CHAPTER 3

MAIN RESULTS

In this chapter, we will study some properties of the set $I_n = \{1, 2, 3, ..., n\}$ and its order preserving transformation semigroup $O(I_n)$.

3.1 Some properties of $(I_n, \leq)$

We start with some definitions.

Definition 3.1.1 [2] A binary relation ω on a set X (that is, a subset ω of $X \times X$) is called a *partial order* if

(1) $(x, x) \in \omega$ for all $x \in X$ (that is, ω is reflexive);

(2) for all $x, y \in X$, $(x, y) \in \omega$ and $(y, x) \in \omega \Rightarrow x = y$ (that is, ω is antisymmetric);

(3) for all $x, y, z \in X$, $(x, y) \in \omega$ and $(y, z) \in \omega \Rightarrow (x, z) \in \omega$ (that is, ω is transitive).

We will write $x \leq y$ rather than $(x, y) \in \omega$. A partial order having the extra property

(4) for all $x, y \in X$, $x \leq y$ or $y \leq x$

will be called a *total order.* We shall refer to $(X, \leq)$, or just to X, as an *(partially) ordered set*, or a *totally ordered set* or *chain.* We shall follow this convention, and also write $a < b$ to mean $a \leq b$ and $a \neq b$.

Let Y be a non-empty subset of a partially ordered set $(X, \leq)$. An element a of Y is called *minimal* of Y if there is no element of Y that is strictly less than a, that is to say, if

$$\text{for all } y \in Y,\ y \leq a \Rightarrow y = a.$$

An element b of Y is called *minimum* if

$$\text{for all } y \in Y,\ b \leq y.$$

It is clear that the minimum element is minimal. An element a of Y is called *maximal* of Y if there is no element of Y that is strictly more than a, that is to say, if

$$\text{for all } y \in Y,\ a \leq y \Rightarrow a = y.$$

An element b of Y is called *maximum* if

$$\text{for all } y \in Y,\ y \leq b.$$

It is clear that the mximum element is maximal. If Y is a non-empty subset of a partially ordered set $(X, \leq)$, we say that an element c of X is a *lower bound* of Y if $c \leq y$ for all $y \in Y$. If the set of lower bounds of Y is non-empty and has the maximum element d, we say that d is the *greatest lower bound*, or *meet*, of Y. The element d is unique if it exists, and we write

$$d = \wedge\{y : y \in Y\}.$$

If $Y = \{a, b\}$ then we write $d = a \wedge b$.

If $(X, \leq)$ is such that $a \wedge b$ exists for all a, b in X, then we say that $(X, \leq)$ is a *lower semilattice*. If we have the stronger property that $\wedge\{y : y \in Y\}$ exists for every non-empty subset Y of X, then we say that $(X, \leq)$ is a *complete lower semilattice*.

If Y is a non-empty subset of a partially ordered set $(X, \leq)$, we say that an element c of X is an *upper bound* of Y if $y \leq c$ for all $y \in Y$. If the set of upper bounds of Y is non-empty and has the minimum element d, we say that d is the *least upper bound*, or *join*, of Y. The element d is unique if it exists, and we write

$$d = \vee\{y : y \in Y\}.$$

If $Y = \{a, b\}$ then we write $d = a \vee b$.

If $(X, \leq)$ is such that $a \vee b$ exists for all a, b in X, then we say that $(X, \leq)$ is an *upper semilattice*. If we have the stronger property that $\vee\{y : y \in Y\}$ exists for every non-empty subset Y of X, then we say that $(X, \leq)$ is a *complete upper semilattice*.

Let $I_n = \{1, 2, 3, ..., n\}$ where $n \in \mathbb{N}$. For each $a, b \in I_n$, define $\leq$ on I_n by

$$a \leq b \text{ if and only if } a|b.$$

Hence $a|b$ means a divides b. And we have that $(I_n, \leq)$ is a partially ordered set.

Proposition 3.1.2 *$(I_n, \leq)$ is a partially ordered set.*

Proof. (1) Since $a|a$ for all $a \in I_n$, $a \leq a$. That is $\leq$ is reflexive.

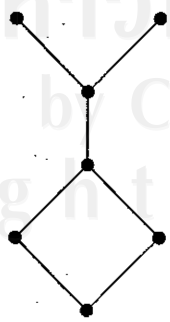
(2) Let $a, b \in I_n$ be such that $a \leq b$ and $b \leq a$. Then $a|b$ and $b|a$. Since a and b are positive integers, $a = b$. That is $\leq$ is antisymmetric.

(3) Let $a, b, c \in I_n$ be such that $a \leq b$ and $b \leq c$. Then $a|b$ and $b|c$ and so $a|c$. Thus $a \leq c$ and that is $\leq$ is transitive.

