Chapter 4

Fixed Point Iterations for Asymptotically Nonexpansive Mappings

4.1 Weak and Strong Convergence to a Fixed Point of Asymptotically Nonexpansive Mapping

A new class of three-step iterative scheme is introduced and studied in this section. The scheme is defined as follows.

Let X be a normed space, C a nonempty convex subset of X, $P : X \to C$ a nonexpansive retraction of X onto C, and $T : C \to X$ a given mapping. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= P((1 - a_n - b_n)x_n + a_nT(PT)^{n-1}x_n + b_nu_n), \\ y_n &= P((1 - c_n - d_n)z_n + c_nT(PT)^{n-1}x_n + d_nv_n), \\ x_{n+1} &= P((1 - \alpha_n - \beta_n)y_n + \alpha_nT(PT)^{n-1}x_n + \beta_nw_n), \quad n \geq 1, \end{aligned} \tag{4.1}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}, \{\beta_n\}$ are appropriate sequences in $[0, 1]$.

The iterative schemes (4.1) are called the new three-step iterations with errors for asymptotically nonexpansive nonself-mappings.

In this section, we prove weak and strong convergence theorems for the new three-step iterative scheme (4.1) for asymptotically nonexpansive nonself-mapping in a uniformly convex Banach space.

Definition 4.1.1 ([7]) Let X be a real normed linear space and let C be a nonempty subset of X. Let $P : X \to C$ be the nonexpansive retraction of X onto C. A map $T : C \to X$ is said to be asymptotically nonexpansive nonself-mapping if there exists a sequence $k_n, k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$, such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x-y\|, \quad \forall x, y \in C,\ n \geq 1.$$

In order to prove our main results, the following lemma is needed.

Lemma 4.1.2 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be an asymptotically nonexpansive nonself-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}$, $\{c_n\}$ and $\{d_n\}$ be real sequences in $[0,1]$ such that $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (4.1).*

(i) If p is a fixed point of T, then $\lim_{n\to\infty}\|x_n - p\|$ exists.

(ii) If $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$, then $\lim_{n\to\infty}\|T(PT)^{n-1}x_n - y_n\| = 0$.

(iii) If $0 < \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$ and $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$, then $\lim_{n\to\infty}\|T(PT)^{n-1}x_n - z_n\| = 0$.

(iv) If $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$ and $\limsup_{n\to\infty} a_n < 1$ then $\lim_{n\to\infty}\|T(PT)^{n-1}x_n - x_n\| = 0$.

Proof.(i) Let $p \in F(T)$, and

$$\begin{aligned} M_1 &= \sup\{\|u_n - p\| : n \geq 1\}, \\ M_2 &= \sup\{\|v_n - p\| : n \geq 1\}, \\ M_3 &= \sup\{\|w_n - p\| : n \geq 1\}, \\ M &= \max\{M_i : i = 1,2,3\}. \end{aligned}$$

Using (4.1) for each $n \geq 1$, we have

$$\begin{aligned} \|z_n - p\| &= \|P((1-a_n-b_n)x_n + a_nT(PT)^{n-1}x_n + b_nu_n) - P(p)\| \\ &\leq \|((1-a_n-b_n)x_n + a_nT(PT)^{n-1}x_n + b_nu_n) - p\| \\ &= \|(1-a_n-b_n)(x_n - p) + a_n(T(PT)^{n-1}x_n - p) \\ &\quad + b_n(u_n - p)\| \end{aligned}$$

$$
\begin{aligned}
&\leq (1-a_n-b_n)\|(x_n-p)\| + a_n\|T(PT)^{n-1}x_n - p\| \\
&\quad +\|b_n(u_n-p)\| \\
&\leq (1-a_n-b_n)\|x_n-p\| + a_nk_n\|x_n-p\| + b_n\|u_n-p\| \\
&\leq (1+a_n(k_n-1))\|x_n-p\| + Mb_n \\
&\leq k_n\|x_n-p\| + Mb_n. \qquad (4.2)
\end{aligned}
$$

From (4.2), we have

$$
\begin{aligned}
\|y_n-p\| &= \|P((1-c_n-d_n)z_n + c_nT(PT)^{n-1}x_n + d_nv_n) - P(p)\| \\
&\leq \|(1-c_n-d_n)(z_n-p) + c_n(T(PT)^{n-1}x_n-p) + d_n(v_n-p)\| \\
&\leq (1-c_n-d_n)\|z_n-p\| + c_n\|T(PT)^{n-1}x_n-p\| + d_n\|v_n-p\| \\
&\leq (1-c_n-d_n)\|z_n-p\| + c_nk_n\|x_n-p\| + d_n\|v_n-p\| \\
&\leq (1-c_n-d_n)(k_n\|x_n-p\| + Mb_n) + c_nk_n\|x_n-p\| + Md_n \\
&\leq k_n\|x_n-p\| + Mb_n + Md_n. \qquad (4.3)
\end{aligned}
$$

From (4.3), we have

$$
\begin{aligned}
\|x_{n+1}-p\| &= \|P((1-\alpha_n-\beta_n)y_n + \alpha_nT(PT)^{n-1}x_n + \beta_nw_n) - P(p)\| \\
&\leq \|(1-\alpha_n-\beta_n)(y_n-p) + \alpha_n(T(PT)^{n-1}x_n-p) + \beta_n(w_n-p)\| \\
&\leq (1-\alpha_n-\beta_n)\|y_n-p\| + \alpha_n\|(T(PT)^{n-1}x_n-p\| + \beta_n\|w_n-p\| \\
&\leq (1-\alpha_n-\beta_n)\|y_n-p\| + \alpha_nk_n\|x_n-p\| + \beta_n\|w_n-p\| \\
&\leq (1-\alpha_n-\beta_n)(k_n\|x_n-p\| + Mb_n + Md_n) + \alpha_nk_n\|x_n-p\| + M\beta_n \\
&\leq k_n\|x_n-p\| + M(b_n+d_n+\beta_n) \\
&= \big(1+(k_n-1)\big)\|x_n-p\| + M(b_n+d_n+\beta_n).
\end{aligned}
$$

Since $\sum_{n=1}^{\infty}(k_n-1) < \infty$, the assertion (i) follows from Lemma 2.1.3.

(ii) By (i), we know that $\lim_{n\to\infty}\|x_n-p\|$ exists for any $p \in F(T)$. It follow that $\{x_n-p\}, \{T(PT)^{n-1}x_n-p\}$, $\{y_n-p\}$ and $\{z_n-p\}$ are bounded. Also, $\{u_n-p\}, \{v_n-p\}$ and $\{w_n-p\}$ are bounded by the assumption. Now we set

$$
\begin{aligned}
r_1 &= \sup\{\|x_n-p\| : n \geq 1\}, \\
r_2 &= \sup\{\|T(PT)^{n-1}x_n-p\| : n \geq 1\}, \\
r_3 &= \sup\{\|y_n-p\| : n \geq 1\}, \\
r_4 &= \sup\{\|z_n-p\| : n \geq 1\}, \\
r_5 &= \sup\{\|u_n-p\| : n \geq 1\}, \\
r_6 &= \sup\{\|v_n-p\| : n \geq 1\}, \\
r_7 &= \sup\{\|w_n-p\| : n \geq 1\}, \\
r &= \max\{r_i : i = 1,2,3,4,5,6,7\}. \qquad (4.4)
\end{aligned}
$$

By using Lemma 2.2.13 and (4.4), we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|P((1-a_n-b_n)x_n + a_nT(PT)^{n-1}x_n + b_nu_n) - P(p)\|^2 \\
&\leq \|(1-a_n-b_n)(x_n-p) + a_n(T(PT)^{n-1}x_n - p) + b_n(u_n-p)\|^2 \\
&\leq (1-a_n-b_n)\|x_n-p\|^2 + a_n\|T(PT)^{n-1}x_n - p\|^2 + b_n\|u_n-p\|^2 \\
&\quad - a_n(1-a_n-b_n)g(\|T(PT)^{n-1}x_n - x_n\|) \\
&\leq (1-a_n-b_n)\|x_n-p\|^2 + a_nk_n^2\|x_n-p\|^2 + b_n\|u_n-p\|^2 \\
&\leq (1-a_n+a_n{k_n}^2)\|x_n-p\|^2 + r^2b_n \\
&\leq (1+a_n(k_n^2-1))\|x_n-p\|^2 + r^2b_n \\
&\leq (1+(k_n^2-1))\|x_n-p\|^2 + r^2b_n \\
&\leq k_n^2\|x_n-p\|^2 + r^2b_n,
\end{aligned}$$

$$\begin{aligned}
\|y_n - p\|^2 &= \|P((1-c_n-d_n)z_n + c_nT(PT)^{n-1}x_n + d_nv_n) - P(p)\|^2 \\
&\leq \|(1-c_n-d_n)(z_n-p) + c_n(T(PT)^{n-1}x_n - p) + d_n(v_n-p)\|^2 \\
&\leq (1-c_n-d_n)\|z_n-p\|^2 + c_n\|T(PT)^{n-1}x_n - p\|^2 + d_n\|v_n-p\|^2 \\
&\quad - c_n(1-c_n-d_n)g(\|T(PT)^{n-1}x_n - z_n\|) \\
&\leq (1-c_n-d_n)\|z_n-p\|^2 + c_nk_n^2\|x_n-p\|^2 + d_n\|v_n-p\|^2 \\
&\quad - c_n(1-c_n-d_n)g(\|T(PT)^{n-1}x_n - z_n\|) \\
&\leq (1-c_n-d_n)(k_n^2\|x_n-p\|^2 + r^2b_n) + c_nk_n^2\|x_n-p\|^2 + r^2d_n \\
&\quad - c_n(1-c_n-d_n)g(\|T(PT)^{n-1}x_n - z_n\|) \\
&\leq \left((1-c_n-d_n)k_n^2 + c_nk_n^2\right)\|x_n-p\|^2 + r^2b_n + r^2d_n \\
&\quad - c_n(1-c_n-d_n)g(\|T(PT)^{n-1}x_n - z_n\|) \\
&= (1-d_n)k_n^2\|x_n-p\|^2 + r^2b_n + r^2d_n \\
&\quad - c_n(1-c_n-d_n)g(\|T(PT)^{n-1}x_n - z_n\|) \\
&\leq k_n^2\|x_n-p\|^2 + r^2b_n + r^2d_n,
\end{aligned}$$

and so

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|P((1-\alpha_n-\beta_n)y_n + \alpha_nT(PT)^{n-1}x_n + \beta_nw_n) - P(p)\|^2 \\
&\leq \|(1-\alpha_n-\beta_n)(y_n-p) + \alpha_n(T(PT)^{n-1}x_n - p) + \beta_n(w_n-p)\|^2 \\
&\leq (1-\alpha_n-\beta_n)\|y_n-p\|^2 + \alpha_n\|T(PT)^{n-1}x_n - p\|^2 + \beta_n\|w_n-p\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|T(PT)^{n-1}x_n - y_n\|) \\
&\leq (1-\alpha_n-\beta_n)\|y_n-p\|^2 + \alpha_nk_n^2\|x_n-p\|^2 + \beta_n\|w_n-p\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|T(PT)^{n-1}x_n - y_n\|) \\
&\leq (1-\alpha_n-\beta_n)(k_n^2\|x_n-p\|^2 + r^2b_n + r^2d_n) + \alpha_nk_n^2\|x_n-p\|^2 + r^2\beta_n \\
&\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|T(PT)^{n-1}x_n - y_n\|)
\end{aligned}$$

