lim_{n\to\infty} \|x_n - q\| = 0$. Since $\|y_n - x_n\| \to 0$ as $n \to \infty$, and

$$\begin{aligned} \|z_n - x_n\| &= \|P((1-a_n-b_n)x_n + a_nTx_n + b_nu_n) - P(x_n)\| \\ &\leq \|(1-a_n-b_n)x_n + a_nTx_n + b_nu_n - x_n\| \\ &\leq a_n\|Tx_n - x_n\| + b_n\|u_n - x_n\| \to 0 \text{ as } n \to \infty, \end{aligned}$$

it follows that $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$. □

For $a_n = b_n \equiv 0$, then Theorem 3.1.2 can be reduced to the two-step iteration with errors.

Corollary 3.1.3 *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T : C \to X$ a completely continuous nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{c_n\}, \{d_n\}, \{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0,1]$ satisfying*
(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$.
For a given $x_1 \in C$, define

$$\begin{aligned} y_n &= P((1-c_n-d_n)x_n + c_nTx_n + d_nv_n), \\ x_{n+1} &= P((1-\alpha_n-\beta_n)y_n + \alpha_nTx_n + \beta_nw_n), \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to a fixed point of T.

In the next result, we prove the weak convergence of the three-step iterative scheme (3.1) for nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.1.4 *Let X be a uniformly convex Banach space which satisfies Opial's condition, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T : C \to X$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ are sequences of real numbers in $[0,1]$ with $c_n + d_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$.

Let $\{x_n\}$ be the sequence defined by three-step iterative scheme (3.1). Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. By using the same proof as in Theorem 3.1.2, it can be shown that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 2.3.5, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 2.3.5, $u, v \in F(T)$. By Lemma 3.1.1(i), $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 2.2.16 that $u = v$. Therefore $\{x_n\}$ converges weakly to fixed point of T. □

When $a_n = b_n \equiv 0$ in Theorem 3.1.4, we obtain the weak convergence theorem of the two-step iteration with errors as follows:

Corollary 3.1.5 *Let X be a uniformly convex Banach space which satisfies Opial's condition, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T : C \to X$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{c_n\}, \{d_n\}, \{\alpha_n\}, \{\beta_n\}$ are sequences of real numbers in $[0,1]$ such that*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$.

For a given $x_1 \in C$, define

$$\begin{aligned} y_n &= P((1 - c_n - d_n)x_n + c_n Tx_n + d_n v_n), \\ x_{n+1} &= P((1 - \alpha_n - \beta_n)y_n + \alpha_n Tx_n + \beta_n w_n), \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

Let X be a normed space, C a nonempty convex subset of X, $P : X \to C$ the nonexpansive retraction of X onto C, and $T : C \to X$ a given mapping. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= P(a_n T x_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n) \\ y_n &= P(b_n T z_n + c_n T x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n) \\ x_{n+1} &= P(\alpha_n T y_n + \beta_n T z_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n), \quad n \geq 1, \end{aligned} \tag{3.8}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}, \{\lambda_n\}$ are appropriate sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}$ *and* $\{w_n\}$ are bounded sequences in C.

If $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then (3.8) reduces to the iteration scheme defined by Shahzad [33]

$$\begin{aligned} y_n &= P(b_n T x_n + (1 - b_n)x_n) \\ x_{n+1} &= P(\alpha_n T y_n + (1 - \alpha_n)x_n), \quad n \geq 1, \end{aligned} \tag{3.9}$$

where $\{b_n\}, \{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

If $T : C \to C$, then the iterative scheme (3.8) reduces to the three-step iterations with errors

$$\begin{aligned} z_n &= a_n T x_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n \\ y_n &= b_n T z_n + c_n T x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n \\ x_{n+1} &= \alpha_n T y_n + \beta_n T z_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n, \quad n \geq 1, \end{aligned} \tag{3.10}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}, \{\lambda_n\}$ are appropriate sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}$ *and* $\{w_n\}$ are bounded sequences in C.

If $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then the iterative scheme (3.10) reduces to the Ishikawa iterative scheme

$$\begin{aligned} y_n &= b_n T x_n + (1 - b_n)x_n \\ x_{n+1} &= \alpha_n T y_n + (1 - \alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{3.11}$$

where $\{b_n\}, \{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

Weak and strong convergence theorems of the new three-step iterative scheme (3.8) for nonexpansive nonself-mapping in a uniformly convex Banach space are given in this section. The following lemma is needed.

Lemma 3.1.6 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let*

$\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$, $\{\beta_n\}, \{\gamma_n\}, \{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n+\gamma_n$, $b_n+c_n+\mu_n$ and $\alpha_n+\beta_n+\lambda_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$, and let $\{u_n\}, \{v_n\}$ and $\{w_n\}$ be bounded sequences in C. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (3.8).

(i) If q is a fixed point of T, then $\lim_{n\to\infty}\|x_n - q\|$ exists.

(ii) If $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$, then $\lim_{n\to\infty}\|Ty_n - x_n\| = 0$.

(iii) If either $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$ or $0 < \liminf_{n\to\infty}\alpha_n$ and $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n+c_n+\mu_n) < 1$, then $\lim_{n\to\infty}\|Tz_n - x_n\| = 0$.

(iv) If the following conditions

(1) $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$ and

(2) either $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$ and $\limsup_{n\to\infty} a_n < 1$ or $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n+c_n+\mu_n) < 1$ are satisfied, then $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.

Proof. Let $q \in F(T)$, by boundedness of the sequence $\{u_n\}, \{v_n\}$ and $\{w_n\}$, we can put

$$M = \max\{\sup_{n\geq 1}\|u_n - q\|,\ \sup_{n\geq 1}\|v_n - q\|,\ \sup_{n\geq 1}\|w_n - q\|\}.$$

(i) For each $n \geq 1$, we have

$$\begin{aligned}\|x_{n+1} - q\| &= \|P(\alpha_n Ty_n + \beta_n Tz_n + (1-\alpha_n-\beta_n-\lambda_n)x_n + \lambda_n w_n) - P(q)\| \\ &= \|\alpha_n Ty_n + \beta_n Tz_n + (1-\alpha_n-\beta_n-\lambda_n)x_n + \lambda_n w_n - q\| \\ &\leq \alpha_n\|Ty_n - q\| + \beta_n\|Tz_n - q\| \\ &\quad + (1-\alpha_n-\beta_n-\lambda_n)\|x_n - q\| + \lambda_n\|w_n - q\| \\ &\leq \alpha_n\|y_n - q\| + \beta_n\|z_n - q\| + (1-\alpha_n-\beta_n-\lambda_n)\|x_n - q\| + M\lambda_n, \end{aligned} \tag{3.12}$$

$$\begin{aligned}\|z_n - q\| &= \|P(a_n Tx_n + (1-a_n-\gamma_n)x_n + \gamma_n u_n) - P(q)\| \\ &\leq a_n\|Tx_n - q\| + (1-a_n-\gamma_n)\|x_n - q\| + \gamma_n\|u_n - q\| \\ &\leq a_n\|x_n - q\| + (1-a_n-\gamma_n)\|x_n - q\| + M\gamma_n \\ &\leq \|x_n - q\| + M\gamma_n \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\|y_n - q\| &= \|P(b_n Tz_n + c_n Tx_n + (1-b_n-c_n-\mu_n)x_n + \mu_n v_n) - P(q)\| \\ &\leq b_n\|Tz_n - q\| + c_n\|Tx_n - q\| \\ &\quad + (1-b_n-c_n-\mu_n)\|x_n - q\| + \mu_n\|v_n - q\| \\ &\leq b_n\|z_n - q\| + c_n\|x_n - q\| + (1-b_n-c_n-\mu_n)\|x_n - q\| + M\mu_n \\ &\leq b_n\|z_n - q\| + (1-b_n)\|x_n - q\| + M\mu_n. \end{aligned}$$