From (1),(2) and (3) we have $(I_n, \leq)$ is a partially ordered set. □

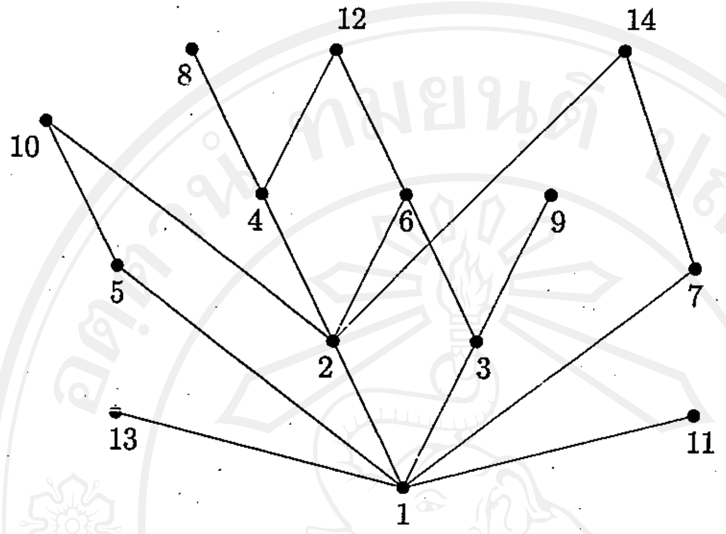
When describing an ordered set $(X, \leq)$, we shall sometimes use so called *Hasse diagrams*. In such a diagram, elements of the set are represented by small black circles, and to elements a and b in X for which $a < b$ and for which there is no $x \in X$ such that $a < x < b$ are depicted thus:

That is, b appears above a and a line connects a and b. Thus we can build up diagrams such as



which we can label if necessary.

The following Hasse diagram is for $(I_{14}, \leq)$.



It is clear that 1 is the minimum element of $(I_n, \leq)$ since 1 divides every element of $(I_n, \leq)$ and it has no the maximum element. Next, we will characterize all minimal elements of $(I_n \setminus \{1\}, \leq)$.

Let $\leq_{nat}$ denote the natural order on I_n. The notation $a <_{nat} b$ will mean that $a \leq_{nat} b$ and $a \neq b$.

Theorem 3.1.3 *For each $m \in I_n$, m is a minimal element of $(I_n \setminus \{1\}, \leq)$ if and only if m is a prime number.*

Proof. Assume that m is a minimal element of $(I_n \setminus \{1\}, \leq)$. Then $1 <_{nat} m \leq_{nat} n$. Suppose that m is not a prime number. Then $m = pq$ where $1 <_{nat} p, q <_{nat} m$. Thus $p \in I_n$ and $p|m$, so $1 < p \leq m$ which contradicts to the minimality of m. Hence m is a prime number.

Conversely, assume that m is prime. Let $1 \neq x \in I_n$ be such that $x \leq m$. Then $x|m$ and so $x = m$ since $1 \neq x$ and m is prime . Therefore, m is a minimal element of $(I_n \setminus \{1\}, \leq)$. $\square$

Example 3.1.4 For $n = 100$, a minimal elements of $(I_{100}, \leq)$ are as follows. By Theorem 3.1.3 minimal elements of $(I_{100} \setminus \{1\}, \leq)$ are prime numbers between 1 and

100. Thus all minimal elements of $(I_{100}, \leq)$ are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$. $\square$

The floor function of real number x, denoted by $\lfloor x \rfloor$, is the largest integer less than or equal to x. We will denote that if $x <_{nat} y$ then $\lfloor x \rfloor \leq_{nat} \lfloor y \rfloor$.

Now, let $D_n = \{x \in I_n : \left\lfloor \frac{n}{2} \right\rfloor <_{nat} x\}$. Then we see that $D_{15} = \{x \in I_{15} : \lfloor 7.5 \rfloor <_{nat} x\} = \{x \in I_{15} : 7 <_{nat} x\} = \{8, 9, 10, 11, 12, 13, 14\}$.

Lemma 3.1.5 *For each $x, y \in I_n$, if $x <_{nat} y$ and $x \in D_n$ then $x \nmid y$.*

Proof. Let $x, y \in I_n$ be such that $x <_{nat} y$ and $x \in D_n$. Suppose that $x|y$. Then $y = xl$ for some $l \in I_n$. Thus $x = \frac{y}{l} \leq_{nat} \frac{y}{2} \leq_{nat} \frac{n}{2}$ since $2 \leq_{nat} l$. Then $x = \lfloor x \rfloor \leq_{nat} \left\lfloor \frac{n}{2} \right\rfloor$, this implies that $x \notin D_n$ which is a contradiction. Therefore, $x \nmid y$. $\square$

Theorem 3.1.6 *For each $M \in I_n$, M is a maximal element of $(I_n, \leq)$ if and only if $M \in D_n$.*

Proof. Assume that M is a maximal element of $(I_n, \leq)$. Suppose that $M \notin D_n$. Then $M \leq_{nat} \left\lfloor \frac{n}{2} \right\rfloor$ and so $M \leq_{nat} \frac{n}{2}$ since $\left\lfloor \frac{n}{2} \right\rfloor \leq_{nat} \frac{n}{2}$. Thus $2M \leq_{nat} n$. Let $k = 2M \in I_n$. Then $M|k$ and hence $M \leq k$ and $M \neq k$. This contradicts to the maximality of M. Therefore, $M \in D_n$.