$$\begin{aligned}
&\le \left((1-\alpha_n-\beta_n)k_n^2+\alpha_n k_n^2\right)\|x_n-p\|^2+r^2b_n+r^2d_n+r^2\beta_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T(PT)^{n-1}x_n-y_n\|) \\
&= (1-\beta_n)k_n^2\|x_n-p\|^2+r^2b_n+r^2d_n+r^2\beta_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T(PT)^{n-1}x_n-y_n\|) \\
&\le k_n^2\|x_n-p\|^2+r^2b_n+r^2d_n+r^2\beta_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T(PT)^{n-1}x_n-y_n\|) \\
&= k_n^2\|x_n-p\|^2+r^2(b_n+d_n+\beta_n) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T(PT)^{n-1}x_n-y_n\|) \\
&= \|x_n-p\|^2+(k_n^2-1)\|x_n-p\|^2+r^2(b_n+d_n+\beta_n) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T(PT)^{n-1}x_n-y_n\|) \\
&\le \|x_n-p\|^2+r^2(k_n^2-1)+r^2(b_n+d_n+\beta_n) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T(PT)^{n-1}x_n-y_n\|),
\end{aligned}$$

which leads to the following:

$$\alpha_n(1-\alpha_n-\beta_n)g(\|T(PT)^{n-1}x_n-y_n\|) \le \|x_n-p\|^2-\|x_{n+1}-p\|^2 +r^2(k_n^2-1)+r^2(b_n+d_n+\beta_n). \quad (4.5)$$

If $0<\liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}(\alpha_n+\beta_n)<1$, then there exists a positive integer n_0 and $\eta,\eta' \in (0,1)$ such that $0<\eta<\alpha_n$ and $\alpha_n+\beta_n<\eta'<1$ for all $n\ge n_0$. Hence, by (4.5), we have

$$\eta(1-\eta')g(\|T(PT)^{n-1}x_n-y_n\|) \le \|x_n-p\|^2-\|x_{n+1}-p\|^2 +r^2(k_n^2-1)+r^2(b_n+d_n+\beta_n), \quad (4.6)$$

for all $n\ge n_0$. Applying (4.6) for $m\ge n_0$, we have

$$\begin{aligned}
\sum_{n=n_0}^{m} g(\|T(PT)^{n-1}x_n-y_n\|) &\le \frac{1}{\eta(1-\eta')}\Bigg(\sum_{n=n_0}^{m}(\|x_n-p\|^2-\|x_{n+1}-p\|^2) \\
&\quad +r^2\sum_{n=n_0}^{m}(b_n+d_n+\beta_n+(k_n^2-1))\Bigg) \\
&\le \frac{1}{\eta(1-\eta')}\Bigg(\|x_{n_0}-p\|^2 \\
&\quad +r^2\sum_{n=n_0}^{m}(b_n+d_n+\beta_n+(k_n^2-1))\Bigg). \quad (4.7)
\end{aligned}$$

Since $0\le t^2-1\le 2t(t-1)$ for all $t\ge 1$, the assumption $\sum_{n=1}^{\infty}(k_n-1)<\infty$ implies that $\sum_{n=1}^{\infty}(k_n^2-1)<\infty$. Let $m\to\infty$ in inequality (4.7) we get that $\sum_{n=n_0}^{\infty} g(\|T(PT)^{n-1}x_n-y_n\|)<\infty$, and therefore $\lim_{n\to\infty} g(\|T(PT)^{n-1}x_n -$

$y_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} \|T(PT)^{n-1}x_n - y_n\| = 0$.

(iii) First, we assume that $0 < \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$ and $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$. By Lemma 2.2.13, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|P((1-\alpha_n-\beta_n)y_n + \alpha_n T(PT)^{n-1}x_n + \beta_n w_n) - P(p)\|^2 \\
&\leq \|(1-\alpha_n-\beta_n)(y_n - p) + \alpha_n(T(PT)^{n-1}x_n - p) + \beta_n(w_n - p)\|^2 \\
&\leq (1-\alpha_n-\beta_n)\|y_n - p\|^2 + \alpha_n\|T(PT)^{n-1}x_n - p\|^2 + \beta_n\|w_n - p\|^2 \\
&\leq (1-\alpha_n-\beta_n)(k_n^2\|x_n - p\|^2 + r^2 b_n + r^2 d_n \\
&\quad -c_n(1-c_n-d_n)g(\|T(PT)^{n-1}x_n - z_n\|)) \\
&\quad +\alpha_n\|T(PT)^{n-1}x_n - p\|^2 + \beta_n\|w_n - p\|^2 \\
&\leq (1-\alpha_n-\beta_n)(k_n^2\|x_n - p\|^2 + r^2 b_n + r^2 d_n \\
&\quad -c_n(1-c_n-d_n)g(\|T(PT)^{n-1}x_n - z_n\|)) \\
&\quad +\alpha_n k_n^2\|x_n - p\|^2 + r^2\beta_n \\
&\leq (1-\alpha_n-\beta_n)k_n^2\|x_n - p\|^2 + r^2 b_n + r^2 d_n \\
&\quad -(1-\alpha_n-\beta_n)c_n(1-c_n-d_n)g(\|T(PT)^{n-1}x_n - z_n\|) \\
&\quad +\alpha_n k_n^2\|x_n - p\|^2 + r^2\beta_n \\
&\leq k_n^2\|x_n - p\|^2 + r^2(b_n + d_n + \beta_n) \\
&\quad -(1-\alpha_n-\beta_n)c_n(1-c_n-d_n)g(\|T(PT)^{n-1}x_n - z_n\|) \\
&= \|x_n - p\|^2 + (k_n^2 - 1)\|x_n - p\|^2 + r^2(b_n + d_n + \beta_n) \\
&\quad -(1-\alpha_n-\beta_n)c_n(1-c_n-d_n)g(\|T(PT)^{n-1}x_n - z_n\|) \\
&\leq \|x_n - p\|^2 + r^2(k_n^2 - 1) + r^2(b_n + d_n + \beta_n) \\
&\quad -(1-\alpha_n-\beta_n)c_n(1-c_n-d_n)g(\|T(PT)^{n-1}x_n - z_n\|). \qquad (4.8)
\end{aligned}$$

Hence, by (4.8), we have

$$\begin{aligned}
(1-\alpha_n-\beta_n)c_n(1-c_n-d_n)g(\|T(PT)^{n-1}x_n - z_n\|) &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad +r^2(k_n^2 - 1) \\
&\quad +r^2(b_n + d_n + \beta_n), \qquad (4.9)
\end{aligned}$$

By our assumption $0 < \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$ and $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$, there exists a positive integer n_0 and $\delta_1, \delta_2, \delta_3 \in (0,1)$ such that $\alpha_n + \beta_n < \delta_1 < 1$, $0 < \delta_2 < c_n$ and $c_n + d_n < \delta_3 < 1$ for all $n \geq n_0$. It follows from (4.9), for $m \geq n_0$,

$$\sum_{n=n_0}^{m} g(\|T(PT)^{n-1}x_n - z_n\|) \leq \frac{1}{(1-\delta_1)\delta_2(1-\delta_3)}\left(\sum_{n=n_0}^{m}(\|x_n-p\|^2 - \|x_{n+1}-p\|^2) + r^2\sum_{n=n_0}^{m}(b_n+d_n+\beta_n+(k_n^2-1))\right)$$
$$\leq \frac{1}{(1-\delta_1)\delta_2(1-\delta_3)}\left(\|x_{n_0}-p\|^2 + r^2\sum_{n=n_0}^{m}(b_n+d_n+\beta_n+(k_n^2-1))\right). \quad (4.10)$$

Since $0 \leq t^2-1 \leq 2t(t-1)$ for all $t \geq 1$, the assumption $\sum_{n=1}^{\infty}(k_n-1) < \infty$ implies that $\sum_{n=1}^{\infty}(k_n^2-1) < \infty$. Let $m \to \infty$ in inequality (4.10) we get that $\sum_{n=n_0}^{\infty} g(\|T(PT)^{n-1}x_n - y_n\|) < \infty$, and therefore $\lim_{n\to\infty} g(\|T(PT)^{n-1}x_n - y_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|T(PT)^{n-1}x_n - z_n\| = 0$.

(iv) If $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$, $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n+d_n) < 1$ and $\limsup_{n\to\infty} a_n < 1$, by (ii) and (iii), we have

$$\lim_{n\to\infty}\|T(PT)^{n-1}x_n - y_n\| = 0 \text{ and } \lim_{n\to\infty}\|T(PT)^{n-1}x_n - z_n\| = 0. \quad (4.11)$$

From $y_n = P((1-c_n-d_n)z_n + c_nT(PT)^{n-1}x_n + d_nv_n)$, we have

$$\begin{aligned}
\|y_n - x_n\| &= \|P((1-c_n-d_n)z_n + c_nT(PT)^{n-1}x_n + d_nv_n) - x_n\| \\
&\leq \|(1-c_n-d_n)z_n + c_nT(PT)^{n-1}x_n + d_nv_n - x_n\| \\
&= \|(z_n-x_n) + c_n(T(PT)^{n-1}x_n - z_n) + d_n(v_n-z_n)\| \\
&\leq \|z_n-x_n\| + c_n\|T(PT)^{n-1}x_n - z_n\| + d_n\|v_n-x_n\| \\
&= \|P((1-a_n-b_n)x_n + a_nT(PT)^{n-1}x_n + b_nu_n) - P(x_n)\| \\
&\quad + c_n\|T(PT)^{n-1}x_n - z_n\| + d_n\|v_n-x_n\| \\
&\leq \|(1-a_n-b_n)x_n + a_nT(PT)^{n-1}x_n + b_nu_n - x_n\| \\
&\quad + c_n\|T(PT)^{n-1}x_n - z_n\| + d_n\|v_n-x_n\| \\
&= \|a_n(T(PT)^{n-1}x_n - x_n) + b_n(u_n-x_n)\| + c_n\|T(PT)^{n-1}x_n - z_n\| \\
&\quad + d_n\|v_n-x_n\| \\
&\leq a_n\|T(PT)^{n-1}x_n - x_n\| + b_n\|u_n-x_n\| + c_n\|T(PT)^{n-1}x_n - z_n\| \\
&\quad + d_n\|v_n-x_n\| \\
&\leq a_n\|T(PT)^{n-1}x_n - x_n\| + c_n\|T(PT)^{n-1}x_n - z_n\| \\
&\quad + 2rb_n + 2rd_n, \qquad (4.12)
\end{aligned}$$

where r is defined by (4.4). From (4.12), we have

$$\begin{aligned}\|T(PT)^{n-1}x_n - x_n\| &\le \|T(PT)^{n-1}x_n - y_n\| + \|y_n - x_n\| \\ &\le \|T(PT)^{n-1}x_n - y_n\| + a_n\|T(PT)^{n-1}x_n - x_n\| \\ &\quad + c_n\|T(PT)^{n-1}x_n - z_n\| + 2rb_n + 2rd_n, \qquad (4.13)\end{aligned}$$

Thus by the inequality (4.13), we have

$$\begin{aligned}(1-a_n)\|T(PT)^{n-1}x_n - x_n\| &\le \|T(PT)^{n-1}x_n - y_n\| + c_n\|T(PT)^{n-1}x_n - z_n\| \\ &\quad + 2rb_n + 2rd_n.\end{aligned}$$

Since $\limsup_{n\to\infty} a_n < 1$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} d_n = 0$, it follows from (4.11) that $\lim_{n\to\infty} \|T(PT)^{n-1}x_n - x_n\| = 0$. □

Theorem 4.1.3 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous asymptotically nonexpansive nonself-map of C with $\{k_n\}$ satisfying $k_n \ge 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \ge 1$, and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} c_n \le \limsup_{n\to\infty}(c_n + d_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$.

Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined by the three-step iterative scheme (4.1). Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

Proof. By Lemma 4.1.2 , we have

$$\begin{aligned}\lim_{n\to\infty} \|T(PT)^{n-1}x_n - y_n\| &= 0, \\ \lim_{n\to\infty} \|T(PT)^{n-1}x_n - z_n\| &= 0, \qquad (4.14) \\ \lim_{n\to\infty} \|T(PT)^{n-1}x_n - x_n\| &= 0.\end{aligned}$$

It follows from (4.12) and (4.14) that $\lim_{n\to\infty} \|y_n - x_n\| = 0$.
Since

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|P((1-\alpha_n-\beta_n)y_n + \alpha_n T(PT)^{n-1}x_n + \beta_n w_n) - P(x_n)\| \\ &\le \|(1-\alpha_n-\beta_n)y_n + \alpha_n T(PT)^{n-1}x_n + \beta_n w_n - x_n\| \\ &= \|(y_n - x_n) + \alpha_n(T(PT)^{n-1}x_n - y_n) + \beta_n(w_n - y_n)\| \\ &\le \|y_n - x_n\| + \alpha_n\|T(PT)^{n-1}x_n - y_n\| + \beta_n\|w_n - y_n\|,\end{aligned}$$

it follows that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Thus

$$\begin{aligned}\|x_{n+1} - T(PT)^{n-1}x_{n+1}\| &\leq \|x_{n+1} - x_n\| + \|T(PT)^{n-1}x_{n+1} - T(PT)^{n-1}x_n\| \\ &\quad + \|T(PT)^{n-1}x_n - x_n\| \\ &\leq \|x_{n+1} - x_n\| + k_n\|x_{n+1} - x_n\| \\ &\quad + \|T(PT)^{n-1}x_n - x_n\| \to 0 \text{ (as } n \to \infty) \quad (4.15)\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - T(PT)^{n-2}x_{n+1}\| &= \|x_{n+1} - x_n + x_n - T(PT)^{n-2}x_n \\ &\quad + T(PT)^{n-2}x_n - T(PT)^{n-2}x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \|T(PT)^{n-2}x_n - x_n\| \\ &\quad + \|T(PT)^{n-2}x_{n+1} - T(PT)^{n-2}x_n\| \\ &\leq \|x_{n+1} - x_n\| + \|T(PT)^{n-2}x_n - x_n\| \\ &\quad + L\|x_{n+1} - x_n\| \to 0 \quad (\text{as } n \to \infty), \qquad (4.16)\end{aligned}$$

where $L = \sup\{k_n : n \geq 1\}$. We denote $(PT)^{1-1}$ to be the identity maps from C onto itself. Thus by the inequality (4.15) and (4.16), we have

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| &\leq \|x_{n+1} - T(PT)^{n-1}x_{n+1}\| + \|T(PT)^{n-1}x_{n+1} - Tx_{n+1}\| \\ &= \|x_{n+1} - T(PT)^{n-1}x_{n+1}\| \\ &\quad + \|T(PT)^{1-1}(PT)^{n-1}x_{n+1} - T(PT)^{1-1}x_{n+1}\| \\ &\leq \|x_{n+1} - T(PT)^{n-1}x_{n+1}\| + L\|(PT)^{n-1}x_{n+1} - x_{n+1}\| \\ &= \|x_{n+1} - T(PT)^{n-1}x_{n+1}\| + L\|(PT)(PT)^{n-2}x_{n+1} - P(x_{n+1})\| \\ &\leq \|x_{n+1} - T(PT)^{n-1}x_{n+1}\| + L\|T(PT)^{n-2}x_{n+1} - x_{n+1}\| \\ &\quad \to 0 \quad (\text{as } n \to \infty),\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \qquad (4.17)$$

Since T is completely continuous and $\{x_n\} \subseteq C$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore from (4.17), $\{x_{n_k}\}$ converges. Let $q = \lim_{k\to\infty} x_{n_k}$. By continuity of T and (4.17) we have that $Tq = q$, so q is a fixed point of T. By Lemma 4.1.2 (i), $\lim_{n\to\infty} \|x_n - q\|$ exists. But $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$. Thus $\lim_{n\to\infty} \|x_n - q\| = 0$. Since $\|y_n - x_n\| \to 0$ as $n \to \infty$, and

$$\begin{aligned}\|z_n - x_n\| &= \|P((1 - a_n - b_n)x_n + a_nT(PT)^{n-1}x_n + b_nu_n) - P(x_n)\| \\ &\leq \|(1 - a_n - b_n)x_n + a_nT(PT)^{n-1}x_n + b_nu_n - x_n\| \\ &\leq a_n\|T(PT)^{n-1}x_n - x_n\| + b_n\|u_n - x_n\| \to 0 \text{ as } n \to \infty,\end{aligned}$$

it follows that $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$. □

For $a_n = b_n \equiv 0$, then Theorem 4.1.3 reduces to the two-step iteration with errors.

Corollary 4.1.4 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous asymptotically nonexpansive nonself-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{c_n\}, \{d_n\}, \{\alpha_n\}, \{\beta_n\}$ be real sequences in $[0,1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *and*

(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$.

For a given $x_1 \in C$, define

$$\begin{aligned} y_n &= P((1 - c_n - d_n)x_n + c_n T(PT)^{n-1}x_n + d_n v_n), \\ x_{n+1} &= P((1 - \alpha_n - \beta_n)y_n + \alpha_n T(PT)^{n-1}x_n + \beta_n w_n), \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to a fixed point of T.

In the next result, we prove weak convergence of the new three-step iterative scheme (4.1) for asymptotically nonexpansive nonself-mapping in a uniformly convex Banach space satisfying Opial's condition.

Theorem 4.1.5 *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be an asymptotically nonexpansive nonself-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *and*

(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$.

Let $\{x_n\}$ be the sequence defined by three-step iterative scheme (4.1). Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. By using the same proof as in Theorem 4.1.3, it can be shown that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 2.4.2, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 2.4.2, $u, v \in F(T)$. By Lemma 4.1.2 (i), $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 2.2.16 that $u = v$. Therefore $\{x_n\}$ converges weakly to a fixed point of T. □

When $a_n = b_n \equiv 0$ in Theorem 4.1.5, we obtain weak convergence theorem of the two-step iteration with errors as follows:

Corollary 4.1.6 *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be an asymptotically nonexpansive nonself-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{c_n\}, \{d_n\}, \{\alpha_n\}, \{\beta_n\}$ be sequences of real numbers in $[0,1]$ such that*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$.

For a given $x_1 \in C$, define

$$\begin{aligned} y_n &= P((1 - c_n - d_n)x_n + c_nT(PT)^{n-1}x_n + d_nv_n), \\ x_{n+1} &= P((1 - \alpha_n - \beta_n)y_n + \alpha_nT(PT)^{n-1}x_n + \beta_nw_n), \quad n \geq 1, \end{aligned}$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

Next, we will consider and study the modified Noor iterations with errors for asymptotically nonexpansive nonself-mapping. This scheme can be viewed as an extension for three-step and two-step iterative schemes of Noor [22, 23], Xu and Noor [45], Suantai [35], Ishikawa [11] and Nammanee, Noor and Suantai [21]. The scheme is defined as follows.

Let X be a normed space, C a nonempty convex subset of X, $P : X \to C$ a nonexpansive retraction of X onto C, and $T : C \to X$ a given mapping. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= P(a_nT(PT)^{n-1}x_n + (1 - a_n - \gamma_n)x_n + \gamma_nu_n) \\ y_n &= P(b_nT(PT)^{n-1}z_n + c_nT(PT)^{n-1}x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_nv_n) \\ x_{n+1} &= P(\alpha_nT(PT)^{n-1}y_n + \beta_nT(PT)^{n-1}z_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_nw_n), \\ & \quad n \geq 1, \end{aligned} \tag{4.18}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}, \{\lambda_n\}$ are appropriate sequences in $[0,1]$ and $\{u_n\}, \{v_n\}$ *and* $\{w_n\}$ are bounded sequences in C.

The iterative schemes (4.18) are called the *modified Noor iterations with errors* for asymptotically nonexpansive nonself mappings.

If $T : C \to C$, then the iterative schemes (4.18) reduces to the modified Noor iterations with errors defined by Nammanee, Noor and Suantai [21],

$$\begin{aligned} z_n &= a_nT^nx_n + (1 - a_n - \gamma_n)x_n + \gamma_nu_n \\ y_n &= b_nT^nz_n + c_nT^nx_n + (1 - b_n - c_n - \mu_n)x_n + \mu_nv_n \\ x_{n+1} &= \alpha_nT^ny_n + \beta_nT^nz_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_nw_n, \quad n \geq 1, \end{aligned} \tag{4.19}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}, \{\lambda_n\}$ are appropriate sequences in $[0,1]$ and $\{u_n\}, \{v_n\}$ *and* $\{w_n\}$ are bounded sequences in C.

If $T : C \to C$ and $\gamma_n = \mu_n = \lambda_n \equiv 0$, then (4.18) reduces to the modified Noor iterations defined by Suantai [35]

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n)x_n \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n)x_n, \quad n \geq 1, \end{aligned} \tag{4.20}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ are appropriate sequences in $[0,1]$.

We note that the usual Ishikawa and Mann iterations are special cases of (4.18) and if $T : C \to C$ and $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then (4.18) reduces to the Noor iterations defined by Xu and Noor [45]

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n)x_n \\ y_n &= b_n T^n z_n + (1 - b_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{4.21}$$

where $\{a_n\}, \{b_n\}, \{\alpha_n\}$ are appropriate sequences in $[0,1]$.

For $T : C \to C$ and $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then (4.18) reduces to the usual Ishikawa iterative scheme

$$\begin{aligned} y_n &= b_n T^n x_n + (1 - b_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{4.22}$$

where $\{b_n\}, \{\alpha_n\}$ are appropriate sequences in $[0,1]$.

In this section, we prove weak and strong convergence theorems of modified Noor iterations with errors for asymptotically nonexpansive nonself-mapping in a Banach space.

Definition 4.1.7 ([7]) Let X be a real normed linear space and let C be a nonempty subset of X. Let $P : X \to C$ be the nonexpansive retraction of X onto C. A map $T : C \to X$ is said to be asymptotically nonexpansive nonself-mapping if there exists a sequence $k_n, k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$, such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\|, \quad \forall x, y \in C, \ n \geq 1.$$

In order to prove our main results, the following lemma is needed.