From (3.13) we get

$$\begin{aligned}\|y_n - q\| &\le b_n(\|x_n - q\| + M\gamma_n) + (1-b_n)\|x_n - q\| + M\mu_n \\ &= \|x_n - q\| + \epsilon^n_{(1)},\end{aligned} \tag{3.14}$$

where $\epsilon^n_{(1)} = Mb_n\gamma_n + M\mu_n$. Since $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\sum_{n=1}^{\infty}\mu_n < \infty$, we have $\sum_{n=1}^{\infty}\epsilon^n_{(1)} < \infty$.

From (3.12), (3.13) and (3.14) we get

$$\begin{aligned}\|x_{n+1} - q\| &\le \alpha_n(\|x_n - q\| + \epsilon^n_{(1)}) + \beta_n(\|x_n - q\| + M\gamma_n) \\ &\quad + (1-\alpha_n-\beta_n-\lambda_n)\|x_n - q\| + M\lambda_n \\ &= \alpha_n\|x_n - q\| + \alpha_n\epsilon^n_{(1)} + \beta_n\|x_n - q\| + M\beta_n\gamma_n \\ &\quad + (1-\alpha_n-\beta_n-\lambda_n)\|x_n - q\| + M\lambda_n \\ &\le \|x_n - q\| + \epsilon^n_{(2)},\end{aligned} \tag{3.15}$$

where $\epsilon^n_{(2)} = \alpha_n\epsilon^n_{(1)} + M\beta_n\gamma_n + M\lambda_n$. Since $\sum_{n=1}^{\infty}\epsilon^n_{(2)} < \infty$ we obtained from (3.15) and Lemma 2.1.3 that $\lim_{n\to\infty}\|x_n - q\|$ exists.

(ii) By (i) we have that $\lim_{n\to\infty}\|x_n - q\|$ exists for any $q \in F(T)$. It follows from (3.13) and (3.14) that $\{x_n - q\}, \{Tx_n - q\}, \{z_n - q\}, \{Tz_n - q\}, \{y_n - q\}$ and $\{Ty_n - q\}$ are bounded sequences. This allows us to put

$$\begin{aligned}K = \max\{M,\ &\sup_{n\ge1}\|x_n - q\|,\ \sup_{n\ge1}\|Tx_n - q\|,\ \sup_{n\ge1}\|z_n - q\|, \\ &\sup_{n\ge1}\|Tz_n - q\|,\ \sup_{n\ge1}\|y_n - q\|,\ \sup_{n\ge1}\|Ty_n - q\|\}.\end{aligned}$$

Since $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$. It follows from (3.13) and (3.14) that

$$\|z_n - q\|^2 \le \|x_n - q\|^2 + \epsilon^n_{(3)} \tag{3.16}$$

$$\|y_n - q\|^2 \le \|x_n - q\|^2 + \epsilon^n_{(4)}, \tag{3.17}$$

where $\epsilon^n_{(3)} = M^2\gamma_n^2 + 2MK\gamma_n$ and $\epsilon^n_{(4)} = (\epsilon^n_{(1)})^2 + 2K\epsilon^n_{(1)}$. Since $\sum_{n=1}^{\infty}\epsilon^n_{(3)} < \infty$ and $\sum_{n=1}^{\infty}\epsilon^n_{(4)} < \infty$, by Lemma 2.2.14, there is a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that

$$\|\lambda x + \beta y + \gamma z + \mu w\|^2 \le \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 + \mu\|w\|^2 - \lambda\beta g(\|x - y\|) \tag{3.18}$$

for all $x, y, z, w \in B_K$ and all $\lambda, \beta, \gamma, \mu \in [0,1]$ with $\lambda + \beta + \gamma = 1$. By (3.16), (3.17) and (3.18), we have

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &= \|P(\alpha_n Ty_n+\beta_n Tz_n \\
&\quad +(1-\alpha_n-\beta_n-\lambda_n)x_n+\lambda_n w_n)-P(q)\|^2 \\
&\leq \|\alpha_n(Ty_n-q)+\beta_n(Tz_n-q) \\
&\quad +(1-\alpha_n-\beta_n-\lambda_n)(x_n-q)+\lambda_n(w_n-q)\|^2 \\
&\leq \alpha_n\|Ty_n-q\|^2+\beta_n\|Tz_n-q\|^2 \\
&\quad +(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+\lambda_n\|w_n-q\|^2 \\
&\quad -\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|Ty_n-x_n\|) \\
&\leq \alpha_n\|y_n-q\|^2+\beta_n\|z_n-q\|^2+(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2 \\
&\quad +K^2\lambda_n-\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|Ty_n-x_n\|) \\
&\leq \alpha_n(\|x_n-q\|^2+\epsilon^n_{(4)})+\beta_n(\|x_n-q\|^2+\epsilon^n_{(3)}) \\
&\quad +(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+K^2\lambda_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|Ty_n-x_n\|) \\
&= \alpha_n\|x_n-q\|^2+\alpha_n\epsilon^n_{(4)}+\beta_n\|x_n-q\|^2+\beta_n\epsilon^n_{(3)} \\
&\quad +(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+K^2\lambda_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|Ty_n-x_n\|) \\
&\leq \|x_n-q\|^2+\epsilon^n_{(5)}-\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|Ty_n-x_n\|), \qquad (3.19)
\end{aligned}
$$

where $\epsilon^n_{(5)}=\alpha_n\epsilon^n_{(4)}+\beta_n\epsilon^n_{(3)}+K^2\lambda_n$. It is worth to note here that $\sum_{n=1}^{\infty}\epsilon^n_{(5)}<\infty$ since $\sum_{n=1}^{\infty}\epsilon^n_{(4)}<\infty, \sum_{n=1}^{\infty}\epsilon^n_{(3)}<\infty$, and $\sum_{n=1}^{\infty}\lambda_n<\infty$. Since $0<\liminf_{n\to\infty}\alpha_n\leq\limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n)<1$, there exists $n_0\in\mathbb{N}$ and $\delta_1,\delta_2\in(0,1)$ such that $0<\delta_1<\alpha_n$ and $\alpha_n+\beta_n+\lambda_n<\delta_2<1$ for all $n\geq n_0$. Hence, by (3.19), we have

$$
\begin{aligned}
\delta_1(1-\delta_2)\sum_{n=n_0}^{m}g(\|Ty_n-x_n\|) &\leq \sum_{n=n_0}^{m}(\|x_n-q\|^2-\|x_{n+1}-q\|^2)+\sum_{n=n_0}^{m}\epsilon^n_{(5)} \\
&= \|x_{n_0}-q\|^2+\sum_{n=n_0}^{m}\epsilon^n_{(5)}. \qquad (3.20)
\end{aligned}
$$

