Chapter 5

Common Fixed Points for a Finite Family of Nonexpansive Mappings in a Banach Space

We introduce and study a new multi-step iterative schemes with errors for a finite family of nonexpansive nonself-mappings. Our schemes can be viewed as an extension for three-step iterative schemes of Thianwan and Suantai[40]. The scheme is defined as follows:

Let C be a nonempty closed convex subset of uniformly convex Banach space X with P as a nonexpansive retraction of X onto C, and let $T_1, T_2, \ldots, T_N : C \to X$ be nonexpansive mappings. Let $x_1 \in C$, the multi-step iteration scheme is defined as follows:

$$\begin{aligned} x_n^1 &= P(\alpha_n^1 T_1 x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1) \\ x_n^2 &= P(\alpha_n^2 T_2 x_n + \beta_n^2 x_n^1 + \gamma_n^2 u_n^2) \\ &\vdots \qquad\qquad \vdots \\ x_{n+1} &= x_n^N = P(\alpha_n^N T_N x_n + \beta_n^N x_n^{N-1} + \gamma_n^N u_n^N), \end{aligned} \tag{5.1}$$

$n \geqslant 1$ where $\{\alpha_n^1\}, \{\alpha_n^2\}, \ldots, \{\alpha_n^N\}$, $\{\beta_n^1\}, \{\beta_n^2\}, \ldots, \{\beta_n^N\}$, $\{\gamma_n^1\}, \{\gamma_n^2\}, \ldots, \{\gamma_n^N\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, \ldots, N$ and $\{u_n^1\}, \{u_n^2\}, \ldots, \{u_n^N\}$ are bounded sequences in C.

For $N = 3$, $T_1 = T_2 = T_3 \equiv T$, $a_n = \alpha_n^1$, $c_n = \alpha_n^2$, $\alpha_n = \alpha_n^2$ and $\gamma_n^1 = \gamma_n^2 = \gamma_n^3 \equiv 0$, then (5.1) reduces to the scheme for a mapping defined by Thianwan and Suantai[40]:

$$\begin{aligned} z_n &= P((1 - a_n - b_n)x_n + a_n T x_n + b_n u_n), \\ y_n &= P((1 - c_n - d_n)z_n + c_n T x_n + d_n v_n), \\ x_{n+1} &= P((1 - \alpha_n - \beta_n)y_n + \alpha_n T x_n + \beta_n w_n), \quad n \geq 1, \end{aligned} \tag{5.2}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

In this section, we prove weak and strong convergence theorems of the iterative scheme given in (5.1) to a common fixed point for a finite family of nonexpansive nonself-mappings in a uniformly convex Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 5.0.6 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \dots, T_N : C \to X$ be nonexpansive mappings and let $\{x_n\}$ be the sequence defined by (5.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ for each $i = 1, 2, \dots, N$. If $F = \cap_{i=1}^{N} F(T_i) \neq \varnothing$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F$.*

Proof. For each $n \geqslant 1$, we note that

$$\begin{aligned}
\|x_n^1 - x^*\| &= \|P(\alpha_n^1 T_1 x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1) - x^*\| \\
&\leqslant \|\alpha_n^1 T_1 x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1 - x^*\| \\
&\leqslant \alpha_n^1 \|T_1 x_n - x^*\| + \beta_n^1 \|x_n - x^*\| + \gamma_n^1 \|u_n^1 - x^*\| \\
&\leqslant \alpha_n^1 \|x_n - x^*\| + \beta_n^1 \|x_n - x^*\| + \gamma_n^1 \|u_n^1 - x^*\| \\
&\leqslant \|x_n - x^*\| + d_n^1,
\end{aligned}$$

where $d_n^1 = \gamma_n^1 \|u_n^1 - x^*\|$. Since $\sum_{n=1}^{\infty} \gamma_n^1 < \infty$, $\sum_{n=1}^{\infty} d_n^1 < \infty$. Next, we note that