Lemma 4.1.8 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be an asymptotically nonexpansive nonself-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n + \gamma_n$, $b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, and let $\{u_n\}, \{v_n\}$ and $\{w_n\}$ be the bounded sequences in C. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (4.18).*

(i) If q is a fixed point of T, then $\lim_{n\to\infty} \|x_n - q\|$ exists.

(ii) If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then $\lim_{n\to\infty} \|T(PT)^{n-1}y_n - x_n\| = 0$.

(iii) If $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ or if $0 < \liminf_{n\to\infty} \alpha_n$ and $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, then $\lim_{n\to\infty} \|T(PT)^{n-1}z_n - x_n\| = 0$.

(iv) If condition in (ii) and (iii) are true, then $\lim_{n\to\infty} \|T(PT)^{n-1}x_n - x_n\| = 0$.

Proof. Let $q \in F(T)$, by boundedness of the sequence $\{u_n\}, \{v_n\}$ and $\{w_n\}$, we can put

$$M = \max\{\sup_{n\geq 1} \|u_n - q\|, \sup_{n\geq 1} \|v_n - q\|, \sup_{n\geq 1} \|w_n - q\|\}.$$

(i) For each $n \geq 1$, we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|P(\alpha_n T(PT)^{n-1}y_n + \beta_n T(PT)^{n-1}z_n \\ &\quad + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n) - P(q)\| \\ &\leq \alpha_n \|T(PT)^{n-1}y_n - q\| + \beta_n \|T(PT)^{n-1}z_n - q\| \\ &\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\| + \lambda_n \|w_n - q\| \\ &\leq \alpha_n k_n \|y_n - q\| + \beta_n k_n \|z_n - q\| \\ &\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\| + M\lambda_n. \end{aligned} \tag{4.23}$$

Consider,

$$\begin{aligned} \|z_n - q\| &= \|P(a_n T(PT)^{n-1}x_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n) - P(q)\| \\ &\leq a_n \|T(PT)^{n-1}x_n - q\| + (1 - a_n - \gamma_n)\|x_n - q\| + \gamma_n \|u_n - q\| \\ &\leq a_n k_n \|x_n - q\| + (1 - a_n - \gamma_n)\|x_n - q\| + M\gamma_n \\ &\leq (a_n k_n + (1 - a_n))\|x_n - q\| + M\gamma_n \\ &= (a_n(k_n - 1) + 1)\|x_n - q\| + M\gamma_n \\ &\leq k_n \|x_n - q\| + M\gamma_n \end{aligned} \tag{4.24}$$

and

$$\begin{aligned}\|y_n - q\| &= \|P(b_nT(PT)^{n-1}z_n + c_nT(PT)^{n-1}x_n \\ &\quad + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n) - P(q)\| \\ &\le b_n\|T(PT)^{n-1}z_n - q\| + c_n\|T(PT)^{n-1}x_n - q\| \\ &\quad + (1 - b_n - c_n - \mu_n)\|x_n - q\| + \mu_n\|v_n - q\| \\ &\le b_nk_n\|z_n - q\| + c_nk_n\|x_n - q\| + (1 - b_n - c_n - \mu_n)\|x_n - q\| + M\mu_n.\end{aligned}$$

From (4.24), we have

$$\begin{aligned}\|y_n - q\| &\le b_nk_n(k_n\|x_n - q\| + M\gamma_n) + c_nk_n\|x_n - q\| \\ &\quad + (1 - b_n - c_n - \mu_n)\|x_n - q\| + M\mu_n \\ &\le (b_nk_n^2 + c_nk_n + (1 - b_n - c_n))\|x_n - q\| + \epsilon_{(1)}^n \\ &\le (b_n(k_n^2 - 1) + c_n(k_n^2 - 1) + 1)\|x_n - q\| + \epsilon_{(1)}^n \\ &= ((k_n^2 - 1)(b_n + c_n) + 1)\|x_n - q\| + \epsilon_{(1)}^n \\ &= k_n^2\|x_n - q\| + \epsilon_{(1)}^n, \end{aligned} \qquad (4.25)$$

where $\epsilon_{(1)}^n = Mb_nk_n\gamma_n + M\mu_n$, and we note here that $\sum_{n=1}^{\infty}\epsilon_{(1)}^n < \infty$ since $\{k_n\}$ is bounded and $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$.

From (4.23), (4.24) and (4.25) we get

$$\begin{aligned}\|x_{n+1} - q\| &\le \alpha_nk_n(k_n^2\|x_n - q\| + \epsilon_{(1)}^n) + \beta_nk_n(k_n\|x_n - q\| + M\gamma_n) \\ &\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\| + M\lambda_n \\ &\le (\alpha_nk_n^3 + \beta_nk_n^2 + (1 - \alpha_n - \beta_n))\|x_n - q\| + \epsilon_{(2)}^n \\ &\le (\alpha_n(k_n^3 - 1) + \beta_n(k_n^3 - 1) + 1)\|x_n - q\| + \epsilon_{(2)}^n \\ &= ((\alpha_n + \beta_n)(k_n^3 - 1) + 1)\|x_n - q\| + \epsilon_{(2)}^n \\ &\le (1 + (k_n^3 - 1))\|x_n - q\| + \epsilon_{(2)}^n, \end{aligned} \qquad (4.26)$$

where $\epsilon_{(2)}^n = \alpha_nk_n\epsilon_{(1)}^n + M\beta_nk_n\gamma_n + M\lambda_n$. Since $\sum_{n=1}^{\infty}(k_n^3 - 1) < \infty$ and $\sum_{n=1}^{\infty}\epsilon_{(2)}^n < \infty$ we obtained by (4.26) and Lemma 2.1.3 that $\lim_{n\to\infty}\|x_n - q\|$ exists.

(ii) By (i), we know that $\lim_{n\to\infty}\|x_n - q\|$ exists for any $q \in F(T)$. It follows from (4.24) and (4.25) that $\{x_n - q\}, \{T(PT)^{n-1}x_n - q\}, \{z_n - q\}, \{T(PT)^{n-1}z_n - q\}, \{y_n - q\}$ and $\{T(PT)^{n-1}y_n - q\}$ are bounded sequence. This allows us to put

$$\begin{aligned}K = \max\{M, &\sup_{n\ge1}\|x_n - q\|, \sup_{n\ge1}\|T(PT)^{n-1}x_n - q\|, \sup_{n\ge1}\|z_n - q\|, \\ &\sup_{n\ge1}\|T(PT)^{n-1}z_n - q\|, \sup_{n\ge1}\|y_n - q\|, \sup_{n\ge1}\|T(PT)^{n-1}y_n - q\|\}.\end{aligned}$$

Since $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$. It follows from (4.24) and (4.25) that

$$\|z_n - q\|^2 \le k_n^2\|x_n - q\|^2 + \epsilon_{(3)}^n \tag{4.27}$$

$$\|y_n - q\|^2 \le k_n^4\|x_n - q\|^2 + \epsilon_{(4)}^n, \tag{4.28}$$

where $\epsilon_{(3)}^n = M^2\gamma_n^2 + 2MK\gamma_n k_n$ and $\epsilon_{(4)}^n = (\epsilon_{(1)}^n)^2 + 2K\epsilon_{(1)}^n k_n^2$ and also observe that $\sum_{n=1}^{\infty} \epsilon_{(3)}^n < \infty$ and $\sum_{n=1}^{\infty} \epsilon_{(4)}^n < \infty$ by bounded of $\{k_n\}$ and $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \epsilon_{(1)}^n < \infty$. By lemma 2.2.14, there is a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that

$$\|\lambda x + \beta y + \gamma z + \mu w\|^2 \le \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 + \mu\|w\|^2 - \lambda\beta g(\|x - y\|) \tag{4.29}$$

for all $x, y, z, w \in B_r$ and all $\lambda, \beta, \gamma, \mu \in [0,1]$ with $\lambda + \beta + \gamma = 1$. It follows that

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|P(\alpha_n T(PT)^{n-1}y_n + \beta_n T(PT)^{n-1}z_n \\ &\quad + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n) - P(q)\|^2 \\ &\le \|\alpha_n(T(PT)^{n-1}y_n - q) + \beta_n(T(PT)^{n-1}z_n - q) \\ &\quad + (1 - \alpha_n - \beta_n - \lambda_n)(x_n - q) + \lambda_n(w_n - q)\|^2. \end{aligned} \tag{4.30}$$

From (4.27), (4.28), (4.29) and (4.30), we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &\le \alpha_n\|T(PT)^{n-1}y_n - q\|^2 + \beta_n\|T(PT)^{n-1}z_n - q\|^2 \\ &\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + \lambda_n\|w_n - q\|^2 \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T(PT)^{n-1}y_n - x_n\|) \\ &\le \alpha_n k_n^2\|y_n - q\|^2 + \beta_n k_n^2\|z_n - q\|^2 + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 \\ &\quad + K^2\lambda_n - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T(PT)^{n-1}y_n - x_n\|) \\ &\le \alpha_n k_n^2(k_n^4\|x_n - q\|^2 + \epsilon_{(4)}^n) + \beta_n k_n^2(k_n^2\|x_n - q\|^2 + \epsilon_{(3)}^n) \\ &\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + K^2\lambda_n \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T(PT)^{n-1}y_n - x_n\|) \\ &\le (\alpha_n k_n^6 + \beta_n k_n^4 + (1 - \alpha_n - \beta_n - \lambda_n))\|x_n - q\|^2 + \epsilon_{(5)}^n \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T(PT)^{n-1}y_n - x_n\|) \\ &\le (\alpha_n(k_n^6 - 1) + \beta_n(k_n^6 - 1) + 1)\|x_n - q\|^2 + \epsilon_{(5)}^n \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T(PT)^{n-1}y_n - x_n\|) \\ &= \|x_n - q\|^2 + (\alpha_n + \beta_n)(k_n^6 - 1)\|x_n - q\|^2 + \epsilon_{(5)}^n \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T(PT)^{n-1}y_n - x_n\|) \\ &\le \|x_n - q\|^2 + K^2(k_n^6 - 1) + \epsilon_{(5)}^n \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T(PT)^{n-1}y_n - x_n\|) \\ &= \|x_n - q\|^2 + \epsilon_{(6)}^n - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T(PT)^{n-1}y_n - x_n\|), \end{aligned} \tag{4.31}$$

where $\epsilon_{(5)}^n = \alpha_n k_n^2 \epsilon_{(4)}^n + \beta_n k_n^2 \epsilon_{(3)}^n + K^2\lambda_n$ and $\epsilon_{(6)}^n = K^2(k_n^6 - 1) + \epsilon_{(5)}^n$ and its worth to note here that $\sum_{n=1}^{\infty} \epsilon_{(6)}^n < \infty$ by boundedness of $\{k_n\}, \sum_{n=1}^{\infty} \epsilon_{(4)}^n < \infty, \sum_{n=1}^{\infty} \epsilon_{(3)}^n < \infty, \sum_{n=1}^{\infty} \lambda_n < \infty$, and $\sum_{n=1}^{\infty}(k_n^6 - 1) < \infty$. Since $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, there exists $n_0 \in \mathbb{N}$ and $\delta_1, \delta_2 \in (0,1)$ such that $0 < \delta_1 < a_n$ and $\alpha_n + \beta_n + \lambda_n < \delta_2 < 1$ for all $n \geq n_0$. Thus we obtain (4.31) that

$$\delta_1(1-\delta_2)\sum_{n=n_0}^{m} g(\|T(PT)^{n-1}y_n - x_n\|) \leq \sum_{n=n_0}^{m}(\|x_n - q\|^2 - \|x_{n+1} - q\|^2) + \sum_{n=n_0}^{m} \epsilon_{(6)}^n$$
$$= \|x_{n_0} - q\|^2 + \sum_{n=n_0}^{m} \epsilon_{(6)}^n. \quad (4.32)$$

Since $\sum_{n=n_0}^{\infty} \epsilon_{(6)}^n < \infty$, by letting $m \to \infty$ in (4.32), we get

$$\sum_{n=n_0}^{\infty} g(\|T(PT)^{n-1}y_n - x_n\|) < \infty,$$

and therefore $\lim_{n\to\infty} g(\|T(PT)^{n-1}y_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} \|T(PT)^{n-1}y_n - x_n\| = 0$.