Since $\sum_{n=n_0}^{\infty}\epsilon^n_{(5)}<\infty$, by letting $m\to\infty$ in (3.20) we get $\sum_{n=n_0}^{\infty}g(\|Ty_n-x_n\|)<\infty$, and therefore $\lim_{n\to\infty}g(\|Ty_n-x_n\|)=0$. Since g is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|Ty_n-x_n\|=0$.

(iii) First, we assume that $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n)<1$. By (3.18), we have

$$
\begin{aligned}
\|x_{n+1}-q\|^2 \le\ & \alpha_n\|y_n-q\|^2+\beta_n\|z_n-q\|^2 \\
& +(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+K^2\lambda_n \\
& -\beta_n(1-\alpha_n-\beta_n-\lambda_n)g(\|Tz_n-x_n\|) \\
\le\ & \alpha_n(\|x_n-q\|^2+\epsilon^n_{(4)})+\beta_n(\|x_n-q\|^2+\epsilon^n_{(3)}) \\
& +(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+K^2\lambda_n \\
& -\beta_n(1-\alpha_n-\beta_n-\lambda_n)g(\|Tz_n-x_n\|) \\
=\ & \alpha_n\|x_n-q\|^2+\alpha_n\epsilon^n_{(4)}+\beta_n\|x_n-q\|^2+\beta_n\epsilon^n_{(3)} \\
& +(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+K^2\lambda_n \\
& -\beta_n(1-\alpha_n-\beta_n-\lambda_n)g(\|Tz_n-x_n\|) \\
\le\ & \|x_n-q\|^2+\epsilon^n_{(5)}-\beta_n(1-\alpha_n-\beta_n-\lambda_n)g(\|Tz_n-x_n\|), \quad (3.21)
\end{aligned}
$$

where $\epsilon^n_{(5)}=\alpha_n\epsilon^n_{(4)}+\beta_n\epsilon^n_{(3)}+K^2\lambda_n$. Since $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n)<1$, there exists $n_0\in\mathbb{N}$ and $\delta_1,\delta_2\in(0,1)$ such that $0<\delta_1<\beta_n$ and $\alpha_n+\beta_n+\lambda_n<\delta_2<1$ for all $n\ge n_0$. Hence, by (3.21), we have $\epsilon^n_{(5)}=\alpha_n\epsilon^n_{(4)}+\beta_n\epsilon^n_{(3)}+K^2\lambda_n$.

$$
\begin{aligned}
\delta_1(1-\delta_2)\sum_{n=n_0}^{m}g(\|Tz_n-x_n\|) &\le \sum_{n=n_0}^{m}(\|x_n-q\|^2-\|x_{n+1}-q\|^2)+\sum_{n=n_0}^{m}\epsilon^n_{(5)} \\
&= \|x_{n_0}-q\|^2+\sum_{n=n_0}^{m}\epsilon^n_{(5)}. \quad (3.22)
\end{aligned}
$$

Since $\sum_{n=n_0}^{\infty}\epsilon^n_{(5)}<\infty$, by letting $m\to\infty$ in (3.22) we get $\sum_{n=n_0}^{\infty}g(\|Tz_n-x_n\|)<\infty$, and therefore $\lim_{n\to\infty}g(\|Tz_n-x_n\|)=0$. Since g is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|Tz_n-x_n\|=0$.

Next, we assume that $0<\liminf_{n\to\infty}\alpha_n$ and $\liminf_{n\to\infty}b_n\le\limsup_{n\to\infty}(b_n+c_n+\mu_n)<1$. By (3.16) and (3.18), we have

$$
\begin{aligned}
\|y_n-q\|^2 =\ & \|P(b_nTz_n+c_nTx_n \\
& +(1-b_n-c_n-\mu_n)x_n+\mu_nv_n)-P(q)\|^2 \\
\le\ & \|b_n(Tz_n-q)+c_n(Tx_n-q) \\
& +(1-b_n-c_n-\mu_n)(x_n-q)+\mu_n(v_n-q)\|^2 \\
\le\ & b_n\|Tz_n-q\|^2+c_n\|Tx_n-q\|^2 \\
& +(1-b_n-c_n-\mu_n)\|x_n-q\|^2+\mu_n\|v_n-q\|^2 \\
& -b_n(1-b_n-c_n-\mu_n)g(\|Tz_n-x_n\|) \\
\le\ & b_n\|z_n-q\|^2+c_n\|x_n-q\|^2 \\
& +(1-b_n-c_n-\mu_n)\|x_n-q\|^2+\mu_nK^2
\end{aligned}
$$

$$
\begin{aligned}
&- b_n(1 - b_n - c_n - \mu_n)g(\|Tz_n - x_n\|) \\
&\le b_n(\|x_n - q\|^2 + \epsilon^n_{(3)}) + c_n\|x_n - q\|^2 \\
&\quad + (1 - b_n - c_n - \mu_n)\|x_n - q\|^2 + \mu_n K^2 \\
&\quad - b_n(1 - b_n - c_n - \mu_n)g(\|Tz_n - x_n\|) \\
&\le \|x_n - q\|^2 + \epsilon^n_{(6)} - b_n(1 - b_n - c_n - \mu_n)g(\|Tz_n - x_n\|), \qquad (3.23)
\end{aligned}
$$

where $\epsilon^n_{(6)} = b_n\epsilon^n_{(3)} + \mu_n K^2$.