Conversely, assume that $M \in D_n$. Then $\left\lfloor \frac{n}{2} \right\rfloor <_{nat} M$. Let $y \in I_n$ be such that $M \leq y$. Then $M|y$ and so $\left\lfloor \frac{n}{2} \right\rfloor <_{nat} M \leq_{nat} y$ which implies that $y \in D_n$. Thus $y \leq_{nat} M$ by Lemma 3.1.5 and hence $y = M$. Therefore, M is a maximal element of $(I_n, \leq)$. $\square$

Let $a, b \in I_n$ with $a <_{nat} b$. Then the number of integers from a to b is $b - a + 1$.

Theorem 3.1.7 *The number of elements in D_n is $n - \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. Since $D_n = \{x \in I_n : \left\lfloor \frac{n}{2} \right\rfloor <_{nat} x\} = \{x \in I_n : \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq_{nat} x\}$, so elements in D_n are all integers from $\left\lfloor \frac{n}{2} \right\rfloor + 1$ to n. Thus $|D_n| = n - (\left\lfloor \frac{n}{2} \right\rfloor + 1) + 1 = n - \left\lfloor \frac{n}{2} \right\rfloor$. □

Example 3.1.8 For $n = 100$, we find a maximal elements of $(I_{100}, \leq)$. By Theorem 3.1.6 we have a maximal element of $(I_{100}, \leq)$ is an element in D_{100} and by Theorem 3.1.7 the number of elements in D_{100} is $100 - \lfloor 50 \rfloor = 50$.

Thus all maximal elements of $(I_{100}, \leq)$ are $51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100$. □

Recall that $(I_n, \leq)$ is a partially ordered set. Let $\emptyset \neq A \subseteq I_n$ and let $c = \vee\{a : a \in A\}$. Then c is the least upper bound of A, that is $a \leq c$ for all $a \in A$ and c is the minimum of the set of upper bounds of A. So $a \leq c$ for all $a \in A$, and if $a \leq d$ for all $a \in A$ then $c \leq d$. Thus $a|c$ for all $a \in A$; and if $a|d$ for all $a \in A$ then $c|d$. Therefore, c is the least common multiple (lcm) of A. Then $(I_n, \leq)$ is not a complete upper semilattice. For example, let $A = \{3, 4, 5\} \subseteq I_6$ we get $lcm(A) = 60 \notin I_6$. But $(I_n, \leq)$ is a complete lower semilattice.

Theorem 3.1.9 *$(I_n, \leq)$ is a complete lower semilattice.*

Proof. Let A be a nonempty subset of $(I_n, \leq)$ and let $gcd(A) = c$, that is $c \leq a$ for all $a \in A$, and if $d \leq a$ for all $a \in A$ then $d \leq c$. We will prove that $c \in I_n$ and $c = \wedge\{a : a \in A\}$. Since $c = gcd(A)$, $c|a$ for all $a \in A$ and $1 \leq_{nat} c$. Then $c \leq_{nat} a \leq_{nat} n$. Thus $c \leq_{nat} n$ and $c \leq a$ for all $a \in A$. Hence $c \in I_n$ and c is a lower bound of A. Next, we show that c is the greatest lower bound of A. Let d be a lower bound of A. Then $d \leq a$ for all $a \in A$ and so $d|a$ for all $a \in A$. Since $c = gcd(A)$, we get $d|c$. Then $d \leq c$ and hence c is the greatest lower bound of A. Therefore, $(I_n, \leq)$ is a complete lower semilattice. □

3.2 Regularity of $O(I_n)$

If X is a nonempty set, we let $T(X)$ denote the semigroup under composition of all total transformations of X. Following standard notation, we let *ran* α denote the *range* of $\alpha \in T(X)$.

If $(X, \leq)$ is a partially ordered set, then we say $\alpha \in T(X)$ is *order-preserving* if for all $x, y \in X$, $x \leq y$ implies $x\alpha \leq y\alpha$; and we let $O(X)$ denote the subsemigroup of $T(X)$ consisting of all order-preserving total transformations of X and we say $a \in X$ is *isolated* if for every $x \in X, x \leq a$ or $x \geq a$ implies $x = a$, and X is *isolated* if all its elements are isolated. Let Y and Z be nonempty subsets of X. We say that Y and Z are *disjoint partially ordered sets* if $Y \cap Z = \emptyset$; and for all $y \in Y$ and $z \in Z$, $y \not< z$ and $z \not< y$.

An element a of a semigroup S is called *regular* if there exists x in S such that $axa = a$. A semigroup S is called *regular* if all its elements are regular. It is known that $O(X)$ is regular if $(X, \leq)$ is a finite chain ([1] page 203, Exercise 6.1.7). However before proving our main theorem, we start with some theorems and lemmas.

Recall that $(I_n, \leq)$ is a partially ordered set, and it is easy to see that $O(I_n)$ is a semigroup under composition of mappings: if $\alpha, \beta \in O(I_n)$ then $\alpha \circ \beta \in O(I_n)$ is defined by

$$x(\alpha \circ \beta) = (x\alpha)\beta, \quad x \in I_n.$$

And for each $x, y \in I_n$ such that $x < y$, we have $x\alpha \leq y\alpha$ and so $(x\alpha)\beta \leq (y\alpha)\beta$. Thus $x(\alpha \circ \beta) \leq y(\alpha \circ \beta)$.