(iii) First, we assume that $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$. By (4.29) and(4.30), we have

$$\begin{aligned}\|x_{n+1} - q\|^2 \leq{}& \alpha_n k_n^2\|y_n - q\|^2 + \beta_n k_n^2\|z_n - q\|^2 \\ &+ (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + K^2\lambda_n \\ &- \beta_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T(PT)^{n-1}z_n - x_n\|).\end{aligned}$$

From this point we follow step by step as in (ii) we will get results

$$\lim_{n\to\infty} \|T(PT)^{n-1}z_n - x_n\| = 0$$

as required. Next, we assume that $0 < \liminf_{n\to\infty} \alpha_n$ and $\liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$.

By (4.27) and (4.29), we have

$$\begin{aligned}\|y_n - q\|^2 ={}& \|P(b_nT(PT)^{n-1}z_n + c_nT(PT)^{n-1}x_n \\ &+ (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n) - P(q)\|^2 \\ \leq{}& \|b_n(T(PT)^{n-1}z_n - q) + c_n(T(PT)^{n-1}x_n - q) \\ &+ (1 - b_n - c_n - \mu_n)(x_n - q) + \mu_n(v_n - q)\|^2 \\ \leq{}& b_n\|T(PT)^{n-1}z_n - q\|^2 + c_n\|T(PT)^{n-1}x_n - q\|^2 \\ &+ (1 - b_n - c_n - \mu_n)\|x_n - q\|^2 + \mu_n\|v_n - q\|^2 \\ &- b_n(1 - b_n - c_n - \mu_n)g(\|T(PT)^{n-1}z_n - x_n\|)\end{aligned}$$

$$\leq b_nk_n^2\|z_n - q\|^2 + c_nk_n^2\|x_n - q\|^2$$
$$+ (1 - b_n - c_n - \mu_n)\|x_n - q\|^2 + \mu_nK^2$$
$$- b_n(1 - b_n - c_n - \mu_n)g(\|T(PT)^{n-1}z_n - x_n\|)$$
$$< b_nk_n^2(k_n^2\|x_n - q\|^2 + \epsilon_{(3)}^n) + c_nk_n^2\|x_n - q\|^2$$
$$+ (1 - b_n - c_n - \mu_n)\|x_n - q\|^2 + \mu_nK^2$$
$$- b_n(1 - b_n - c_n - \mu_n)g(\|T(PT)^{n-1}z_n - x_n\|)$$
$$\leq (b_nk_n^4 + c_nk_n^2 + (1 - b_n - c_n))\|x_n - q\|^2 + \epsilon_{(7)}^n$$
$$- b_n(1 - b_n - c_n - \mu_n)g(\|T(PT)^{n-1}z_n - x_n\|)$$
$$\leq k_n^4\|x_n - q\|^2 + \epsilon_{(7)}^n - b_n(1 - b_n - c_n - \mu_n)g(\|T(PT)^{n-1}z_n - x_n\|), \quad (4.33)$$

where $\epsilon_{(7)}^n = b_nk_n^2\epsilon_{(3)}^n + \mu_nK^2$.

By (4.27), (4.29) and (4.33), we also have

$$\|x_{n+1} - q\|^2 \leq \alpha_nk_n^2\|y_n - q\|^2 + \beta_nk_n^2\|z_n - q\|^2$$
$$+ (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + K^2\lambda_n$$
$$\leq \alpha_nk_n^2(k_n^4\|x_n - q\|^2 + \epsilon_{(7)}^n$$
$$- b_n(1 - b_n - c_n - \mu_n)g(\|T(PT)^{n-1}z_n - x_n\|))$$
$$+ \beta_nk_n^2(k_n^2\|x_n - q\|^2 + \epsilon_{(3)}^n) + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + K^2\lambda_n$$
$$\leq (\alpha_nk_n^6 + \beta_nk_n^4 + (1 - \alpha_n - \beta_n - \lambda_n))\|x_n - q\|^2 + \epsilon_{(8)}^n$$
$$- \alpha_nk_n^2b_n(1 - b_n - c_n - \mu_n)g(\|T(PT)^{n-1}z_n - x_n\|)$$
$$\leq \|x_n - q\|^2 + (\alpha_n(k_n^6 - 1) + \beta_n(k_n^6 - 1))\|x_n - q\|^2 + \epsilon_{(8)}^n$$
$$- \alpha_nb_n(1 - b_n - c_n - \mu_n)g(\|T(PT)^{n-1}z_n - x_n\|)$$
$$\leq \|x_n - q\|^2 + \epsilon_{(9)}^n - \alpha_nb_n(1 - b_n - c_n - \mu_n)g(\|T(PT)^{n-1}z_n - x_n\|), \quad (4.34)$$

where $\epsilon_{(8)}^n = \alpha_nk_n^2\epsilon_{(7)}^n + \beta_nk_n^2\epsilon_{(3)}^n + K^2\lambda_n$ and $\epsilon_{(9)}^n = \epsilon_{(8)}^n + K^2(k_n^6 - 1)$. It is worth to note here that $\sum_{n=1}^{\infty}\epsilon_{(8)}^n < \infty$ and $\sum_{n=1}^{\infty}\epsilon_{(9)}^n < \infty$ since $\{k_n\}$ is bounded, $\sum_{n=1}^{\infty}\epsilon_{(7)}^n < \infty$, $\sum_{n=1}^{\infty}\epsilon_{(3)}^n < \infty$ and $\sum_{n=1}^{\infty}\lambda_n < \infty$.

By our assumption $0 < \liminf_{n\to\infty}\alpha_n$ and $0 < \liminf_{n\to\infty}b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, there exists $n_0 \in \mathbb{N}$ and $\delta_1, \delta_2 \in (0,1)$ such that $0 < \delta_1 < \alpha_n$, $0 < \delta_1 < b_n$ and $b_n + c_n + \mu_n < \delta_2 < 1$ for all $n \geq n_0$. Hence, by (4.34), we have

$$\delta_1^2(1 - \delta_2)\sum_{n=n_0}^{m} g(\|T(PT)^{n-1}z_n - x_n\|) \leq \sum_{n=n_0}^{m}(\|x_n - q\|^2 - \|x_{n+1} - q\|^2) + \sum_{n=n_0}^{m}\epsilon_{(9)}^n$$
$$= \|x_{n_0} - q\|^2 + \sum_{n=n_0}^{m}\epsilon_{(9)}^n.$$

Since $\sum_{n=n_0}^{\infty} \epsilon_{(9)}^{n} < \infty$, by letting $m \to \infty$, we get $\sum_{n=n_0}^{\infty} g(\|T(PT)^{n-1} z_n - x_n\|) < \infty$, and therefore $\lim_{n\to\infty} g(\|T(PT)^{n-1}z_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} \|T(PT)^{n-1}z_n - x_n\| = 0$.

(iv) Suppose that the conditions (ii) and (iii) are satisfied, we have

$$\lim_{n\to\infty} \|T(PT)^{n-1}y_n - x_n\| = 0 \text{ and } \lim_{n\to\infty} \|T(PT)^{n-1}z_n - x_n\| = 0. \tag{4.35}$$

From $y_n = P(b_nT(PT)^{n-1}z_n + c_nT(PT)^{n-1}x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n)$, we have

$$\|y_n - x_n\| \leq b_n\|T(PT)^{n-1}z_n - x_n\| + c_n\|T(PT)^{n-1}x_n - x_n\| + \mu_n\|v_n - x_n\|.$$

It follows that

$$\begin{aligned}\|T(PT)^{n-1}x_n - x_n\| &\leq \|T(PT)^{n-1}x_n - T(PT)^{n-1}y_n\| + \|T(PT)^{n-1}y_n - x_n\| \\ &\leq k_n\|x_n - y_n\| + \|T(PT)^{n-1}y_n - x_n\| \\ &\leq k_n(b_n\|T(PT)^{n-1}z_n - x_n\| + c_n\|T(PT)^{n-1}x_n - x_n\| \\ &\quad + \mu_n\|v_n - x_n\|) + \|T(PT)^{n-1}y_n - x_n\| \\ &= k_nb_n\|T(PT)^{n-1}z_n - x_n\| + c_nk_n\|T(PT)^{n-1}x_n - x_n\| \\ &\quad + \mu_nk_n\|v_n - x_n\| + \|T(PT)^{n-1}y_n - x_n\|. \end{aligned} \tag{4.36}$$

By Lemma 2.2.17, there exists positive integer n_1 and $\gamma \in (0, 1)$ such that $c_nk_n < \gamma$ for all $n \geq n_1$. This together with (4.36) implies that for $n \geq n_1$

$$\begin{aligned}\|T(PT)^{n-1}x_n - x_n\| \leq k_nb_n\|T(PT)^{n-1}z_n - x_n\| + \gamma\|T(PT)^{n-1}x_n - x_n\| \\ + \mu_nk_n\|v_n - x_n\| + \|T(PT)^{n-1}y_n - x_n\|.\end{aligned}$$

Hence

$$\begin{aligned}(1-\gamma)\|T(PT)^{n-1}x_n - x_n\| \leq k_nb_n\|T(PT)^{n-1}z_n - x_n\| + \mu_nk_n\|v_n - x_n\| \\ + \|T(PT)^{n-1}y_n - x_n\|.\end{aligned}$$

It follows from (4.35) that $\lim_{n\to\infty} \|T(PT)^{n-1}x_n - x_n\| = 0$. □

Theorem 4.1.9 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous asymptotically nonexpansive nonself-map of C with $\{k_n\}$ satisfying $k_n \geq 1, \sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}$ and $\{\lambda_n\}$ be sequences of real numbers in $[0, 1]$ with $a_n + \gamma_n \in [0, 1], b_n + c_n + \mu_n \in [0, 1]$ and $\alpha_n + \beta_n + \lambda_n \in [0, 1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ and* *(i)* $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$, *or*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$ *and* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$.

Let $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be the sequences defined by the modified Noor iterations with errors (4.18). Then* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *converge strongly to a fixed point of* T.