By (3.16), (3.18) and (3.23), we also have

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &= \|P(\alpha_n Ty_n + \beta_n Tz_n \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n) - P(q)\|^2 \\
&\le \|\alpha_n(Ty_n - q) + \beta_n(Tz_n - q) \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)(x_n - q) + \lambda_n(w_n - q)\|^2 \\
&\le \alpha_n\|y_n - q\|^2 + \beta_n\|z_n - q\|^2 + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + K^2\lambda_n \\
&\le \alpha_n(\|x_n - q\|^2 + \epsilon^n_{(6)} - b_n(1 - b_n - c_n - \mu_n)g(\|Tz_n - x_n\|)) \\
&\quad + \beta_n(\|x_n - q\|^2 + \epsilon^n_{(3)}) + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + K^2\lambda_n \\
&= \alpha_n\|x_n - q\|^2 + \alpha_n\epsilon^n_{(6)} - \alpha_n b_n(1 - b_n - c_n - \mu_n)g(\|Tz_n - x_n\|) \\
&\quad + \beta_n\|x_n - q\|^2 + \beta_n\epsilon^n_{(3)} + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + K^2\lambda_n \\
&\le \|x_n - q\|^2 + \epsilon^n_{(7)} - \alpha_n b_n(1 - b_n - c_n - \mu_n)g(\|Tz_n - x_n\|), \qquad (3.24)
\end{aligned}
$$

where $\epsilon^n_{(7)} = \alpha_n\epsilon^n_{(6)} + \beta_n\epsilon^n_{(3)} + K^2\lambda_n$.

It is worth to note here that $\sum_{n=1}^{\infty}\epsilon^n_{(7)} < \infty$ since $\sum_{n=1}^{\infty}\epsilon^n_{(6)} < \infty, \sum_{n=1}^{\infty}\epsilon^n_{(3)} < \infty$, and $\sum_{n=1}^{\infty}\lambda_n < \infty$.

By our assumption $0 < \liminf_{n\to\infty}\alpha_n$ and $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, there exists $n_0 \in \mathbb{N}$ and $\delta_1, \delta_2 \in (0,1)$ such that $0 < \delta_1 < \alpha_n$, $0 < \delta_1 < b_n$ and $b_n + c_n + \mu_n < \delta_2 < 1$ for all $n \ge n_0$. Hence, by (3.24), we have

$$
\begin{aligned}
\delta_1^2(1 - \delta_2)\sum_{n=n_0}^{m} g(\|Tz_n - x_n\|) &\le \sum_{n=n_0}^{m}(\|x_n - q\|^2 - \|x_{n+1} - q\|^2) + \sum_{n=n_0}^{m}\epsilon^n_{(7)} \\
&= \|x_{n_0} - q\|^2 + \sum_{n=n_0}^{m}\epsilon^n_{(7)}. \qquad (3.25)
\end{aligned}
$$

Since $\sum_{n=n_0}^{\infty}\epsilon^n_{(7)} < \infty$, by letting $m \to \infty$ in (3.25) we get $\sum_{n=n_0}^{\infty} g(\|Tz_n - x_n\|) < \infty$, and therefore $\lim_{n\to\infty} g(\|Tz_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty}\|Tz_n - x_n\| = 0$.

(iv) Suppose that the conditions (1) and (2) are satisfied. Then by (ii) and (iii), we have

$$\lim_{n\to\infty} \|Ty_n - x_n\| = 0 \text{ and } \lim_{n\to\infty} \|Tz_n - x_n\| = 0. \tag{3.26}$$

From $z_n = P(a_nTx_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n)$ and $y_n = P(b_nTz_n + c_nTx_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n)$, we have $\|z_n - x_n\| \leq a_n\|Tx_n - x_n\| + \gamma_n\|u_n - x_n\|$ and $\|y_n - x_n\| \leq b_n\|Tz_n - x_n\| + c_n\|Tx_n - x_n\| + \mu_n\|v_n - x_n\|$.

It follows that

$$\begin{aligned}
\|Tx_n - x_n\| &\leq \|Tx_n - Tz_n\| + \|Tz_n - x_n\| \\
&\leq \|x_n - z_n\| + \|Tz_n - x_n\| \\
&\leq a_n\|Tx_n - x_n\| + \gamma_n\|u_n - z_n\| + \|Tz_n - x_n\|,
\end{aligned}$$

which implies

$$(1 - a_n)\|Tx_n - x_n\| \leq \gamma_n\|u_n - z_n\| + \|Tz_n - x_n\|.$$

If $\limsup_{n\to\infty} a_n < 1$, this together with (3.26) and $\lim_{n\to\infty} \gamma_n = 0$ imply that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.

If $\limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, there exists a positive integer N_0 and $\eta \in (0, 1)$ such that

$$c_n \leq b_n + c_n + \mu_n < \eta \quad \forall n \geq N_0.$$

Then for $n \geq N_0$, we have

$$\begin{aligned}
\|Tx_n - x_n\| &\leq \|Tx_n - Ty_n\| + \|Ty_n - x_n\| \\
&\leq \|x_n - y_n\| + \|Ty_n - x_n\| \\
&\leq b_n\|Tz_n - x_n\| + c_n\|Tx_n - x_n\| \\
&\quad + \mu_n\|v_n - x_n\| + \|Ty_n - x_n\| \\
&\leq b_n\|Tz_n - x_n\| + \eta\|Tx_n - x_n\| \\
&\quad + \mu_n\|v_n - x_n\| + \|Ty_n - x_n\|.
\end{aligned}$$

Hence

$$(1 - \eta)\|Tx_n - x_n\| \leq b_n\|Tz_n - x_n\| + \mu_n\|v_n - x_n\| + \|Ty_n - x_n\|.$$

This together with (3.26) and the fact that $\mu_n \to 0$ as $n \to \infty$ imply $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. □

Theorem 3.1.7 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be sequences of real numbers in $[0,1]$ with $a_n + \gamma_n \in [0,1], b_n + c_n + \mu_n \in [0,1]$ and $\alpha_n + \beta_n + \lambda_n \in [0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \lambda_n < \infty$. If*

(i) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$ *or*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$ *and* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$,

then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by the iterative scheme (3.8) converge strongly to a fixed point of T.

Proof. It follows from Lemma 3.1.6(i) that $\{x_n\}$ is bounded. Again by Lemma 3.1.6 , we have

$$\begin{aligned} \lim_{n\to\infty} \|Ty_n - x_n\| &= 0, \\ \lim_{n\to\infty} \|Tz_n - x_n\| &= 0, \\ \lim_{n\to\infty} \|Tx_n - x_n\| &= 0. \end{aligned} \tag{3.27}$$

Since T is completely continuous and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Hence, by $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, it follows that $\{x_{n_k}\}$ converges. Let $\lim_{n\to\infty} x_{n_k} = q$. By continuity of T and (3.27) we have that $Tq = q$, so q is a fixed point of T. By Lemma 3.1.6 (i), $\lim_{n\to\infty} \|x_n - q\|$ exists. But $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$, so $\lim_{n\to\infty} \|x_n - q\| = 0$. By (3.27), we have

$$\begin{aligned} \|y_n - x_n\| &= \|P(b_nTz_n + c_nTx_n \\ &\quad + (1 - b_n - c_n - \mu_n)x_n + \mu_nv_n) - P(x_n)\| \\ &\leq b_n\|Tz_n - x_n\| + c_n\|Tx_n - x_n\| + \mu_n\|v_n - x_n\| \\ &\to 0 \ \ (\text{as } n \to \infty), \end{aligned}$$

and

$$\begin{aligned} \|z_n - x_n\| &= \|P(a_nTx_n + (1 - a_n - \gamma_n)x_n + \gamma_nu_n) - P(x_n)\| \\ &\leq a_n\|Tx_n - x_n\| + \gamma_n\|u_n - x_n\| \\ &\to 0 \ \ (\text{as } n \to \infty). \end{aligned}$$

It follows that $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$. □

If T is a self-mapping, then the iterative scheme (3.8) reduces to that of (3.10) and the following result is directly obtained by Theorem 3.1.7.