$$\begin{aligned}
\|x_n^2 - x^*\| &= \|P(\alpha_n^2 T_2 x_n + \beta_n^2 x_n^1 + \gamma_n^2 u_n^2) - x^*\| \\
&\leqslant \|\alpha_n^2 T_2 x_n + \beta_n^2 x_n^1 + \gamma_n^2 u_n^2 - x^*\| \\
&\leqslant \alpha_n^2 \|T_2 x_n - x^*\| + \beta_n^2 \|x_n^1 - x^*\| + \gamma_n^2 \|u_n^2 - x^*\| \\
&\leqslant \alpha_n^2 \|x_n - x^*\| + \beta_n^2 \|x_n^1 - x^*\| + \gamma_n^2 \|u_n^2 - x^*\| \\
&\leqslant \alpha_n^2 \|x_n - x^*\| + \beta_n^2 (\|x_n - x^*\| + d_n^1) + \gamma_n^2 \|u_n^2 - x^*\| \\
&= \alpha_n^2 \|x_n - x^*\| + \beta_n^2 \|x_n - x^*\| + \beta_n^2 d_n^1 + \gamma_n^2 \|u_n^2 - x^*\| \\
&= (\alpha_n^2 + \beta_n^2) \|x_n - x^*\| + \beta_n^2 d_n^1 + \gamma_n^2 \|u_n^2 - x^*\| \\
&\leqslant \|x_n - x^*\| + d_n^2,
\end{aligned}$$

where $d_n^2 = \beta_n^2 d_n^1 + \gamma_n^2 \|u_n^2 - x^*\|$. Since $\sum_{n=1}^{\infty} d_n^1 < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^2 < \infty$, $\sum_{n=1}^{\infty} d_n^2 < \infty$. Similarly, we have

$$\begin{aligned}
\|x_n^3 - x^*\| &= \|P(\alpha_n^3 T_3 x_n + \beta_n^3 x_n^2 + \gamma_n^3 u_n^3) - x^*\| \\
&\leqslant \|\alpha_n^3 T_3 x_n + \beta_n^3 x_n^2 + \gamma_n^3 u_n^3 - x^*\| \\
&\leqslant \alpha_n^3 \|T_3 x_n - x^*\| + \beta_n^3 \|x_n^2 - x^*\| + \gamma_n^3 \|u_n^3 - x^*\| \\
&\leqslant \alpha_n^3 \|x_n - x^*\| + \beta_n^3 (\|x_n - x^*\| + d_n^2) + \gamma_n^3 \|u_n^3 - x^*\| \\
&= \alpha_n^3 \|x_n - x^*\| + \beta_n^3 \|x_n - x^*\| + \beta_n^3 d_n^2 + \gamma_n^3 \|u_n^3 - x^*\| \\
&= (\alpha_n^3 + \beta_n^3) \|x_n - x^*\| + \beta_n^3 d_n^2 + \gamma_n^3 \|u_n^3 - x^*\| \\
&\leqslant \|x_n - x^*\| + d_n^3
\end{aligned}$$

where $d_n^3 = \beta_n^3 d_n^2 + \gamma_n^3 \|u_n^3 - x^*\|$, and we also have that $\sum_{n=1}^{\infty} d_n^3 < \infty$.

By continuing the above method, there exists a nonnegative real sequences $\{d_n^i\}$ such that $\sum_{n=1}^{\infty} d_n^i < \infty$ and

$$\|x_n^i - x^*\| \leqslant \|x_n - x^*\| + d_n^i, \quad \forall n \geqslant 1, \quad \forall i = 1, 2, \ldots, N. \tag{5.3}$$

Thus $\|x_{n+1} - x^*\| = \|x_n^N - x^*\| \leqslant \|x_n - x^*\| + d_n^N$ for all $n \in \mathbb{N}$. Hence, by Lemma 2.1.3, $\lim_{n\to\infty} \|x_n - x^*\|$ exist. This completes the proof. □

Lemma 5.0.7 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be nonexpansive mappings and let $\{x_n\}$ be the sequence defined by (5.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \ldots, N$ for some $\varepsilon \in (0,1)$. If $F = \cap_{i=1}^N F(T_i) \neq \varnothing$ and $\limsup_{n\to\infty}(\alpha_n^i + \beta_n^i) < 1$ for all $i = 1, 2, \ldots, N$, then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \ldots, N$.*