Proof. By Lemma 4.1.8 , we have

$$\begin{aligned} \lim_{n\to\infty} \|T(PT)^{n-1}y_n - x_n\| &= 0, \\ \lim_{n\to\infty} \|T(PT)^{n-1}z_n - x_n\| &= 0, \\ \lim_{n\to\infty} \|T(PT)^{n-1}x_n - x_n\| &= 0. \end{aligned} \tag{4.37}$$

Since $x_{n+1} = P(\alpha_n(T(PT)^{n-1}y_n + \beta_n(T(PT)^{n-1}z_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n)$. By (4.37), we have

$$\begin{aligned} \|x_{n+1} - x_n\| = \|&P(\alpha_n(T(PT)^{n-1}y_n + \beta_n(T(PT)^{n-1}z_n \\ &+ (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n) - P(x_n)\| \\ &\leq \alpha_n\|T(PT)^{n-1}y_n - x_n\| + \beta_n\|T(PT)^{n-1}z_n - x_n\| \\ &+ \lambda_n\|w_n - x_n\| \to 0 \quad (\text{as} \quad n \to \infty). \end{aligned}$$

Thus

$$\begin{aligned} \|x_{n+1} - T(PT)^{n-1}x_{n+1}\| &\leq \|x_{n+1} - x_n\| + \|T(PT)^{n-1}x_{n+1} - T(PT)^{n-1}x_n\| \\ &\quad + \|T(PT)^{n-1}x_n - x_n\| \\ &\leq \|x_{n+1} - x_n\| + k_n\|x_{n+1} - x_n\| + \|T(PT)^{n-1}x_n - x_n\| \\ &= (1 + k_n)\|x_{n+1} - x_n\| + \|T(PT)^{n-1}x_n - x_n\| \\ &\quad \to 0 \quad (\text{as} \quad n \to \infty). \end{aligned}$$

Hence

$$\begin{aligned} \|x_{n+1} - T(PT)^{n-2}x_{n+1}\| &\leq \|x_{n+1} - x_n + x_n - T(PT)^{n-2}x_n \\ &\quad + T(PT)^{n-2}x_n - T(PT)^{n-2}x_{n+1}\| \\ &\quad \|x_{n+1} - x_n\| + \|T(PT)^{n-2}x_n - x_n\| \\ &\quad + \|T(PT)^{n-2}x_{n+1} - T(PT)^{n-2}x_n\| \\ &\leq \|x_{n+1} - x_n\| + \|T(PT)^{n-2}x_n - x_n\| \\ &\quad + L\|x_{n+1} - x_n\| \to 0 \quad (\text{as} \quad n \to \infty), \end{aligned}$$

where $L = \sup_{n\geq 1} k_n$. We denote $(PT)^{1-1}$ to be the identity maps from C onto itself. It follows that

$$\begin{aligned}\|x_{n+1}-Tx_{n+1}\| &\le \|x_{n+1}-T(PT)^{n-1}x_{n+1}\| \\ &\quad + \|T(PT)^{n-1}x_{n+1}-Tx_{n+1}\| \\ &= \|x_{n+1}-T(PT)^{n-1}x_{n+1}\| \\ &\quad + \|T(PT)^{1-1}(PT)^{n-1}x_{n+1}-T(PT)^{1-1}x_{n+1}\| \\ &\le \|x_{n+1}-T(PT)^{n-1}x_{n+1}\| + L\|(PT)^{n-1}x_{n+1}-x_{n+1}\| \\ &= \|x_{n+1}-T(PT)^{n-1}x_{n+1}\| + L\|(PT)(PT)^{n-2}x_{n+1}-P(x_{n+1})\| \\ &\le \|x_{n+1}-T(PT)^{n-1}x_{n+1}\| + L\|T(PT)^{n-2}x_{n+1}-x_{n+1}\| \\ &\to 0 \quad (\text{as} \quad n\to\infty),\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{4.38}$$

Since T is completely continuous and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore from (4.38), $\{x_{n_k}\}$ converges. Let $\lim_{k\to\infty} x_{n_k} = q$. By continuity of T and (4.38) we have that $Tq = q$, so q is a fixed point of T. By Lemma 4.1.8 (i), $\lim_{n\to\infty}\|x_n - q\|$ exists. But $\lim_{k\to\infty}\|x_{n_k} - q\| = 0$. Thus $\lim_{n\to\infty}\|x_n - q\| = 0$. Since

$$\begin{aligned}\|y_n - x_n\| &= \|P(b_nT(PT)^{n-1}z_n + c_nT(PT)^{n-1}x_n \\ &\quad + (1-b_n-c_n-\mu_n)x_n + \mu_n v_n) - P(x_n)\| \\ &\le b_n\|T(PT)^{n-1}z_n - x_n\| + c_n\|T(PT)^{n-1}x_n - x_n\| \\ &\quad + \mu_n\|v_n - x_n\| \to 0 \ (\text{as } n\to\infty),\end{aligned}$$

and

$$\begin{aligned}\|z_n - x_n\| &= \|P(a_nT(PT)^{n-1}x_n + (1-a_n-\gamma_n)x_n + \gamma_n u_n) - P(x_n)\| \\ &\le a_n\|T(PT)^{n-1}x_n - x_n\| + \gamma_n\|u_n - x_n\| \to 0 \ (\text{as } n\to\infty),\end{aligned}$$

it follows that $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$. □

For $\gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 4.1.9 , we obtain the following result.

Theorem 4.1.10 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous asymptotically nonexpansive nonself-map of C with $\{k_n\}$ satisfying $k_n \ge 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \ne \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \ge 1$, and*

(i) $0 < \min\{\liminf_{n\to\infty}\alpha_n, \liminf_{n\to\infty}\beta_n\} \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *or*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$ *and* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$.

Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined by the modified Noor iterations with errors (4.18). Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

For $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 4.1.9 , we obtain the following result.

Theorem 4.1.11 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous asymptotically nonexpansive nonself-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{\alpha_n\}$ be real sequences in $[0, 1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

For a given $x_1 \in C$, define

$$\begin{aligned} z_n &= P(a_nT(PT)^{n-1}x_n + (1 - a_n)x_n) \\ y_n &= P(b_nT(PT)^{n-1}z_n + (1 - b_n)x_n) \\ x_{n+1} &= P(\alpha_nT(PT)^{n-1}y_n + (1 - \alpha_n)x_n), \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

For $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 4.1.9 , we obtain the following result.

Theorem 4.1.12 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous asymptotically nonexpansive nonself-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{b_n\}, \{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$,*and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

For a given $x_1 \in C$, define

$$\begin{aligned} y_n &= P(b_nT(PT)^{n-1}z_n + (1 - b_n)x_n) \\ x_{n+1} &= P(\alpha_nT(PT)^{n-1}y_n + (1 - \alpha_n)x_n), \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to a fixed point of T.

If T is a self-mapping, then the iterative scheme (4.18) reduces to that of (4.19) and the following result is directly obtained by Theorem 4.1.9.

Theorem 4.1.13 (**[21, Theorem 2.3]**) *Let X be a uniformly convex Banach space, and C a nonempty closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}$ and $\{\lambda_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n + \mu_n \in [0,1]$ and $\alpha_n + \beta_n + \lambda_n \in [0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \lambda_n < \infty$ and*

(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$.

Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined by (4.19). Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

For T is a self-mapping and $\gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 4.1.9 , we obtain the following result.

Theorem 4.1.14 (**[35, Theorem 2.3]**) *Let X be a uniformly convex Banach space, and C a nonempty closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \geq 1$, and*

(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$.

Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined by (4.20). Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

For T is a self-mapping and $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then the iterative scheme (4.18) reduces to that of (4.21) and the following result is directly obtained by Theorem 4.1.9.

Theorem 4.1.15 (**[45, Theorem 2.1]**) *Let X be a uniformly convex Banach space, and let C be a closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{\alpha_n\}$ be real sequences in $[0,1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

For a given $x_1 \in C$, *define*

$$\begin{aligned} z_n &= a_n T^n x_n + (1-a_n)x_n \\ y_n &= b_n T^n z_n + (1-b_n)x_n, \quad n \geq 1 \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n. \end{aligned}$$

Then $\{x_n\}, \{y_n\}$, *and* $\{z_n\}$ *converge strongly to a fixed point of* T.

When T is a self-mapping and $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 4.1.9 , we can obtain Ishikawa-type convergence result which is a generalization of Theorem 3 in [29].

Theorem 4.1.16 *Let* X *be a uniformly convex Banach space, and let* C *be a closed, bounded and convex subset of* X. *Let* T *be a completely continuous asymptotically nonexpansive self-map of* C *with* $\{k_n\}$ *satisfying* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{b_n\}, \{\alpha_n\}$ *be a real sequence in* $[0,1]$ *satisfying*

(*i*) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$,*and*
(*ii*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

For a given $x_1 \in C$, *define*

$$\begin{aligned} y_n &= b_n T^n z_n + (1-b_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n, \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to a fixed point of* T.

In the next result, we prove weak convergence of the modified Noor iterations with errors for asymptotically nonexpansive nonself-mapping in a uniformly convex Banach space satisfying Opial's condition.

Theorem 4.1.17 *Let* X *be a uniformly convex Banach space which satisfies Opial's condition, and* C *a nonempty closed convex nonexpansive retract of* X *with* P *as a nonexpansive retraction. Let* $T : C \to X$ *be an asymptotically nonexpansive nonself-map of* C *with* $\{k_n\}$ *satisfying* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ *and* $F(T) \neq \emptyset$. *Let* $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\lambda_n\}$ *be sequences of real numbers in* $[0,1]$ *with* $a_n + \gamma_n, b_n + c_n + \mu_n$ *and* $\alpha_n + \beta_n + \lambda_n$ *are in* $[0,1]$ *for all* $n \geq 1$, *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and*

(*i*) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$, *or*

(*ii*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$ *and* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$.

Let $\{x_n\}$ *be the sequence defined by modified Noor iterations with errors (4.18). Then* $\{x_n\}$ *converges weakly to a fixed point of* T.

Proof. It follows from Lemma 4.1.8 (iv) that $\lim_{n\to\infty}\|T(PT)^{n-1}x_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 2.4.2, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 2.4.2, $u, v \in F(T)$. By Lemma 4.1.8 (i), $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. It follows from Lemma 2.2.16 that $u = v$. Therefore $\{x_n\}$ converges weakly to fixed point of T. □

For $\gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 4.1.17, we obtain the following result.

Corollary 4.1.18 *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be an asymptotically nonexpansive nonself-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be sequences of real numbers in $[0, 1]$ with $b_n + c_n \in [0, 1]$ and $\alpha_n + \beta_n \in [0, 1]$ for all $n \geq 1$, and*

(i) $0 < \min\{\liminf_{n\to\infty}\alpha_n, \liminf_{n\to\infty}\beta_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *or*
(ii) $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$ *and* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$.

Let $\{x_n\}$ be the sequence defined by modified Noor iterations with errors (4.18). Then $\{x_n\}$ converges weakly to a fixed point of T.

For $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 4.1.17, we obtain the following result.

Corollary 4.1.19 *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be an asymptotically nonexpansive nonself-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{\alpha_n\}$ be real sequences in $[0, 1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}\alpha_n < 1$,

For a given $x_1 \in C$, define

$$
\begin{aligned}
z_n &= P(a_nT(PT)^{n-1}x_n + (1 - a_n)x_n) \\
y_n &= P(b_nT(PT)^{n-1}z_n + (1 - b_n)x_n), \quad n \geq 1 \\
x_{n+1} &= P(\alpha_nT(PT)^{n-1}y_n + (1 - \alpha_n)x_n).
\end{aligned}
$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

For $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 4.1.17, we obtain the following result.