Theorem 3.1.8 *Let X be a uniformly convex Banach space, and C a nonempty closed convex subset of X. Let T be a completely continuous nonexpansive self-mapping of C with $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \geq 1$. If*

(i) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$ *or*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$ *and* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$,

then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by the iterations (3.10) converge strongly to a fixed point of T.

When $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 3.1.7 , the following result is obtained.

Theorem 3.1.9 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{\alpha_n\}$ be real sequences in $[0,1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, *and*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

For a given $x_1 \in C$, define

$$\begin{aligned} z_n &= P(a_n T x_n + (1 - a_n)x_n) \\ y_n &= P(b_n T z_n + (1 - b_n)x_n), \quad n \geq 1 \\ x_{n+1} &= P(\alpha_n T y_n + (1 - \alpha_n)x_n). \end{aligned}$$

Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

When $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 3.1.7 , we obtain the following result.

Theorem 3.1.10 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{b_n\}, \{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$,*and*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

For a given $x_1 \in C$, define

$$\begin{aligned} y_n &= P(b_n T x_n + (1 - b_n)x_n) \\ x_{n+1} &= P(\alpha_n T y_n + (1 - \alpha_n)x_n), \qquad n \geq 1. \end{aligned}$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to a fixed point of T.

The mapping $T : C \to X$ with $F(T) \neq \emptyset$ is said to satisfy *condition*(A) [32] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in C$

$$\|x - Tx\| \geq f(d(x, F(T))).$$

The following result gives strong convergence theorem for nonexpansive nonself-mapping in a uniformly convex Banach space satisfying the *condition*(A).

Theorem 3.1.11 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}$ and $\{\lambda_n\}$ be sequences of real numbers in $[0,1]$ with $a_n+\gamma_n \in [0,1], b_n+c_n+\mu_n \in [0,1]$ and $\alpha_n+\beta_n+\lambda_n \in [0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$. Suppose that T satisfies condition(A). If*

(i) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \ \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$ *or*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$ *and* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$,

then the sequences $\{x_n\}$ defined by the iterative scheme (3.8) converge strongly to some fixed point of T.

Proof. Let $q \in F(T)$. Then, as in Lemma 3.1.6, $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - q\|$ exists and

$$\|x_{n+1} - q\| \leq \|x_n - q\| + \epsilon_{(2)}^n,$$

where $\sum_{n=1}^{\infty} \epsilon_{(2)}^n < \infty$ for all $n \geq 1$. This implies that $d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + \epsilon_{(2)}^n$ and so, by Lemma 2.1.3, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Also, by Lemma 3.1.6, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since T satisfies *condition*(A), we conclude that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence.

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} \epsilon_{(2)}^n < \infty$, given any $\epsilon > 0$, there exists a natural number n_0 such that $d(x_n, F(T)) < \frac{\epsilon}{4}$ and $\sum_{k=n_0}^{n} \epsilon_{(2)}^k < \frac{\epsilon}{2}$ for all $n \geq n_0$. So we can find $y^* \in F(T)$ such that $\|x_{n_0} - y^*\| < \frac{\epsilon}{4}$. For $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &= \|x_{n+m} - y^*\| + \|x_n - y^*\| \\ &\leq \|x_{n_0} - y^*\| + \|x_{n_0} - y^*\| + \sum_{k=n_0}^{n} \epsilon_{(2)}^k \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since X is complete. Let $\lim_{n\to\infty} x_n = u$. Then $d(u, F(T)) = 0$. It follows that $u \in F(T)$. This completes the proof. □

For $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, the iterative scheme (3.8) reduces to that of (3.9) and the following result is directly obtained by Theorem 3.1.11.

Theorem 3.1.12 ([33, Theorem 3.6]) *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{b_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Suppose that T satisfies condition(A). Then the sequences $\{x_n\}$ defined by the iterative scheme (3.9) converge strongly to some fixed point of T.*

In the next result, we prove weak convergence of the iterative scheme (3.8) for nonexpansive nonself-mapping in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.1.13 *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\lambda_n\}$ be sequences of real numbers in $[0,1]$ with $a_n + \gamma_n, b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$. If*

(i) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \ \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$ *or*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$ *and* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$,

then the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by the iterative scheme (3.8) converge weakly to a fixed point of T.

Proof. It follows from Lemma 3.1.6 that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ and $\lim_{n\to\infty} \|Tz_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 2.3.5, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 2.3.5, $u, v \in F(T)$. By Lemma 3.1.6 (i), $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 2.2.16 that $u = v$. Therefore $\{x_n\}$ converges weakly to a fixed point u of T. Since $\|y_n - x_n\| \leq b_n\|Tz_n - x_n\| + c_n\|Tx_n - x_n\| + \mu_n\|v_n - x_n\| \to 0$ (as $n \to \infty$) and $\|z_n - x_n\| \leq a_n\|Tx_n - x_n\| + \gamma_n\|u_n - x_n\| \to 0$ (as $n \to \infty$) and $x_n \to u$ weakly as $n \to \infty$, it follows that $y_n \to u$ and $z_n \to u$ weakly as $n \to \infty$. □

3.2 Common Fixed Points of Nonexpansive Mappings

We introduce the following implicit iterative scheme for a finite family of nonexpansive mappings in a Banach space. The scheme is defined as follows:

Let X be a normed linear space, let C be a nonempty convex subset of X, let $\{T_i : i \in J\}$ (here $J = \{1, 2, \dots, N\}$) be a finite family of nonexpansive self-mappings of C. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0, 1]$. Then for an arbitrary $x_0 \in C$, the sequence $\{x_n\}$ is generated as follows:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + \beta_1 T_1 x_0 + (1 - \alpha_1 - \beta_1) T_1 x_1, \\ x_2 &= \alpha_2 x_1 + \beta_2 T_2 x_1 + (1 - \alpha_2 - \beta_2) T_2 x_2, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + \beta_N T_N x_{N-1} + (1 - \alpha_N - \beta_N) T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} T_{N+1} x_N + (1 - \alpha_{N+1} - \beta_{N+1}) T_{N+1} x_{N+1}, \\ &\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_{n-1} + (1 - \alpha_n - \beta_n) T_n x_n, \ \forall n \geq 1, \qquad (3.28)$$

where $T_n = T_{n(mod N)}$ (here the $modN$ function takes values in J).

We note that Xu and Ori's iteration is a special case of the above implicit iterative scheme. If $\beta_n \equiv 0$, then (3.28) reduces to Xu and Ori's iteration [46].