Proof. Let $x^* \in F$. Then, by Lemma 5.0.6, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Let $\lim_{n\to\infty} \|x_n - x^*\| = r$. If $r = 0$, then by the continuity of each T_i the conclusion follows. Suppose that $r > 0$. Since $\{x_n\}$ and $\{u_n^i\}$ are bounded for all $i = 1, 2, \ldots, N$, there exists $R > 0$ such that $x_n^{i-1} - x^* + \gamma_n^i(u_n^i - x_n^{i-1})$, $T_i x_n - x^* + \gamma_n^i(u_n^i - x_n^{i-1}) \in B_R(0)$ for all $i = 1, 2, \ldots, N$ for all $n \geqslant 1$. Using Lemma 2.2.12 and (5.3), we have

$$\begin{aligned}
\|x_n^i - x^*\|^2 &= \|P(\alpha_n^i T_i x_n + \beta_n^i x_n^{i-1} + \gamma_n^i u_n^i) - x^*\|^2 \\
&\leqslant \|\alpha_n^i T_i x_n + \beta_n^i x_n^{i-1} + \gamma_n^i u_n^i - x^*\|^2 \\
&= \|\alpha_n^i (T_i x_n - x^* + \gamma_n^i(u_n^i - x_n^{i-1})) \\
&\quad + (1-\alpha_n^i)(x_n^{i-1} - x^* + \gamma_n^i(u_n^i - x_n^{i-1}))\|^2 \\
&\leqslant \alpha_n^i \|T_i x_n - x^* + \gamma_n^i(u_n^i - x_n^{i-1})\|^2 \\
&\quad + (1-\alpha_n^i)\|x_n^{i-1} - x^* + \gamma_n^i(u_n^i - x_n^{i-1})\|^2 \\
&\quad - W_2(\alpha_n^i) g(\|T_i x_n - x_n^{i-1}\|) \\
&\leqslant \alpha_n^i (\|x_n - x^*\| + \gamma_n^i \|u_n^i - x_n^{i-1}\|)^2 \\
&\quad + (1-\alpha_n^i)(\|x_n^{i-1} - x^*\| + \gamma_n^i \|u_n^i - x_n^{i-1}\|)^2 \\
&\quad - W_2(\alpha_n^i) g(\|T_i x_n - x_n^{i-1}\|) \\
&\leqslant \alpha_n^i (\|x_n - x^*\| + d_n^{i-1} + \gamma_n^i \|u_n^i - x_n^{i-1}\|)^2 \\
&\quad + (1-\alpha_n^i)(\|x_n - x^*\| + d_n^{i-1} + \gamma_n^i \|u_n^i - x_n^{i-1}\|)^2 \\
&\quad - W_2(\alpha_n^i) g(\|T_i x_n - x_n^{i-1}\|) \\
&= (\|x_n - x^*\| + \lambda_n^{i-1})^2 - W_2(\alpha_n^i) g(\|T_i x_n - x_n^{i-1}\|),
\end{aligned} \tag{5.4}$$

where $\lambda_n^{i-1} := d_n^{i-1} + \gamma_n^i \|u_n^i - x_n^{i-1}\|$. Since $\sum_{n=1}^{\infty} d_n^{i-1} < \infty$, $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\|u_n^i - x_n^{i-1}\|\}$ is bounded, we have that $\sum_{n=1}^{\infty} \lambda_n^{i-1} < \infty$. Since $\alpha_n^i \in [\varepsilon, 1-\varepsilon]$, it follows that $\lambda = \varepsilon^2 \leq W_2(\alpha_n^i)$ for all $n \in N$. This together with (5.4) imply that

$$\begin{aligned} \lambda g(\|T_i x_n - x_n^{i-1}\|) &\leqslant (\|x_n - x^*\| + \lambda_n^{i-1})^2 - \|x_n^i - x^*\|^2 \\ &= \|x_n - x^*\|^2 + 2\lambda_n^{i-1}\|x_n - x^*\| + (\lambda_n^{i-1})^2 \\ &\quad -\|x_{n+1} - x^*\|^2 \\ &= \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \rho_n^{i-1}, \end{aligned}$$

where $\rho_n^{i-1} := 2\lambda_n^{i-1}\|x_n - x^*\| + (\lambda_n^{i-1})^2$. Since $\sum_{n=1}^{\infty} d_n^{i-1} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^i < \infty$, we get $\sum_{n=1}^{\infty} \rho_n^{i-1} < \infty$. This implies that $\lim_{n\to\infty} g(\|T_i x_n - x_n^{i-1}\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, we have $\lim_{n\to\infty} \|T_i x_n - x_n^{i-1}\| = 0$ for all $i = 1, 2, \ldots, N$. Note that,