Corollary 4.1.20 *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be an asymptotically nonexpansive nonself-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{b_n\}, \{\alpha_n\}$ be real sequences in $[0,1]$ satisfying*

(*i*) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1,$ *and*
(*ii*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$

For a given $x_1 \in C$, define

$$\begin{aligned} y_n &= P(b_nT(PT)^{n-1}z_n + (1-b_n)x_n) \\ x_{n+1} &= P(\alpha_nT(PT)^{n-1}y_n + (1-\alpha_n)x_n), \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

4.2 Common Fixed Points of Asymptotically Nonexpansive Mappings

In 2001, Xu and Ori [46] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in J\}$ (here $J = \{1, 2, \ldots, N\}$) with $\{\alpha_n\}$ is a real sequence in $(0, 1)$, and an initial point $x_0 \in C$:

$$\begin{aligned} x_1 &= \alpha_1x_0 + (1-\alpha_1)T_1x_1, \\ x_2 &= \alpha_2x_1 + (1-\alpha_2)T_2x_2, \\ &\vdots \\ x_N &= \alpha_Nx_{N-1} + (1-\alpha_N)T_Nx_N, \\ x_{N+1} &= \alpha_{N+1}x_N + (1-\alpha_{N+1})T_{N+1}x_{N+1}, \\ &\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_nx_{n-1} + (1-\alpha_n)T_nx_n, \quad \forall n \geqslant 1, \tag{4.39}$$

where $T_n = T_{n(mod\ N)}$ (here the *mod* N function takes values in J). Xu and Ori also proved the weak convergence of this process to a common fixed point of the finite family of nonexpansive mappings in a Hilbert space.

In [50], Zhou and Chang studied the weak and strong convergence of this implicit process to a common fixed point for a finite family of nonexpansive mappings in a uniformly convex Banach space. Recently, Chidume and Shahzad [8] proved that Xu and Ori's iteration process converges strongly to a common fixed point for a finite family of nonexpansive mappings if one of the mappings is semi-compact. Inspired and motivated by these facts, we will extend the process (4.39) to a process for a finite family of asymptotically quasi-nonexpansive mappings in a uniformly convex Banach spaces, which is defined as follows:

Let X be a normed linear space, C a nonempty convex subset of X, $\{T_i : i \in J\}$ a finite families of asymptotically quasi-nonexpansive mappings of C. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$ such that $\sup\{k_n(1-\alpha_n-\beta_n) : n \geq 1\} \leq 1$. Then for arbitrary $x_0 \in C$, the sequence $\{x_n\}$ is generated as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + \beta_1 T_1 x_0 + (1-\alpha_1-\beta_1)T_1 x_1, \\
x_2 &= \alpha_2 x_1 + \beta_2 T_2 x_1 + (1-\alpha_2-\beta_2)T_2 x_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + \beta_N T_N x_{N-1} + (1-\alpha_N-\beta_N)T_N x_N, \\
x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} T_1^2 x_N + (1-\alpha_{N+1}-\beta_{N+1})T_1^2 x_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} T_N^2 x_{2N-1} + (1-\alpha_{2N}-\beta_{2N})T_N^2 x_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} T_1^3 x_{2N+1} + (1-\alpha_{2N+1}-\beta_{2N+1})T_1^3 x_{2N+1}, \\
&\vdots
\end{aligned}$$

is called the implicit iterative sequence for a finite family of asymptotically quasi-nonexpansive mappings $\{T_1, T_2, \ldots, T_N\}$. Since for each $n \geq 1$, it can be written as $n = (k-1)N + i$, where $i = i(n) \in J$, $k = k(n) \geq 1$ is positive integer and $k(n) \to \infty$, as $n \to \infty$. Hence we can write the above table in the following compact form:

$$x_n = \alpha_n x_{n-1} + \beta_n T_i^k x_{n-1} + (1-\alpha_n-\beta_n)T_i^k x_n, \;\; \forall n \geq 1. \qquad (4.40)$$

In this section, we prove weak and strong convergence of the implicit iteration process (4.40) to a common fixed point for a finite family of asymptotically quasi-nonexpansive mappings in a uniformly convex Banach space.

Theorem 4.2.1 *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in J\}$ be N asymptotically quasi-nonexpansive self-mappings of C, i.e., $\|T_i^n x - q\| \leq (1+u_{in})\|x-q\|$ for all $x \in C$, $q \in F(T_i)$, $i \in J$ with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ (here $F(T_i)$ denotes the set of fixed points of T_i). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that α_n, $\alpha_n + \beta_n$ are in $(s, 1-s)$ for some $s \in (0,1)$, for all $n \geq 1$, and $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in J$. Then the implicitly iterative sequence $\{x_n\}$ generated by (4.40)* *converges to a common fixed point in F if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$, *where* $d(x,F)$ *denotes the distance of* x *to set* F, *i.e.,* $d(x,F) = \inf_{y\in F} d(x,y)$.

Proof. The necessity of the conditions is obvious. Thus, we will only prove the sufficiency. For any $p \in F$, from

$$x_n = \alpha_n x_{n-1} + \beta_n T_i^k x_{n-1} + (1-\alpha_n-\beta_n)T_i^k x_n,$$

where $n = (k-1)N + i, T_n = T_n(mod\, N) = T_i, i \in J$, it follows that

$$\begin{aligned}
\|x_n - p\| &= \|\alpha_n x_{n-1} + \beta_n T_i^k x_{n-1} + (1-\alpha_n-\beta_n)T_i^k x_n - p\| \\
&\le \alpha_n\|x_{n-1}-p\| + \beta_n\|T_i^k x_{n-1} - p\| + (1-\alpha_n-\beta_n)\|T_i^k x_n - p\| \\
&\le \alpha_n\|x_{n-1}-p\| + \beta_n(1+u_{ik})\|x_{n-1}-p\| \\
&\quad + (1-\alpha_n-\beta_n)(1+u_{ik})\|x_n - p\| \\
&\le (\alpha_n+\beta_n+\beta_n u_{ik})\|x_{n-1}-p\| + (1-\alpha_n-\beta_n-\beta_n u_{ik}+u_{ik})\|x_n-p\|.
\end{aligned}$$

Transposing and simplifying above inequality, and noticing that $s < \alpha_n + \beta_n < 1 - s < 1$, we have

$$\begin{aligned}
(\alpha_n+\beta_n+\beta_n u_{ik})\|x_n-p\| &\le (\alpha_n+\beta_n+\beta_n u_{ik})\|x_{n-1}-p\| + u_{ik}\|x_n-p\| \\
&\le (\alpha_n+\beta_n+\beta_n u_{ik})\|x_{n-1}-p\| \\
&\quad + u_{ik}(\frac{\alpha_n+\beta_n+\beta_n u_{ik}}{s})\|x_n-p\|.
\end{aligned}$$

Hence

$$\frac{s-u_{ik}}{s}\|x_n-p\| \;\le\; \|x_{n-1}-p\|. \tag{4.41}$$

Since $\sum_{k=1}^{\infty} u_{ik} < \infty$ for all $i \in J$, thus $\lim_{k\to\infty} u_{ik} = 0$, there exists a natural number n_0, as $k > n_0/N + 1$, i.e., $n > n_0$ such that $s - u_{ik} > 0$ and $u_{ik} < \frac{s}{2}$. Then (4.41) becomes

$$\|x_n - p\| \;\le\; \frac{s}{s-u_{ik}}\|x_{n-1}-p\|. \tag{4.42}$$

Let $1 + v_{ik} = \frac{s}{s-u_{ik}} = 1 + \frac{u_{ik}}{s-u_{ik}}$. Then $v_{ik} = (\frac{1}{s-u_{ik}})u_{ik} < \frac{2}{s}u_{ik}$, therefore $\sum_{k=1}^{\infty} v_{ik} < \frac{2}{s}\sum_{k=1}^{\infty} u_{ik} < \infty$ for all $i \in J$ and (4.42) becomes

$$\|x_n - p\| \;\le\; (1+v_{ik})\|x_{n-1}-p\|, \forall p \in F. \tag{4.43}$$

This implies that $d(x_n, F) \le (1+v_{ik})d(x_{n-1}, F)$. From Lemma 2.1.3 we have $\lim_{n\to\infty} d(x_n, F) = 0$. Hereafter, we will prove that $\{x_n\}$ is a Cauchy sequence. Notice that when $x > 0$, $1 + x \le e^x$, from (4.43) we have

$$\begin{aligned}
\|x_{n+m} - p\| &\le \exp\{\sum_{i=1}^{n}\sum_{k=1}^{\infty} v_{ik}\}\|x_n - p\| \\
&< M\|x_n - p\|, \quad \forall p \in F,
\end{aligned} \tag{4.44}$$

for all natural number m, n, where $M = \exp\{\sum_{i=1}^{n}\sum_{k=1}^{\infty} v_{ik}\} + 1 < \infty$. For all $\epsilon > 0$, there exists a natural number n_1 such that when $n \geq n_1$, $d(x_n, F) \leq \frac{\epsilon}{2M}$ as $\lim_{n\to\infty} d(x_n, F) = 0$; specifically, $d(x_{n_1}, F) \leq \frac{\epsilon}{2M}$. Thus there exists a point $p_1 \in F$ such that $\|x_{n_1} - p_1\| \leq d(x_{n_1}, p_1) \leq \frac{\epsilon}{2M}$ by the definition of $d(x_n, F)$. It follows, from (4.44), that when $n \geq n_1$, for all m,

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\ &< M\|x_{n_1} - p_1\| + M\|x_{n_1} - p_1\| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Because the space is complete, the sequence $\{x_n\}$ is convergent. Let $\lim_{n\to\infty} x_n = p$. Moreover, since the set of fixed points of an asymptotically quasi-nonexpansive mapping is closed, so is F, thus $p \in F$ from $\lim_{n\to\infty} d(x_n, F) = 0$, i.e., p is a common point of $F(T_i)$, for all $i \in J$. This completes the proof. □

Corollary 4.2.2 *Suppose the conditions are as same as in Theorem 4.2.1. Then the implicitly iterative sequence $\{x_n\}$ generated by (4.40) converges to a common fixed point $p \in F$ if and only if there exists some infinite subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p.*

The main purpose of this paper is to prove the following convergent result for the process (4.40).