Theorem 3.2.1 [3] *Suppose that X is a partially ordered set. Then $O(X)$ is not regular if X contains a partially ordered subset of the form*

$$\{a, b, c, d : \ d < c < a, \ d < c < b \text{ and } \{a, b\} \text{ is isolated}\}$$

or

$$\{a, b, c, d : \ d < c < a, \ d < b; \text{ and } \{a, b\}, \ \{b, c\} \text{ are isolated}\}.$$

Theorem 3.2.2 [3] *Suppose that X is a partially ordered set and let $m(X)$ [$M(X)$] denote the set of all minimal [maximal] elements of X. Then $O(X)$ is regular if $X = m(X) \cup M(X)$ and $x < y$ for all $x \in m(X)$ and $y \in M(X)$.*

Lemma 3.2.3 *If $n \leq_{nat} 3$, then $O(I_n)$ is regular.*

Proof. If $n = 1$ or $n = 2$, then I_n is a finite chain. Thus by [1, p.203, Exercise 6.1.7] $O(I_n)$ is regular. If $n = 3$, then I_n is a partially ordered set of the form $\{1, 2, 3 : 1 < 2,\ 1 < 3$ and $\{2, 3\}$ is isolated$\}$. Hence $m(I_n) = \{1\}$, $M(I_n) = \{2, 3\}$, $X = m(I_n) \cup M(I_n)$ and $x < y$ for all $x \in m(I_n)$ and $y \in M(I_n)$. Therefore, by Lemma 3.2.2 $O(I_n)$ is regular. □

Lemma 3.2.4 *If $n \geq_{nat} 4$, then $O(I_n)$ is not regular.*

Proof. Let $I_n = \{1, 2, ..., n\}$ where $n \geq_{nat} 4$. Then I_n contains a partially ordered subset of the form $\{1, 2, 3, 4 : 1 < 2 < 4,\ 1 < 3$; and $\{4, 3\}$, $\{3, 2\}$ are isolated $\}$. Thus by Theorem 3.2.1 $O(I_n)$ is not regular. □

Theorem 3.2.5 *$O(I_n)$ is regular if and only if $n \leq_{nat} 3$.*

Proof. By Lemma 3.2.3 and Lemma 3.2.4. □

Lemma 3.2.6 *If X is isolated, then $O(X)$ is regular.*

Proof. Assume that X is an isolated set. Then $O(X) = T(X)$. Since $T(X)$ is regular, $O(X)$ is also regular. □

Lemma 3.2.7 *Let X be a partially ordered set such that $X = Y \cup Z$ where $|Y| \geq_{nat} 2$ and there exist $a, m \in Y$ with $a < m$; and Y, Z are disjoint partially ordered sets. Then O(X) is not regular.*

Proof. Let $\alpha \in T(X)$ be such that

$$x\alpha = \begin{cases} m & \text{if } x \in Y, \\ a & \text{if } x \in Z. \end{cases}$$

Then $\alpha \in O(X)$ since Y and Z are isolated. Suppose that $\alpha\beta\alpha = \alpha$ for some $\beta \in O(X)$. First, we show that $a\beta \in Z$ and $m\beta \in Y$. Suppose this is not true. Then $a\beta \notin Z$ or $m\beta \notin Y$.

Case 1:. $a\beta \notin Z$. Then $a\beta\alpha = m$ and so $a = x\alpha = x\alpha\beta\alpha = a\beta\alpha = m$ for some $x \in Z$ which is a contradiction since $a < m$.

Case 2:. $m\beta \notin Y$. Then $m\beta\alpha = a$ and so $m = x\alpha = x\alpha\beta\alpha = m\beta\alpha = a$ for some $x \in Y$ which is a contradiction since $a < m$.

Thus $a\beta \in Z$ and $m\beta \in Y$. Since $a < m$ and β is order preserving, $a\beta \leq m\beta$. This contradicts to the fact that Y and Z are disjoint partially ordered sets. Therefore, $O(X)$ is not regular. □

Theorem 3.2.8 *Let X be a proper partially ordered subset of I_4. Then $O(X)$ is regular if and only if X is one of the following forms :*

(1) $\prod_1$ *is a chain,*

(2) $\prod_2 = \{a_1, a_2, a_3 : \ a_1 < a_2$ *and* $a_1 < a_3$ *and* $\{a_2, a_3\}$ *is isolated* $\}$,

(3) $\prod_3$ *is isolated.*

Proof. Assume that X is of the form $\prod_1$ or $\prod_2$ or $\prod_3$.
If X is of the form $\prod_1$, then by [1, p.203, Exercise 6.1.7] $O(X)$ is regular.
If X is of the form $\prod_2$, then by Theorem 3.2.2 $O(X)$ is regular.
If X is of the form $\prod_3$, then by Lemma 3.2.6 $O(X)$ is regular.