$$\begin{aligned} \|x_n^{i-1} - x_n^{i-2}\| &= \|P(\alpha_n^{i-1}T_{i-1}x_n + \beta_n^{i-1}x_n^{i-2} + \gamma_n^{i-1}u_n^{i-1}) - x_n^{i-2}\| \\ &\leqslant \|\alpha_n^{i-1}(T_{i-1}x_n - x_n^{i-2}) + \gamma_n^{i-1}(u_n^{i-1} - x_n^{i-2}))\| \\ &\leqslant \alpha_n^{i-1}\|T_{i-1}x_n - x_n^{i-2}\| + \gamma_n^{i-1}\|u_n^{i-1} - x_n^{i-2}\|. \end{aligned}$$

Since $\lim_{n\to\infty} \|T_{i-1}x_n - x_n^{i-2}\| = 0$ and $\sum_{n=1}^{\infty} \gamma_n^{i-1} < \infty$, it follows that

$$\lim_{n\to\infty} \|x_n^{i-1} - x_n^{i-2}\| = 0.$$

For all $i = 1, 2, \ldots, N$, we have

$$\begin{aligned} \|x_n^{i-1} - x_n\| &= \|P(\alpha_n^{i-1}T_{i-1}x_n + \beta_n^{i-1}x_n^{i-2} + \gamma_n^{i-1}u_n^{i-1}) - x_n\| \\ &\leqslant \|\alpha_n^{i-1}T_{i-1}x_n + \beta_n^{i-1}x_n^{i-2} + \gamma_n^{i-1}u_n^{i-1} - x_n\| \\ &= \|\alpha_n^{i-1}(T_{i-1}x_n - x_n) + \beta_n^{i-1}(x_n^{i-2} - x_n) \\ &\quad +\gamma_n^{i-1}(u_n^{i-1} - x_n)\| \\ &= \|\alpha_n^{i-1}(T_{i-1}x_n - x_n^{i-2} + x_n^{i-2} - x_n) \\ &\quad +\beta_n^{i-1}(x_n^{i-2} - x_n) + \gamma_n^{i-1}(u_n^{i-1} - x_n)\| \\ &\leqslant \alpha_n^{i-1}\|T_{i-1}x_n - x_n^{i-2}\| + \alpha_n^{i-1}\|x_n^{i-2} - x_n\| \\ &\quad +\beta_n^{i-1}\|x_n^{i-2} - x_n\| + \gamma_n^{i-1}\|u_n^{i-1} - x_n\| \\ &\leqslant \alpha_n^{i-1}\|T_{i-1}x_n - x_n^{i-2}\| + \alpha_n^{i-1}\|x_n^{i-2} - x_n^{i-1} \\ &\quad +x_n^{i-1} - x_n\| + \beta_n^{i-1}\|x_n^{i-2} - x_n^{i-1} + x_n^{i-1} - x_n\| \\ &\quad +\gamma_n^{i-1}\|u_n^{i-1} - x_n\| \\ &\leqslant \alpha_n^{i-1}\|T_{i-1}x_n - x_n^{i-2}\| + \alpha_n^{i-1}\|x_n^{i-2} - x_n^{i-1}\| \\ &\quad +\alpha_n^{i-1}\|x_n^{i-1} - x_n\| + \beta_n^{i-1}\|x_n^{i-2} - x_n^{i-1}\| \\ &\quad +\beta_n^{i-1}\|x_n^{i-1} - x_n\| + \gamma_n^{i-1}\|u_n^{i-1} - x_n\|. \end{aligned}$$

This implies that

$$\begin{aligned} (1 - \alpha_n^{i-1} - \beta_n^{i-1})\|x_n^{i-1} - x_n\| &\leqslant \alpha_n^{i-1}\|T_{i-1}x_n - x_n^{i-2}\| \\ &\quad + \alpha_n^{i-1}\|x_n^{i-2} - x_n^{i-1}\| \\ &\quad + \beta_n^{i-1}\|x_n^{i-2} - x_n^{i-1}\| \\ &\quad + \gamma_n^{i-1}\|u_n^{i-1} - x_n\| \end{aligned}$$

Since $\limsup_{n\to\infty}(\alpha_n^i + \beta_n^i) < 1$ for all $i = 1, 2, \ldots, N$, there exists a positive integer n_0 and $\eta \in (0,1)$ such that $\alpha_n^i + \beta_n^i < \eta < 1$ for all $n \geq n_0$.