In this section, we prove weak and strong convergence of the implicit iteration process (3.28) to a common fixed point for a finite family of nonexpansive mappings in a uniformly convex Banach space.

Lemma 3.2.1 *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in J\}$ be N nonexpansive self-mappings of C with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ (here $F(T_i)$ denotes the set of fixed points of T_i). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ such that $\alpha_n + \beta_n$ is in $[0, 1]$ for all $n \geq 1$ and $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (3.28).*

(i) If $x^* \in F$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists.
(ii) For all $l \in J$, $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$.

Proof. Let $x^* \in F$. (i) For each $n \geq 1$, we have

$$\begin{aligned}\|x_n - x^*\| &= \|\alpha_n x_{n-1} + \beta_n T_n x_{n-1} + (1-\alpha_n-\beta_n)T_n x_n - x^*\| \\ &\le \alpha_n\|x_{n-1}-x^*\| + \beta_n\|T_n x_{n-1} - x^*\| + (1-\alpha_n-\beta_n)\|T_n x_n - x^*\| \\ &\le \alpha_n\|x_{n-1}-x^*\| + \beta_n\|x_{n-1} - x^*\| + (1-\alpha_n-\beta_n)\|x_n - x^*\| \\ &= (\alpha_n+\beta_n)\|x_{n-1}-x^*\| + (1-\alpha_n-\beta_n)\|x_n - x^*\|.\end{aligned}$$

This implies that

$$\|x_n - x^*\| \le \|x_{n-1} - x^*\|.$$

It implies by Lemma 2.1.3 that $\lim_{n\to\infty}\|x_n - x^*\|$ exists.

(ii) We shall show that $\lim_{n\to\infty}\|x_{n-1} - T_n x_n\| = 0$. Using Lemma 2.2.13, we have

$$\begin{aligned}\|x_n - x^*\|^2 &= \|\alpha_n x_{n-1} + \beta_n T_n x_{n-1} + (1-\alpha_n-\beta_n)T_n x_n - x^*\|^2 \\ &= \|\alpha_n(x_{n-1}-x^*) + \beta_n(T_n x_{n-1} - x^*) + (1-\alpha_n-\beta_n)(T_n x_n - x^*)\|^2 \\ &\le \alpha_n\|x_{n-1}-x^*\|^2 + \beta_n\|T_n x_{n-1} - x^*\|^2 + (1-\alpha_n-\beta_n)\|T_n x_n - x^*\|^2 \\ &\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|x_{n-1} - T_n x_n\|) \\ &\le \alpha_n\|x_{n-1}-x^*\|^2 + \beta_n\|x_{n-1} - x^*\|^2 + (1-\alpha_n-\beta_n)\|x_n - x^*\|^2 \\ &\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|x_{n-1} - T_n x_n\|).\end{aligned}$$

Hence

$$\begin{aligned}\alpha_n(1-\alpha_n-\beta_n)g(\|x_{n-1} - T_n x_n\|)\| &\le \alpha_n\|x_{n-1}-x^*\|^2 + \beta_n\|x_{n-1}-x^*\|^2 \\ &\quad + (1-\alpha_n-\beta_n)\|x_n - x^*\|^2 - \|x_n - x^*\|^2 \\ &\le \alpha_n\|x_{n-1}-x^*\|^2 + \beta_n\|x_{n-1}-x^*\|^2 \\ &\quad + (1-\alpha_n-\beta_n)\|x_{n-1} - x^*\|^2 - \|x_n - x^*\|^2 \\ &= \|x_{n-1}-x^*\|^2 - \|x_n - x^*\|^2.\end{aligned}$$

If $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$, then there exists a positive integer n_0 and $\eta, \eta' \in (0,1)$ such that $0 < \eta < \alpha_n$ and $\alpha_n+\beta_n < \eta' < 1$, $\forall n \ge n_0$. Hence

$$\eta(1-\eta')g(\|x_{n-1} - T_n x_n\|) \le \|x_{n-1}-x^*\|^2 - \|x_n - x^*\|^2, \quad \forall n \ge n_0.$$

It follows that for $m \ge n_0$,

$$\sum_{n=n_0}^{m} g(\|x_{n-1} - T_n x_n\|) \le \frac{1}{\eta(1-\eta')}\sum_{n=n_0}^{m}(\|x_{n-1}-x^*\|^2 - \|x_n - x^*\|^2).$$

We get $\sum_{n=n_0}^{\infty} g(\|x_{n-1} - T_n x_n\|) < \infty$ as $m \to \infty$. This implies that $\lim_{n\to\infty} g(\|x_{n-1} - T_n x_n\|) = 0$. Since g is strictly increasing, continuous and $g(0) = 0$, we have $\lim_{n\to\infty}\|x_{n-1} - T_n x_n\| = 0$. Since T_n is nonexpansive, we have

$$\begin{aligned}\|x_n - x_{n-1}\| &= \|\alpha_n x_{n-1} + \beta_n T_n x_{n-1} + (1-\alpha_n-\beta_n)T_n x_n - x_{n-1}\| \\ &= \|\alpha_n(x_{n-1}-x_{n-1}) + \beta_n(T_n x_{n-1} - x_{n-1}) \\ &\quad + (1-\alpha_n-\beta_n)(T_n x_n - x_{n-1})\| \\ &= \|\beta_n(T_n x_{n-1} - T_n x_n + T_n x_n - x_{n-1}) \\ &\quad + (1-\alpha_n-\beta_n)(T_n x_n - x_{n-1})\| \\ &\leq \beta_n\|T_n x_{n-1} - T_n x_n\| + \beta_n\|T_n x_n - x_{n-1}\| \\ &\quad + (1-\alpha_n-\beta_n)\|T_n x_n - x_{n-1}\| \\ &\leq \beta_n\|x_{n-1} - x_n\| + (1-\alpha_n)\|T_n x_n - x_{n-1}\| \\ &\leq \beta_n\|x_{n-1} - x_n\| + \|T_n x_n - x_{n-1}\| \\ &= \beta_n\|x_{n-1} - x_n\| + \|x_{n-1} - T_n x_n\|.\end{aligned}$$

This implies that

$$(1-\beta_n)\|x_n - x_{n-1}\| \leq \|x_{n-1} - T_n x_n\|.$$

By $\limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$, there exists a positive integer n_0 and $\beta \in (0,1)$ such that $\beta_n \leq \alpha_n + \beta_n < \beta$, $\forall n \geq n_0$. Hence, we have

$$(1-\beta)\|x_n - x_{n-1}\| \leq \|x_{n-1} - T_n x_n\|.$$

Let $n \to \infty$. It follows that $\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0$. Also $\lim_{n\to\infty}\|x_n - x_{n+l}\| = 0$ for all $l \in J$. Since $\|x_n - T_n x_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|$, we have $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. Now since for all $l \in J$

$$\begin{aligned}\|x_n - T_{n+l}x_n\| &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|T_{n+l}x_{n+l} - T_{n+l}x_n\| \\ &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|x_{n+l} - x_n\|,\end{aligned}$$

we have that $\lim_{n\to\infty}\|x_n - T_{n+l}x_n\| = 0$ for all $l \in J$. Since for each $l \in J$, $\{\|x_n - T_l x_n\|\}$ is a subset of $\cup_{l=1}^N\{\|x_n - T_{n+l}x_n\|\}$, we have $\lim_{n\to\infty}\|x_n - T_l x_n\| = 0$ for all $l \in J$. This completes the proof. □