Conversely, assume that $O(X)$ is regular. Since X is a proper subset of I_4, $|X| \leq_{nat} 3$. Consider the following cases :
Case 1: $|X| = 1$. Then $X = \prod_1$.
Case 2: $|X| = 2$. Then $X = \prod_1$ or $\prod_3$.
Case 3: $|X| = 3$. Then $X = \{2,3,4\}$ or $\{1,3,4\}$ or $\{1,2,4\}$ or $\{1,2,3\}$.
Since $O(X)$ is regular, X can not be $\{2,3,4\}$ (if $X = \{2,3,4\}$ then $O(X)$ is not

regular by Lemma 3.2.7). Thus $X = \{1,3,4\}$ or $\{1,2,4\}$ or $\{1,2,3\}$ and therefore, X is of the form $\prod_1$ or $\prod_2$. □

Theorem 3.2.9 *Let X be a proper partially ordered subset of I_5. Then $O(X)$ is regular if and only if X is one of the following forms :*

(1) $\prod_1$ *is a chain,*

(2) $\prod_2 = \{a_1, a_2, a_3 : \ a_1 < a_2, \ a_1 < a_3$ *and* $\{a_2, a_3\}$ *is isolated* $\}$,

(3) $\prod_3 = \{a_1, a_2, a_3, a_4 : \ a_1 < a_i$ *for all* $i = 2, 3, 4$ *and* $\{a_2, a_3, a_4\}$ *is isolated* $\}$,

(4) $\prod_4$ *is isolated.*

Proof. Assume that X is of the form $\prod_1$ or $\prod_2$ or $\prod_3$ or $\prod_4$.
If X is of the form $\prod_1$, then by [1, p.203, Exercise 6.1.7] $O(X)$ is regular.
If X is of the form $\prod_2$, then by Theorem 3.2.2 $O(X)$ is regular.
If X is of the form $\prod_3$, then by Theorem 3.2.2 $O(X)$ is regular.
If X is of the form $\prod_4$, then by Lemma 3.2.6 $O(X)$ is regular.

Conversely, assume that $O(X)$ is regular. Since X is a proper subset of I_5, $|X| \leq_{nat} 4$. Consider the following cases :
Case 1: $|X| \leq 2$. Then $X = \prod_1$ or $\prod_4$.
Case 2: $|X| = 3$. If X has no isolated elements, then $X = \prod_1$ or $\prod_2$. But, if X is isolated then $X = \prod_4$, otherwise X is of the form $\{a_1, a_2, a_3 : \ a_1 < a_2$ and $\{a_3\}$ is isolated $\}$ and $O(X)$ is not regular by Lemma 3.2.7.
Case 3: $|X| = 4$. Then $X = \{1,2,3,4\}$ or $\{1,2,3,5\}$ or $\{1,2,4,5\}$ or $\{1,3,4,5\}$ or $\{2,3,4,5\}$. Since $O(X)$ is regular, X can not be $\{1,2,3,4\}$ or $\{1,2,4,5\}$ or $\{2,3,4,5\}$ (if $X = \{1,2,3,4\}$ or $\{1,2,4,5\}$ then $O(X)$ is not regular by Theorem 3.2.1 and if $X = \{2,3,4,5\}$ then $O(X)$ is not regular by Lemma 3.2.7). Thus $X = \{1,2,3,5\}$ or $\{1,3,4,5\}$ and therefore, X is of the form $\prod_3$. □

Lemma 3.2.10 *Let $X = \{a_1, a_2, a_3, a_4 : \ a_1 < a_3, \ a_1 < a_4, \ a_2 < a_4$; and $\{a_1, a_2\}$, $\{a_3, a_4\}$ are isolated $\}$ be a partially ordered set. Then any order preserving permutation of X equals to 1_X, the identity map on X.*

Proof. Assume that α is an order preserving permutation of X. Suppose that α is not the identity map on X. Then there exists $x \in X$ such that $x\alpha \neq x$.

Case 1: $x = a_1$ and $a_1\alpha = a_2$. Then $a_3\alpha = a_2$ or $a_3\alpha = a_4$ and $a_4\alpha = a_2$ or $a_4\alpha = a_4$, since $a_1 < a_3$ and $a_1 < a_4$ and $a_2 < a_4$. So, in this case we see that $ran\ \alpha \subseteq \{a_2, a_4\} \cup \{a_2\alpha\}$ which is a proper subset of X. Thus α is not one-to-one which is a contradiction.

Case 2: $x = a_1$ and $a_1\alpha = a_3$. Then $a_3\alpha = a_3$ and $a_4\alpha = a_3$ since $a_1 < a_3$ and $a_1 < a_4$. This contradicts to that α is not one-to-one.

Case 3: $x = a_1$ and $a_1\alpha = a_4$. Then $a_3\alpha = a_4$ and $a_4\alpha = a_4$ since $a_1 < a_3$ and $a_1 < a_4$. This contradicts to that α is not one-to-one.

Case 4: $x = a_2$ and $a_2\alpha = a_1$. Then $a_4\alpha = a_1$ or $a_4\alpha = a_3$ or $a_4\alpha = a_4$ since $a_2 < a_4$ and $a_1 < a_3$ and $a_1 < a_4$. Consider the following cases :

If $a_4\alpha = a_1$, then $a_2\alpha = a_1 = a_4\alpha$ and so α is not one-to-one which is a contradiction.

If $a_4\alpha = a_3$, then $a_1\alpha = a_1$ or $a_1\alpha = a_3$ and thus $ran\ \alpha \subseteq \{a_1, a_3\} \cup \{a_3\alpha\}$ which is a proper subset of X. This contradicts to that α is not one-to-one.