Theorem 4.2.3 *Let X be a uniformly convex Banach space and let C be a bounded closed convex subset of X. Let $\{T_i : i \in J\}$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive self-mappings of C with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ and there exists one member T in $\{T_i : i \in J\}$ to be semi-compact. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ such that α_n, $\alpha_n + \beta_n$ are in $(s, 1-s)$ for some $s \in (0, 1)$, for all $n \geq 1$, and $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in J$. Then sequence $\{x_n\}$ defined by the implicit iteration process (4.40) strongly converges to a common fixed point of the mappings $\{T_i : i \in J\}$.*

Proof. Since C is bounded, take $r > 0$ such that $C \subset B(0, r)$, where $B(0, r)$ is the closed ball of X with center zero and radius r. By Lemma 2.2.13, we get for any $q \in F$

$$\begin{aligned} \|x_n - q\|^2 &= \|\alpha_n x_{n-1} + \beta_n T_i^k x_{n-1} + (1 - \alpha_n - \beta_n) T_i^k x_n - q\|^2 \\ &= \|\alpha_n (x_{n-1} - q) + \beta_n (T_i^k x_{n-1} - q) + (1 - \alpha_n - \beta_n)(T_i^k x_n - q)\|^2 \\ &\leq \alpha_n \|x_{n-1} - q\|^2 + \beta_n \|T_i^k x_{n-1} - q\|^2 \\ &\quad + (1 - \alpha_n - \beta_n)\|T_i^k x_n - q\|^2 - \alpha_n(1 - \alpha_n - \beta_n) g(\sigma_n), \end{aligned}$$

where $\sigma_n = \|T_i^k x_n - x_{n-1}\| = \|T_n^k x_n - x_{n-1}\|, \quad n = (k-1)N + i, \ i \in J.$

Since T_n is asymptotically quasi-nonexpansive, it follows that

$$\begin{aligned}\|x_n - q\|^2 &\le \alpha_n\|x_{n-1} - q\|^2 + \beta_n(1+u_{ik})^2\|x_{n-1} - q\|^2 \\ &\quad + (1-\alpha_n-\beta_n)(1+u_{ik})^2\|x_n - q\|^2 - \alpha_n(1-\alpha_n-\beta_n)g(\sigma_n) \\ &\le \alpha_n\|x_{n-1} - q\|^2 + \beta_n\|x_{n-1} - q\|^2 + \beta_n v_{ik}\|x_{n-1} - q\|^2 \\ &\quad + (1-\alpha_n-\beta_n-\beta_n v_{ik} + v_{ik})\|x_n - q\|^2 - \alpha_n(1-\alpha_n-\beta_n)g(\sigma_n),\end{aligned} \tag{4.45}$$

where $v_{ik} = 2u_{ik} + u_{ik}^2$. Hence $\sum_{k=1}^{\infty} v_{ik} < \infty$ for all $i \in J$. Thus, from (4.45) and $s < \alpha_n + \beta_n \le \alpha_n + \beta_n + \beta_n v_{ik}$, we have

$$\begin{aligned}(\alpha_n + \beta_n + \beta_n v_{ik})\|x_n - q\|^2 \le (\alpha_n + \beta_n + \beta_n v_{ik})\|x_{n-1} - q\|^2 \\ + v_{ik}\frac{(\alpha_n + \beta_n + \beta_n v_{ik})}{s}\|x_n - q\|^2 \\ - \alpha_n(1-\alpha_n-\beta_n)g(\sigma_n).\end{aligned} \tag{4.46}$$

Hence $\|x_n - q\|^2 \le \|x_{n-1} - q\|^2 + \frac{v_{ik}}{s}\|x_n - q\|^2$. Therefore, as in Theorem 4.2.1, we can show that $\lim_{n\to+\infty}\|x_n - q\|^2$ exists and let $\lim_{n\to+\infty}\|x_n - q\|^2 = d$. From (4.46) and $s < \alpha_n \le \alpha_n + \beta_n < 1 - s$, $\forall n \in \mathbb{N}$, we have

$$\begin{aligned}\frac{s^2}{1-s+\beta_n v_{ik}}g(\sigma_n) &< \frac{\alpha_n(1-\alpha_n-\beta_n)}{\alpha_n+\beta_n+\beta_n v_{ik}}g(\sigma_n) \\ &\le \|x_{n-1} - q\|^2 - \|x_n - q\|^2 + \frac{v_{ik}}{s}\|x_n - q\|^2.\end{aligned}$$

Thus

$$g(\sigma_n) \le \frac{1-s+\beta_n v_{ik}}{s^2}(\|x_{n-1} - q\|^2 - \|x_n - q\| + \frac{v_{ik}}{s}\|x_n - q\|^2).$$

Since $\sum_{k=1}^{\infty} v_{ik} < \infty$, there exists $K > 0$ such that

$$g(\sigma_n) \le \frac{1-s+K}{s^2}(\|x_{n-1} - q\|^2 - \|x_n - q\|^2 + \frac{v_{ik}}{s}\|x_n - q\|^2).$$

Hence

$$\sum_{n=1}^{m} g(\sigma_n) \le \frac{1-s+K}{s^2}\sum_{n=1}^{m}(\|x_{n-1} - q\|^2 - \|x_n - q\|^2 + v_{ik}M), \tag{4.47}$$

where $M = \frac{2r}{s} < \infty$, r is the ball radius. Since $\sum_{k=1}^{\infty} v_{ik} < \infty$, by letting $m \to \infty$ in (4.47) we get $\sum_{n=1}^{\infty} g(\sigma_n) < \infty$, and therefore $\lim_{n\to\infty} g(\sigma_n) = 0$. Since g is strictly increasing and continuous at 0 with $g(0)$, it follows that

$$\lim_{n\to\infty}\sigma_n = \lim_{n\to\infty}\|T_n^k x_n - x_{n-1}\| = 0.$$

Since T_i^k is asymptotically quasi-nonexpansive, we have

$$\begin{aligned}\|x_n - x_{n-1}\| &= \|\beta_n(T_i^k x_{n-1} - x_{n-1}) + (1-\alpha_n-\beta_n)(T_i^k x_n - x_{n-1})\| \\ &\le \beta_n\|T_i^k x_{n-1} - x_{n-1}\| + (1-\alpha_n-\beta_n)\|T_i^k x_n - x_{n-1}\| \\ &\le \beta_n\|T_i^k x_{n-1} - T_i^k x_n\| + \beta_n\|T_i^k x_n - x_{n-1}\| \\ &\quad + (1-\alpha_n-\beta_n)\|T_i^k x_n - x_{n-1}\| \\ &\le \beta_n(1+u_{ik})\|x_{n-1} - x_n\| + \beta_n\|T_i^k x_n - x_{n-1}\| \\ &\quad + (1-\alpha_n-\beta_n)\|T_i^k x_n - x_{n-1}\| \\ &\le \beta_n(1+\frac{s}{2})\|x_{n-1} - x_n\| + \beta_n\|T_n^k x_n - x_{n-1}\| \\ &\quad + (1-\alpha_n-\beta_n)\|T_n^k x_n - x_{n-1}\|.\end{aligned}$$

This implies that

$$(1-\beta_n-\beta_n\frac{s}{2})\|x_n - x_{n-1}\| \le \beta_n\|T_n^k x_n - x_{n-1}\| + (1-\alpha_n-\beta_n)\|T_n^k x_n - x_{n-1}\|.$$

From $\beta_n \le \alpha_n + \beta_n < 1-s$, $\forall n \in \mathbb{N}$, we have

$$(s-(1-s)\frac{s}{2})\|x_n - x_{n-1}\| \le (1-\alpha_n)\|T_n^k x_n - x_{n-1}\|.$$

It follows that $\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0$. Also $\lim_{n\to\infty}\|x_n - x_{n+l}\| = 0$ for all $l < 2N$. Hence, when $n > N$, we have

$$\begin{aligned}\|x_{n-1} - T_n x_n\| &\le \|x_{n-1} - T_n^k x_n\| + \|T_n^k x_n - T_n x_n\| \\ &\le \sigma_n + L(\|T_n^{k-1}x_n - T_{n-N}^{k-1}x_{n-N}\| + \\ &\quad \|T_{n-N}^{k-1}x_{n-N} - x_{(n-N)-1}\| + \|x_{(n-N)-1} - x_n\|).\end{aligned}$$

Notice that $n \equiv (n-N)(mod\, N)$. Thus $T_n = T_{n-N}$ and above inequality becomes

$$\|x_{n-1} - T_n x_n\| \le \sigma_n + L^2\|x_n - x_{n-N}\| + L\sigma_{n-N} + L\|x_n - x_{(n-N)-1}\|,$$

which yields $\lim_{n\to\infty}\|x_{n-1} - T_n x_n\| = 0$. From

$$\|x_n - T_n x_n\| \le \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|,$$

it follows that $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. Hence for all $l \in J$

$$\begin{aligned}\|x_n - T_{n+l}x_n\| &\le \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|T_{n+l}x_{n+l} - T_{n+l}x_n\| \\ &\le (1+L)\|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\|,\end{aligned}$$

we have that $\lim_{n\to\infty}\|x_n - T_{n+l}x_n\| = 0 \quad (\forall l \in J)$. Since for each $l \in J$, $\{\|x_n - T_{n+l}x_n\|\}$ is a subset of $\cup_{l=1}^N\{\|x_n - T_{n+l}x_n\|\}$, we have

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0 \quad (\forall l \in J). \tag{4.48}$$

By hypothesis that there exists T in $\{T_i : i \in J\}$ to be semi-compact, we may assume that T_1 is semi-compact without loss of generality. Therefore by (4.48), we have $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$ and by the definition of semi-compact there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^* \in C$. By (4.48) again, we have

$$\|x^* - T_l x^*\| = \lim_{n_j\to\infty} \|x_{n_j} - T_l x_{n_j}\| = 0 \quad (\forall 1 \le l \le N).$$

It shows that $x^* \in F$ and $\liminf_{n\to+\infty} d(x_n, F) = 0$, therefore by Theorem 4.2.1 and Corollary 4.2.2 we have that x_n converges to a common fixed point q in F. This completes the proof. □

Theorem 4.2.4 *Let X be a uniformly convex Banach space and let C be a bounded closed convex subset of X. Let $T_i, i \in, J$ be N asymptotically nonexpansive self-mappings of C with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ and there exists one member T in $\{T_i : i \in J\}$ to be semi-compact. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that α_n, $\alpha_n + \beta_n$ are in $(s, 1-s)$ for some $s \in (0,1)$, for all $n \ge 1$, and $\sum_{n=1}^\infty u_{in} < \infty$ for all $i \in J$. Then the sequence $\{x_n\}$ defined by the implicit iteration process (4.40) strongly converges to a common fixed point of the mappings $\{T_i : i \in J\}$.*

In the next results, we prove weak convergence of the sequence $\{x_n\}$ defined by (4.40) in uniformly convex Banach space satisfying *Opial's condition.*

Lemma 4.2.5 *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C be a nonempty closed convex subset of X. Let $\{T_i, i \in, J\}$ be N asymptotically nonexpansive self-mappings of C with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that α_n, $\alpha_n + \beta_n$ are in $(s, 1-s)$ for some $s \in (0,1)$, for all $n \ge 1$, and $\sum_{n=1}^\infty u_{in} < \infty$ for all $i \in J$. Then the sequence $\{x_n\}$ defined by the implicit iteration process (4.40) weakly converges to a common fixed point of the mappings $\{T_i : i \in J\}$.*

Proof. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to x^*$ weakly as $n \to \infty$, with out loss of generality. By Lemma 2.4.1, we have $x^* \in F(T_i)$ for all $i \in J$. Hence $x^* \in F$. Suppose that there exist subsequence $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to y^* and z^* respectively. By Lemma 2.4.1, $y^*, z^* \in F$. As in Theorem 4.2.1, we have $\lim_{n\to\infty} \|x_n - y^*\|$ and $\lim_{n\to\infty} \|x_n - z^*\|$ exists. It follows from Lemma 2.2.16 we have $y^* = z^*$. Therefore $\{x_n\}$ converges weakly to a common fixed point x^* in F. □