Theorem 3.2.2 *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in J\}$ be N nonexpansive self-mappings of C with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Suppose that $\{T_i : i \in J\}$ satisfies condition(B). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n$ is in $[0,1]$ for all $n \geq 1$ and $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (3.28). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$.*

Proof. Let $x^* \in F$. By Lemma 3.2.1 (i), we have $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - x^*\|$ exists and $\|x_n - x^*\| \leq \|x_{n-1} - x^*\|$ for all $n \geq 1$. This implies that $d(x_n, F) \leq d(x_{n-1}, F)$, so $\lim_{n\to\infty} d(x_n, F)$ exists. Also, by Lemma 3.2.1 (ii), $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$ for all $l \in J$. Since $\{T_i : i \in J\}$ satisfies *condition*(B), we conclude that $\lim_{n\to\infty} d(x_n, F) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F) = 0$, for any $\epsilon > 0$, there exists a natural number n_0 such that $d(x_n, F) < \frac{\epsilon}{2}$ for all $n \geq n_0$. So we can find $y^* \in F$ such that $\|x_{n_0} - y^*\| < \frac{\epsilon}{2}$. For all $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - y^*\| + \|x_n - y^*\| \\ &\leq \|x_{n_0} - y^*\| + \|x_{n_0} - y^*\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since X is complete. Let $\lim_{n\to\infty} x_n = z^*$. Then $z^* \in C$. It remains to show that $z^* \in F$. Let $\epsilon' > 0$ be given. Then there exists $n_1 \in \mathbb{N}$ such that $\|x_n - z^*\| < \frac{\epsilon'}{4}$, $\forall n \geq n_1$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists $n_2 \in \mathbb{N}$ and $n_2 \geq n_1$ such that for all $n \geq n_2$ we have $d(x_n, F) < \frac{\epsilon'}{4}$ and in particular we have $d(x_{n_2}, F) < \frac{\epsilon'}{4}$. Therefore, there exists $w^* \in F$ such that $\|x_{n_2} - w^*\| < \frac{\epsilon'}{4}$. For any $i \in J$ and $n \geq n_2$, we have

$$\begin{aligned}\|T_i z^* - z^*\| &\leq \|T_i z^* - w^*\| + \|w^* - z^*\| \\ &\leq 2\|w^* - z^*\| \\ &\leq 2(\|w^* - x_{n_2}\| + \|x_{n_2} - z^*\|) \\ &< 2(\frac{\epsilon}{4} + \frac{\epsilon}{4}) = \epsilon.\end{aligned}$$

This implies that $T_i z^* = z^*$. Hence $z^* \in F(T_i)$ for all $i \in J$ and so $z^* \in F$. This completes the proof. □

We recall that a mapping $T : C \to C$ is called semi-compact(or hemicompact) if any sequence $\{x_n\}$ in C satisfying $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ has a convergent subsequence.

Theorem 3.2.3 *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in J\}$ be N nonexpansive self-mappings of C with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Suppose that one of the mappings in $\{T_i : i \in J\}$ is semi-compact. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n$ is in $[0,1]$ for all $n \geq 1$ and $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (3.28). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$.*

Proof. Suppose that T_{i_0} is semi-compact for some $i_0 \in J$. By Lemma 3.2.1 (ii), we have $\lim_{n\to\infty} \|x_n - T_{i_0}x_n\| = 0$. So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^* \in C$ as $j \to \infty$. Now Lemma 3.2.1 (ii) guarantees that $\lim_{n\to\infty} \|x_{n_j} - T_l x_{n_j}\| = 0$ for all $l \in J$ and so $\|x^* - T_l x^*\| = 0$ for all $l \in J$. This implies that $x^* \in F$. By Lemma 3.2.1 (i), $\lim_{n\to\infty} \|x_n - x^*\|$ exists and then

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{j\to\infty} \|x_{n_j} - x^*\| = 0.$$

This completes the proof. □

For $\beta_n \equiv 0$, the iterative scheme (3.28) reduces to that of (1.3) and the following results are directly obtained by Theorem 3.2.2 and Theorem 3.2.3, respectively.

Theorem 3.2.4 ([8, Theorem 3.2]) *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in J\}$ be N nonexpansive self-mappings of C with $F = \cap_{i=1}^{N} F(T_i) \neq \varnothing$. Suppose that $\{T_i : i \in J\}$ satisfies condition (B). Let $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad \forall n \geqslant 1.$$

Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$.

Theorem 3.2.5 ([8, Theorem 3.3]) *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in J\}$ be N nonexpansive self-mappings of C with $F = \cap_{i=1}^{N} F(T_i) \neq \varnothing$. Suppose that one of the mappings in $\{T_i : i \in J\}$ is semi-compact. Let $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad \forall n \geqslant 1.$$

Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$.

In the next results, we prove weak convergence of the sequence $\{x_n\}$ defined by (3.28) in uniformly convex Banach space satisfying *Opial's condition.*

Lemma 3.2.6 *Let X be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in J\}$ be N nonexpansive self-mappings of C with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n$ is in $[0,1]$ for all $n \geq 1$ and $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (3.28). Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i : i \in J\}$.*

Proof. It follows from Lemma 3.2.1 (ii) that $\lim_{n\to\infty}\|x_n - T_l x_n\| = 0$ for all $l \in J$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to x^*$ weakly as $n \to \infty$, without loss of generality. By Lemma 2.3.5, we have $x^* \in F(T_i)$ for all $i \in J$. Hence $x^* \in F$. Suppose that there exist subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to y^* and z^*, respectively. By Lemma 2.3.5, $y^*, z^* \in F$. By Lemma 3.2.1 (i), we have $\lim_{n\to\infty}\|x_n - y^*\|$ and $\lim_{n\to\infty}\|x_n - z^*\|$ exist. It follows from Lemma 2.2.16 we have $y^* = z^*$. Therefore $\{x_n\}$ converges weakly to a common fixed point x^* in F. □

Finally, we will prove weak convergence of the sequence $\{x_n\}$ defined by (3.28) in a uniformly convex Banach space X whose its dual X^* has the Kadec-Klee property.