If $a_4\alpha = a_4$, then since $a_1 < a_4$ we must have $a_1\alpha = a_1$ or a_2 or a_4. The case $a_1\alpha = a_1$ or $a_1\alpha = a_4$ gives $ran\ \alpha \subseteq \{a_1, a_4\} \cup \{a_3\alpha\}$ which contradicts to that α is not one-to-one. The case $a_1\alpha = a_2$ implies $a_3\alpha = a_2$ or a_4 since $a_1 < a_3$ and thus $ran\ \alpha \subseteq \{a_1, a_2, a_4\}$ which is a contradiction.

Case 5: $x = a_2$ and $a_2\alpha = a_3$. Then $a_4\alpha = a_3$ because $a_2 < a_4$. Thus α is not one-to-one which is a contradiction.

Case 6: $x = a_2$ and $a_2\alpha = a_4$. Then $a_4\alpha = a_4$ because $a_2 < a_4$. Thus α is not one-to-one which is a contradiction.

Case 7: $x = a_3$ and $a_3\alpha = a_1$. Then $a_1\alpha = a_1$ because $a_1 < a_3$. Thus α is not one-to-one which is a contradiction.

Case 8: $x = a_3$ and $a_3\alpha = a_2$. Then $a_1\alpha = a_2$ because $a_1 < a_3$. Thus α is not one-to-one which is a contradiction.

Case 9: $x = a_3$ and $a_3\alpha = a_4$. Then $a_1\alpha = a_1$ or $a_1\alpha = a_2$ or $a_1\alpha = a_4$ because $a_1 < a_3$ and $a_1 < a_4$ and $a_2 < a_4$. Consider the following cases :

If $a_1\alpha = a_1$ then since $a_1 < a_4$ we must have $a_4\alpha = a_1$ or a_3 or a_4. The case

$a_4\alpha = a_1$ or $a_4\alpha = a_4$ gives $ran\ \alpha \subseteq \{a_1, a_4\} \cup \{a_2\alpha\}$ which contradicts to α is one-to-one. The case $a_4\alpha = a_3$ implies $a_2\alpha = a_1$ or a_3 since $a_2 < a_4$ and thus $ran\ \alpha \subseteq \{a_1, a_3, a_4\}$ which is a contradiction.

If $a_1\alpha = a_2$, then $a_4\alpha = a_2$ or a_4 and thus $ran\ \alpha \subseteq \{a_2, a_4\} \cup \{a_2\alpha\}$ which is a proper subset of X. This contradics to that α is one-to-one.

If $a_1\alpha = a_4$, then $a_1\alpha = a_4 = a_3\alpha$ and so α is not one-to-one which is a contradiction.

Case 10: $x = a_4$ and $a_4\alpha = a_1$. Then $a_1\alpha = a_1$ because $a_1 < a_4$, thus α is not one-to-one which is a contradiction.

Case 11: $x = a_4$ and $a_4\alpha = a_2$. Then $a_1\alpha = a_2$ because $a_1 < a_4$, thus α is not one-to-one which is a contradiction.

Case 12: $x = a_4$ and $a_4\alpha = a_3$. Then $a_1\alpha = a_1$ or $a_1\alpha = a_3$ and $a_2\alpha = a_1$ or $a_2\alpha = a_3$ because $a_1 < a_4$ and $a_2 < a_4$ and $a_1 < a_3$. So there are four possible cases: $a_1\alpha = a_1$ and $a_2\alpha = a_1$; or $a_1\alpha = a_1$ and $a_2\alpha = a_3$; or $a_1\alpha = a_3$ and $a_2\alpha = a_1$; or $a_1\alpha = a_3$ and $a_2\alpha = a_3$. In all cases give α is not one-to-one which is a contradiction.

Therefore, α is the identity map on X. □

Theorem 3.2.11 *Let $X = \{a_1, a_2, a_3, a_4 : a_1 < a_3,\ a_1 < a_4,\ a_2 < a_4;$ and $\{a_1, a_2\}$, $\{a_3, a_4\}$ are isolated $\}$ be a partially ordered set. Then $O(X)$ is regular.*

Proof. Let $\alpha \in O(X)$.

If $|ran\ \alpha| = 1$ then choose $\beta = \alpha$ and hence $\alpha\beta\alpha = \alpha$.

If $ran\ \alpha = X$ then by Lemma 3.2.10, α is the identity map on X. Thus we choose $\beta = \alpha$ and hence $\alpha\beta\alpha = \alpha$.

If $ran\ \alpha = \{a_i, a_j\}$ for some i, j then since $a_i\alpha^{-1} \cup a_j\alpha^{-1} = X$ can not be partitioned into two disjoint partially ordered sets we must have $a_i < a_j$ or $a_j < a_i$. Suppose that $a_i < a_j$ so $a_i = a_1$ or a_2 and we can choose $p \in a_i\alpha^{-1}$ and $q \in a_j\alpha^{-1}$ such that $p < q$ and define $\beta : X \to X$ by

$$x\beta = \begin{cases} p & \text{if } x = a_i, \\ q & \text{otherwise.} \end{cases}$$

To show that β is order preserving, let $x, y \in I_n$ be such that $x < y$. Then $x = a_1$ or a_2. If $x = a_1$ then $y = a_3$ or a_4 and $x\beta = a_1\beta = p$ or $q \leq q = y\beta$. If $x = a_2$ then $y = a_4$ and $x\beta = a_2\beta = p$ or $q \leq q = y\beta$.