Hence,

$$(1-\eta)\|x_n^{i-1} - x_n\| \leq \|T_{i-1}x_n - x_n^{i-2}\| + \gamma_n^{i-1}\|u_n^{i-1} - x_n\| + 2\|x_n^{i-2} - x_n^{i-1}\|.$$

Since $\lim_{n\to\infty}\|T_{i-1}x_n - x_n^{i-2}\| = 0$, $\lim_{n\to\infty}\|x_n^{i-2} - x_n^{i-1}\| = 0$ and $\sum_{n=1}^{\infty}\gamma_n^{i-1} < \infty$, it follows that $\lim_{n\to\infty}\|x_n - x_n^{i-1}\| = 0$.
Thus for all $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}\|x_n - T_i x_n\| &= \|x_n - x_n^{i-1} + x_n^{i-1} - T_i x_n\| \\ &\leq \|x_n - x_n^{i-1}\| + \|T_i x_n - x_n^{i-1}\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

This completes the proof. □

Theorem 5.0.8 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be nonexpansive mappings which satisfying condition (B) and let $\{x_n\}$ be the sequence defined by (5.1) with $\sum_{n=1}^{\infty}\gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \ldots, N$ for some $\varepsilon \in (0,1)$. If $F = \cap_{i=1}^{N}F(T_i) \neq \varnothing$ and $\limsup_{n\to\infty}(\alpha_n^i + \beta_n^i) < 1$ for all $i = 1, 2, \ldots, N$, then $\{x_n\}$ converges strongly to a common fixed point in F.*

Proof. By Lemma 5.0.7, $\lim_{n\to\infty}\|T_i x_n - x_n\| = 0$ for all $i = 1, 2, \ldots, N$. Now by condition (B), $f(d(x_n, F)) \leqslant M_n := \max_{1\leqslant i\leqslant N}\{\|T_i x_n - x_n\|\}$ for all $n \in N$. Hence $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$.

Now we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{y_j\} \in F$ such that $\|x_{n_j} - y_j\| < 2^{-j}$. By the following method of the proof of Tan and Xu [39] we get that $\{y_j\}$ is a Cauchy sequence in F and so it converges. Let $y_j \to y$. Since F is closed, therefore $y \in F$ and then $x_{n_j} \to y$. By Lemma 5.0.6, $\lim_{n\to\infty}\|x_n - x^*\|$ exists for all $x^* \in F$, $x_n \to y \in F$. □

When $N = 3$, $T_1 = T_2 = T_3 \equiv T$, $a_n = \alpha_n^1$, $c_n = \alpha_n^2$, $\alpha_n = \alpha_n^3$ and $\gamma_n^1 = \gamma_n^2 = \gamma_n^3 \equiv 0$, the following results is obtained from Theorem 5.0.8.

Corollary 5.0.9 ([40, Theorem 2.2]) *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T : C \to X$ a completely continuous nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}$ and $\{d_n\}$ are sequences of real numbers in $[0,1]$ with $c_n + d_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty}\beta_n < \infty$, and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$.

Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined by the three-step iterative scheme (5.2). Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

We recall that a mapping $T : C \to C$ is called semi-compact(or hemicompact) if any sequence $\{x_n\}$ in C satisfying $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ has a convergent subsequence.

Theorem 5.0.10 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be nonexpansive mappings. Suppose that one of the mappings in $\{T_i : i = 1, 2, \ldots, N\}$ is semi-compact. Let $\{x_n\}$ be the sequence defined by (5.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \ldots, N$ for some $\varepsilon \in (0,1)$. If $F = \cap_{i=1}^{N} F(T_i) \neq \varnothing$ and $\limsup_{n\to\infty}(\alpha_n^i + \beta_n^i) < 1$ for all $i = 1, 2, \ldots, N$, then $\{x_n\}$ converges strongly to a common fixed point in F.*