Theorem 3.2.7 *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in J\}$ be N nonexpansive self-mappings of C with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n$ is in $[0,1]$ for all $n \geq 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (3.28). Then for all $y^*, z^* \in F$, the limit $\lim_{n\to\infty}\|tx_n + (1-t)y^* - z^*\|$ exists for all $t \in [0,1]$.*

Proof. It follows from Lemma 3.2.1 (i) that the sequence $\{x_n\}$ is bounded. Then there exists $R > 0$ such that $\{x_n\} \subset B_R \cap C$. Let $a_n(t) = \|tx_n - (1-t)y^* - z^*\|$, where $t \in (0,1)$. Then $\lim_{n\to\infty} a_n(0) = \|y^* - z^*\|$ and by Lemma 3.2.1 (i), $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty}\|x_n - z^*\|$ exists. So we let $\lim_{n\to\infty}\|x_n - z^*\| = r$ for some positive number r. Let $x \in C$. We note that for all $i = 1, 2, \cdots, N, N+1$, the mappings

$$S_{x,i-1} := \alpha_i x + \beta_i T_i x + (1 - \alpha_i - \beta_i) T_i$$

are contractions. It follows from the Banach contraction principle that there exists a unique fixed point $y_{x,i-1}$ of $S_{x,i-1}$ for each i. Hence, we can define $G_n : C \to C$ by

$$G_n x = y_{x,n}, \quad \forall x \in C,\ n \geq 0.$$

Using G_n, which can be written the following compact form:

$$G_n x = \alpha_{n+1} x + \beta_{n+1} T_{n+1} x + (1 - \alpha_{n+1} - \beta_{n+1}) T_{n+1} G_n x,$$

where $T_n = T_{n (mod\ N)}$. By the definition of G_n, it easy to see that $\|G_n w - G_n z\| \leq \|w - z\|$ for each $w, z \in C$. This implies that G_n is a nonexpansive mapping for all $n \geq 0$. Moreover, we have

$$\begin{aligned} \|G_n x_n - x_{n+1}\| &= \|\alpha_{n+1} x_n + \beta_{n+1} T_{n+1} x_n + (1 - \alpha_{n+1} - \beta_{n+1}) T_{n+1} G_n x_n - x_{n+1}\| \\ &= \|(1 - \alpha_{n+1} - \beta_{n+1}) T_{n+1} G_n x_n - (1 - \alpha_{n+1} - \beta_{n+1}) T_{n+1} x_{n+1}\| \\ &\leq (1 - \alpha_{n+1} - \beta_{n+1}) \|G_n x_n - x_{n+1}\|. \end{aligned}$$

This implies that $G_n x_n = x_{n+1}$ for all $n \geq 0$. Now, for $x^* \in F$, we have

$$\begin{aligned}
\|G_n x^* - x^*\| &= \|\alpha_{n+1}x^* + \beta_{n+1}T_{n+1}x^* + (1-\alpha_{n+1}-\beta_{n+1})T_{n+1}G_n x^* - x^*\| \\
&\le \beta_{n+1}\|T_{n+1}x^* - x^*\| + (1-\alpha_{n+1}-\beta_{n+1})\|T_{n+1}G_n x^* - x^*\| \\
&\le (1-\alpha_{n+1}-\beta_{n+1})\|G_n x^* - x^*\|
\end{aligned}$$

and so $G_n x^* = x^*$ for all $n \ge 0$. Set $H_{n,m} := G_{n+m-1}G_{n+m-2}\cdots G_n$, $n, m \ge 1$ and $b_{n,m} = \|H_{n,m}(tx_n + (1-t)y^*) - (tH_{n,m}x_n + (1-t)y^*)\|$, where $0 \le t \le 1$. It is easy to see that $H_{n,m}x_n = x_{n+m}$ and $H_{n,m}x^* = x^*$ for all $x^* \in F$. It follows from Lemma 2.3.7 that

$$\begin{aligned}
b_{n,m} &= \|H_{n,m}(tx_n + (1-t)y^*) - (tH_{n,m}x_n + (1-t)y^*)\| \\
&\le \gamma^{-1}(\|x_n - y^*\| - \|H_{n,m}x_n - H_{n,m}y^*\|) \\
&= \gamma^{-1}(\|x_n - y^*\| - \|x_{n+m} - y^*\|).
\end{aligned}$$

Hence $\gamma(b_{n,m}) \le \|x_n - y^*\| - \|x_{n+m} - y^*\|$. This implies that $\lim_{n,m\to\infty}\gamma(b_{n,m}) = 0$. By the property of γ, we obtain that $\lim_{n,m\to\infty} b_{n,m} = 0$. Observe that

$$\begin{aligned}
a_{n+m}(t) &= \|tx_{n+m} + (1-t)y^* - z^*\| \\
&\le \|H_{n,m}(tx_n + (1-t)y^*) - (tH_{n,m}x_n + (1-t)y^*)\| \\
&+ \|H_{n,m}(tx_n + (1-t)y^*) - z^*\| \\
&\le b_{n,m} + \|tx_n + (1-t)y^* - z^*\| = b_{n,m} + a_n(t).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\limsup_{m\to\infty} a_m(t) &= \limsup_{m\to\infty} a_{n+m}(t) \\
&\le \limsup_{m\to\infty}(b_{n,m} + a_n(t)) \\
&\le \gamma^{-1}(\|x_n - y^*\| - \lim_{m\to\infty}\|x_m - y^*\|) + a_n(t)
\end{aligned}$$

and $\limsup_{n\to\infty} a_n(t) \le \liminf_{n\to\infty} a_n(t)$. This implies that $\lim_{n\to\infty} a_n(t)$ exists for all $t \in [0,1]$. This completes the proof. □

Theorem 3.2.8 *Let X be a uniformly convex Banach space such that its dual X^* has the Kadec-Klee property and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in J\}$ be N nonexpansive self-mappings of C with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n$ is in $[0,1]$ for all $n \ge 1$ and $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (3.28). Then $\{x_n\}$ converges weakly to some common fixed point of $\{T_i : i \in J\}$.*

Proof. It follows from Lemma 3.2.1 (i) that the sequence $\{x_n\}$ is bounded. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to a point $z^* \in C$. By the Lemma 3.2.1 (ii), we have $\lim_{k\to\infty}\|x_{n_k} - T_l x_{n_k}\| = 0$. Now using

Lemma 2.3.5, we have $(I - T_l)z^* = 0$, that is $T_l z^* = z^*$ for all $l \in J$. Thus $z^* \in F$. Next we prove that $\{x_n\}$ converges weakly to z^*. Suppose that $\{x_{n_j}\}$ is another subsequence of $\{x_n\}$ converging weakly to some y^*. Then $y^* \in C$ and so $z^*, y^* \in \omega_w(x_n) \cap F$. By Theorem 3.2.7, $\lim_{n\to\infty} \|tx_n + (1-t)y^* - z^*\|$ exists for all $t \in [0, 1]$. It follows from Lemma 2.3.6, we have $z^* = y^*$. As a result, $\omega_w(x_n)$ is a singleton, and so $\{x_n\}$ converges weakly to some fixed point in F.

□