For each $x \in X, x\alpha = a_i$ or $x\alpha = a_j$, so $a_i\beta\alpha = a_i$ or $a_j\beta\alpha = a_j$. Thus $x\alpha\beta\alpha = x\alpha$ for all $x \in X$. Hence $\alpha\beta\alpha = \alpha$.

If $ran\ \alpha = \{a_i, a_j, a_k\}$ for some i, j, k then we consider in four cases:

Case 1: $ran\ \alpha = \{a_1, a_3, a_4\}$. Since $a_1 < a_3$ and $a_1 < a_4$, so $a_1 \in a_1\alpha^{-1}$ and there exist $p \in a_3\alpha^{-1}$ and $q \in a_4\alpha^{-1}$ such that $a_1 < p$ and $a_1 < q$. Now, define $\beta : X \to X$ by

$$x\beta = \begin{cases} a_1 & \text{if } x = a_1, \\ p & \text{if } x = a_3, \\ q & \text{if } x \in \{a_2, a_4\}. \end{cases}$$

Then β is order preserving and $\alpha\beta\alpha = \alpha$.

Case 2: $ran\ \alpha = \{a_1, a_2, a_4\}$. Since $a_1 < a_4$ and $a_2 < a_4$, so $a_4 \in a_4\alpha^{-1}$ and there exist $p \in a_1\alpha^{-1}$ and $q \in a_2\alpha^{-1}$ such that $p < a_4$ and $q < a_4$. Now, define $\beta : X \to X$ by

$$x\beta = \begin{cases} p & \text{if } x = a_1, \\ q & \text{if } x = a_2, \\ a_4 & \text{if } x \in \{a_3, a_4\}. \end{cases}$$

Then β is order preserving and $\alpha\beta\alpha = \alpha$.

Case 3: $ran\ \alpha = \{a_2, a_3, a_4\}$. Since $\{a_2, a_4\}$ and $\{a_3\}$ are disjoint partially ordered sets, so $a_2\alpha^{-1} \cup a_4\alpha^{-1}$ and $a_3\alpha^{-1}$ must be disjoint partially ordered sets. This implies $X = (a_2\alpha^{-1} \cup a_4\alpha^{-1}) \cup a_3\alpha^{-1}$ can be partitioned into two disjoint partially ordered sets which is a contradiction. Thus this case can not be occurred.

Case 4: $ran\ \alpha = \{a_1, a_2, a_3\}$. By using the same arguments as given in case 3, but starting with $\{a_1, a_3\}$ and $\{a_2\}$ will lead to a contradiction. Thus this case can not be occurred.

Therefore, $O(X)$ is regular as required. □

Theorem 3.2.12 *Let* $X = \{a_1, a_2, a_3, a_4, a_5 : \ a_1 < a_4, \ a_2 < a_4, \ a_2 < a_5, \ a_3 < a_5;$ *and* $\{a_1, a_2, a_3\}, \ \{a_1, a_5\}, \ \{a_3, a_4\}$ *are isolated* $\}$ *be a partially ordered set. Then* $O(X)$ *is not regular.*

Proof. We prove by contradiction, suppose that $O(X)$ is regular and let $\alpha \in O(X)$ be defined by

$$\alpha = \begin{pmatrix} a_1 & \{a_2, a_4, a_5\} & a_3 \\ a_2 & a_4 & a_1 \end{pmatrix}.$$

Then $\alpha\beta\alpha = \alpha$ for some $\beta \in O(X)$. Thus $x\alpha\beta\alpha = x\alpha$ for all $x \in X$ and so $a_1\beta \in a_1\alpha^{-1}$, $a_2\beta \in a_2\alpha^{-1}$ and $a_4\beta \in a_4\alpha^{-1}$. Then we get $a_1\beta = a_3$, $a_2\beta = a_1$; and $a_4\beta = a_2$ or $a_4\beta = a_4$ or $a_4\beta = a_5$. Thus we consider the following cases :

Case 1: $a_1\beta = a_3$, $a_2\beta = a_1$ and $a_4\beta = a_2$. Then $\beta \notin O(X)$ since $a_1 < a_4$ but $a_1\beta \not\leq a_4\beta$.

Case 2: $a_1\beta = a_3$, $a_2\beta = a_1$ and $a_4\beta = a_4$. Then $\beta \notin O(X)$ since $a_1 < a_4$ but $a_1\beta \not\leq a_4\beta$.

Case 3: $a_1\beta = a_3$, $a_2\beta = a_1$ and $a_4\beta = a_5$. Then $\beta \notin O(X)$ since $a_2 < a_4$ but $a_2\beta \not\leq a_4\beta$.