Proof. Suppose that T_{i_0} is *semi-compact* for some $i_0 \in \{1, 2, \ldots, N\}$. By Lemma 5.0.6, we have $\lim_{n\to\infty} \|x_n - T_{i_0}x_n\| = 0$. Since T_{i_0} is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $x^* \in C$ such that $x_{n_j} \to x^*$ as $j \to \infty$. Now Lemma 5.0.7 guarantees that $\lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$ for all $i = 1, 2, \ldots, N$. Hence $\|x^* - T_i x^*\| = 0$ for all $i = 1, 2, \ldots, N$. This implies that $x^* \in F$. By Lemma 5.0.6, $\lim_{n\to\infty} \|x_n - x^*\|$ exists and then $\lim_{n\to\infty} \|x_n - x^*\| = \lim_{j\to\infty} \|x_{n_j} - x^*\| = 0$. This completes the proof. □

In the next result, we prove weak convergence of the sequence $\{x_n\}$ defined by (5.1) in a uniformly convex Banach space satisfying *Opial's condition.*

Theorem 5.0.11 *Let X be a uniformly convex Banach space satisfying the Opial's condition, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be nonexpansive mappings and let $\{x_n\}$ be a sequence defined by (5.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \ldots, N$ for some $\varepsilon \in (0,1)$. If $F = \cap_{i=1}^{N} F(T_i) \neq \varnothing$ and $\limsup_{n\to\infty}(\alpha_n^i + \beta_n^i) < 1$, then $\{x_n\}$ converges weakly to a common fixed point in F.*

Proof. By Lemma 5.0.6, $\lim_{x\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F$. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in F. To prove this, let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ and $z_1, z_2 \in C$ be such that $x_{n_i} \to z_1$ weakly as $i \to \infty$ and $x_{n_j} \to z_2$ weakly as $j \to \infty$. By Lemma 5.0.7,

$$\lim_{i\to\infty} \|x_{n_i} - T_k x_{n_i}\| = 0 = \lim_{j\to\infty} \|x_{n_j} - T_k x_{n_j}\|$$

for all $k = 1, 2, \ldots, N$ and Lemma 2.3.5 insures that $I - T_k$ are demi-closed at zero for all $k = 1, 2, \ldots, N$. Therefore we obtain $T_k z_1 = z_1$ and $T_k z_2 = z_2$ for all $k = 1, 2, \ldots, N$. Thus $z_1, z_2 \in F$. It follows from Lemma 2.2.16, we have that $z_1 = z_2$. Hence $\{x_n\}$ converges weakly to a common fixed point in F. □

When $N = 3$, $T_1 = T_2 = T_3 \equiv T$, $a_n = \alpha_n^1$, $c_n = \alpha_n^2$, $\alpha_n = \alpha_n^3$ and $\gamma_n^1 = \gamma_n^2 = \gamma_n^3 \equiv 0$, the following results is obtained from Theorem 5.0.11.

Corollary 5.0.12 ([40, Theorem 2.4]) *Let X be a uniformly convex Banach space which satisfies Opial's condition, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T : C \to X$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ are sequences of real numbers in $[0,1]$ with $c_n + d_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$.

Let $\{x_n\}$ be the sequence defined by three-step iterative scheme (5.2). Then $\{x_n\}$ converges weakly to a fixed point of T.

Finally, we prove weak convergence of the sequence $\{x_n\}$ defined by (5.1) in a uniformly convex Banach space X whose its dual X^* has the Kadec-Klee property. The following lemma is needed.

Lemma 5.0.13 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be nonexpansive mappings with $F = \cap_{i=1}^{N} F(T_i) \neq \varnothing$. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the iterative scheme (5.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ for $i = 1, 2, \ldots, N$. Then for all $u, v \in F$, the limit*

$$\lim_{n\to\infty} \|tx_n + (1-t)u - v\|$$

exists for all $t \in [0,1]$.

Proof. By Lemma 5.0.6, we have $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F$. This implies that $\{x_n\}$ is bounded. Let $a_n(t) = \|tx_n + (1-t)u - v\|$, where $t \in (0,1)$. Then $\lim_{n\to\infty} a_n(0) = \|u - v\|$, and from Lemma 5.0.6, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - v\|$ exists. So, we let $\lim_{n\to\infty} \|x_n - u\| = r$ for some positive number r. For any $n \geqslant 1$ and for all $i = 1, 2, \ldots, N$, we define $A_n^i : C \to C$ by

$$A_n^i := P(\alpha_n^i T_i + \beta_n^i A_n^{i-1} + \gamma_n^i u_n^i),$$

where $A_n^0 = I$, the identity operator on C. For $x, y \in C$, we have $\|A_n^i x - A_n^i y\| \leqslant \alpha_n^i \|x - y\| + \beta_n^i \|A_n^{i-1} x - A_n^{i-1} y\|$ for all $i = 2, \ldots, N$, and $\|A_n^1 x - A_n^1 y\| \leqslant \alpha_n^1 \|x -$

$y\| + \beta_n^1\|x-y\| \leq \|x-y\|$. This imply, by induction, that A_n^i is nonexpansive for all $i = 1, 2, \ldots, N$ and all $n \in N$. Set $S_{n,m} := A_{n+m-1}^N A_{n+m-2}^N \cdots A_n^N$, $n, m \geqslant 1$ and $b_{n,m} := \|S_{n,m}(tx_n + (1-t)u) - (tS_{n,m}x_n + (1-t)S_{n,m}u)\|$. It easy to see that $A_n^N x_n = x_{n+1}$, $S_{n,m}x_n = x_{n+m}$ and $S_{n,m}$ is nonexpansive for all $m, n \in N$.

We show first that, for any $x^* \in F$, $\|S_{n,m}x^* - x^*\| \to 0$ uniformly for all $m \geq 1$ as $n \to \infty$. Indeed, for any $x^* \in F$, we have

$$\|A_n^i x^* - x^*\| \leqslant \beta_n^i \|A_n^{i-1}x^* - x^*\| + \gamma_n^i \|u_n^i - x^*\|$$

for all $i = 2, \ldots, N$, and $\|A_n^1 x^* - x^*\| \leqslant \gamma_n^1 \|u_n^1 - x^*\|$. Therefore

$$\begin{aligned} \|A_n^N x^* - x^*\| &\leqslant \sigma_n^2 \gamma_n^1 \|u_n^1 - x^*\| + \sigma_n^3 \gamma_n^2 \|u_n^2 - x^*\| + \cdots + \sigma_n^N \gamma_n^{N-1} \|u_n^{N-1} - x^*\| \\ &\quad + \gamma_n^N \|u_n^N - x^*\| \leqslant M \textstyle\sum_{i=1}^N \gamma_n^i, \quad \text{for all } n \geqslant 1, \end{aligned}$$

where

$M = \max\{\sup\{\|u_n^1 - x^*\|\}, \ldots, \sup\{\|u_n^N - x^*\|\}\}$ and $\sigma_n^k = \prod_{i=k}^N \beta_n^i$. Hence

$$\begin{aligned} \|S_{n,m}x^* - x^*\| &\leqslant \|A_{n+m-1}^N A_{n+m-2}^N \cdots A_n^N x^* - A_{n+m-1}^N A_{n+m-2}^N \cdots A_{n+1}^N x^*\| \\ &\quad + \|A_{n+m-1}^N A_{n+m-2}^N \cdots A_{n+1}^N x^* - A_{n+m-1}^N A_{n+m-2}^N \cdots A_{n+2}^N x^*\| \\ &\quad \vdots \qquad\qquad \vdots \\ &\quad + \|A_{n+m-1}^N A_{n+m-2}^N x^* - A_{n+m-1}^N x^*\| + \|A_{n+m-1}^N x^* - x^*\| \\ &\leqslant \|A_n^N x^* - x^*\| + \|A_{n+1}^N x^* - x^*\| + \ldots + \|A_{n+m-1}^N x^* - x^*\| \\ &\leqslant M \textstyle\sum_{i=1}^N (\gamma_n^i + \gamma_{n+1}^i + \ldots + \gamma_{n+m-1}^i) \\ &\leqslant \delta_n^{x^*}, \end{aligned}$$

where $\delta_n^{x^*} := M \sum_{i=1}^N \sum_{k=n}^\infty \gamma_k^i$. Since $\sum_{n=1}^\infty \gamma_n^i < \infty$ for all $i = 1, 2, \ldots, N$, we have $\delta_n^{x^*} \to 0$ as $n \to \infty$. By Weierstrass M-test, we can conclude that $\|S_{n,m}x^* - x^*\| \to 0$ uniformly on $m \in N$ as $n \to \infty$. By Lemma 2.3.7, there exists $\varphi \in \Gamma$ such that

$$\begin{aligned} b_{n,m} &\leq \varphi^{-1}(\|x_n - u\| - \|S_{n,m}x_n - S_{n,m}u\|) \\ &= \varphi^{-1}(\|x_n - u\| - \|x_{n+m} - u + u - S_{n,m}u\|) \\ &\leq \varphi^{-1}(\|x_n - u\| - (\|x_{n+m} - u\| - \|S_{n,m}u - u\|)), \end{aligned} \tag{5.5}$$

and so the sequence $\{b_{n,m}\}$ converges uniformly to 0 on $m \in N$ as $n \to \infty$. By (5.5), for $m, n \in N$, we have

$$\begin{aligned} a_{n+m}(t) &= \|tx_{n+m} + (1-t)u - v\| \\ &= \|tS_{n,m}x_n + (1-t)u - v\| \\ &\leq \|tS_{n,m}x_n + (1-t)u - S_{n,m}(tx_n + (1-t)u)\| \\ &\quad + \|S_{n,m}(tx_n + (1-t)u) - v\| \\ &= \|tS_{n,m}x_n + (1-t)S_{n,m}u - S_{n,m}(tx_n + (1-t)u) + (1-t)(u - S_{n,m}u)\| \\ &\quad + \|S_{n,m}(tx_n + (1-t)u) - v\| \end{aligned}$$

$\leq b_{n,m} + \|S_{n,m}(tx_n + (1-t)u) - v\| + (1-t)\|u - S_{n,m}u\|$
$\leq b_{n,m} + \|S_{n,m}(tx_n + (1-t)u) - S_{n,m}v\| + \|S_{n,m}v - v\|$
$\quad + (1-t)\|u - S_{n,m}u\|$
$\leq b_{n,m} + a_n(t) + \|S_{n,m}v - v\| + (1-t)\|u - S_{n,m}u\|$
$\leq b_{n,m} + a_n(t) + \delta_n^v + (1-t)\delta_n^u$
$\leq \varphi^{-1}(\|x_n - u\| - (\|x_{n+m} - u\| - \|S_{n,m}u - u\|)) + a_n(t) + \delta_n^v + (1-t)\delta_n^u. \qquad (5.6)$

Thus fixing n and letting $m \to \infty$ in (5.6), we have

$$\limsup_{m\to\infty} a_{n+m}(t) \leqslant \varphi^{-1}(\|x_n - u\| - (\lim_{m\to\infty} \|x_m - u\| - \delta_n^u)) + a_n(t) + \delta_n^v + (1-t)\delta_n^u$$

and again letting $n \to \infty$,

$$\limsup_{n\to\infty} a_n(t) \leqslant \varphi^{-1}(0) + \liminf_{n\to\infty} a_n(t) = \liminf_{n\to\infty} a_n(t).$$

This completes the proof. □

Theorem 5.0.14 *Let X be a real uniformly convex Banach space such that its dual X^* has the Kaded-Klee property and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be a nonexpansive mappings with $F = \cap_{i=1}^N F(T_i) \neq \varnothing$ and $\limsup_{n\to\infty}(\alpha_n^i + \beta_n^i) < 1$. From arbitrary $x_1 \in C$ define the sequence $\{x_n\}$ by the iterative scheme (5.1) with $\sum_{n=1}^\infty \gamma_n^i < \infty$ and $\alpha_n^i \in [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \ldots, N$ for some $\varepsilon \in (0,1)$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Lemma 5.0.6 guarantees that $\{x_n\}$ is bounded. Since X is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to some $x^* \in C$. By Lemma 5.0.7, we have $\lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$ for all $i = 1, 2, \ldots, N$. Now Lemma 2.3.5 guarantees that $I - T_i$ is demiclosed at zero for all $i = 1, 2, \ldots, N$. This implies that $T_i x^* = x^*$ for all $i = 1, 2, \ldots, N$ this means that $x^* \in F$. It remains to show that $\{x_n\}$ converges weakly to x^*. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converging weakly to some y^*. Then $y^* \in F$ and so $x^*, y^* \in \omega_w(x_n) \cap F$. By Lemma 5.0.13, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\|$$

exists for all $t \in [0,1]$. By Lemma 2.3.6, we have $x^* = y^*$. As a result, $\omega_w(x_n) \cap F$ is a singleton, and so $\{x_n\}$ converges weakly to a fixed point of T. □