In all cases, there are a contradictions. Therefore, $O(X)$ is not regular. □

Theorem 3.2.13 *Let* $X = \{a_1, a_2, a_3, a_4, a_5 : \ a_1 < a_3, \ a_1 < a_4, \ a_2 < a_4, \ a_2 < a_5;$ *and* $\{a_3, a_4, a_5\}, \ \{a_1, a_5\}, \ \{a_2, a_3\}$ *are isolated* $\}$ *be a partially ordered set. Then* $O(X)$ *is not regular.*

Proof. We prove by contradiction, suppose that $O(X)$ is regular and let $\alpha \in O(X)$ be defined by

$$\alpha = \begin{pmatrix} \{a_1, a_2, a_4\} & a_3 & a_5 \\ a_2 & a_5 & a_4 \end{pmatrix}.$$

Then $\alpha\beta\alpha = \alpha$ for some $\beta \in O(X)$. Thus $x\alpha\beta\alpha = x\alpha$ for all $x \in X$ and so $a_2\beta \in a_2\alpha^{-1}$, $a_4\beta \in a_4\alpha^{-1}$ and $a_5\beta \in a_5\alpha^{-1}$. Then we get $a_2\beta = a_1$ or $a_2\beta = a_2$ or $a_2\beta = a_4$; and $a_4\beta = a_5$, $a_5\beta = a_3$. Thus we consider the following cases :

Case 1: $a_2\beta = a_1$, $a_4\beta = a_5$ and $a_5\beta = a_3$. Then $\beta \notin O(X)$ since $a_2 < a_4$ but

$a_2\beta \nleq a_4\beta$.

Case 2: $a_2\beta = a_2$, $a_4\beta = a_5$ and $a_5\beta = a_3$. Then $\beta \notin O(X)$ since $a_2 < a_5$ but $a_2\beta \nleq a_5\beta$.

Case 3: $a_2\beta = a_4$, $a_4\beta = a_5$ and $a_5\beta = a_3$. Then $\beta \notin O(X)$ since $a_2 < a_5$ but $a_2\beta \nleq a_5\beta$.

In all cases, there are a contradictions. Therefore, $O(X)$ is not regular. □

Lemma 3.2.14 *Let $\alpha \in O(I_n)$. If there exist $x, y \in ran\ \alpha$ such that $x < y$ and $x\alpha^{-1}$ and $y\alpha^{-1}$ are disjoint partially ordered sets, then α is not regular.*

Proof. Assume that there exist $x, y \in ran\ \alpha$ such that $x < y$ and $x\alpha^{-1}$ and $y\alpha^{-1}$ are disjoint partially ordered sets. Suppose that α is regular in $O(I_n)$. Then there exists $\beta \in O(I_n)$ such that $\alpha\beta\alpha = \alpha$. Since $x, y \in ran\ \alpha$, there exist $x', y' \in I_n$ such that $x'\alpha = x$ and $y'\alpha = y$. Since $\alpha\beta\alpha = \alpha$, $x'\alpha\beta\alpha = x'\alpha$. Then $x\beta\alpha = x$ which implies that $x\beta \in x\alpha^{-1}$. Similarly, we have that $y\beta \in y\alpha^{-1}$. Since $x < y$ and β is order preserving, $x\beta \leq y\beta$. But $x\alpha^{-1}$ and $y\alpha^{-1}$ are disjoint partially ordered sets, so it is a contradiction. Hence α is not regular. □

Remark 3.2.15 Let $\alpha \in O(I_n)$.

(1) If α is regular, then for all $x, y \in ran\ \alpha, x < y$ implies $x\alpha^{-1}$ and $y\alpha^{-1}$ are not disjoint partially ordered sets.

(2) If $1 \in ran\ \alpha$, then $1\alpha = 1$.

Proof. Assume that $1 \in ran\ \alpha$. Then $1\alpha = x$ for some $x \in I_n$. Since $1 \leq a$ for all $a \in I_n$ and α is order preserving, $x = 1\alpha \leq a\alpha$ for all $a \in I_n$. Then x is the minimum of $ran\ \alpha$. Since 1 is the minimum of $ran\ \alpha$, $1\alpha = x = 1$ □

Before proving Theorem 3.2.16, we need some notations. For $\alpha \in O(I_n)$ with $ran\ \alpha = \{a_1, a_2, .., a_m\}$, choose $b_i \in a_i\alpha^{-1}$ for all $i = 1, 2, ..., m$. Let

$$A_\alpha = \{b_1, b_2, ..., b_m\}$$

and

$$B_\alpha = \{x \in I_n \setminus ran\ \alpha : \ a_i < x \text{ for some } a_i \neq 1\}.$$

If $B_\alpha \neq \emptyset$, then for each $x \in B_\alpha$, we set $A_\alpha(x) = \{b_i \in a_i\alpha^{-1} : a_i < x\}$.

Example 3.2.16 Let $\alpha = \begin{pmatrix} \{1\} & \{2,4\} & \{3,5,6\} \\ 1 & 2 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} \{1\} & \{2,3,4,5,6\} \\ 1 & 5 \end{pmatrix}$ be two elements in $O(I_6)$. Let $A_\alpha = \{1,2,6\}$, then $B_\alpha = \{6\}$ and $A_\alpha(6) = \{1,2\}$. But, if we let $A_\beta = \{1,5\}$ then $B_\beta = \emptyset$. □

Theorem 3.2.17 *Let $\alpha \in O(I_n)$ with $ran\ \alpha = \{a_1, a_2, ..., a_m\}$. Then α is regular if and only if the following conditions hold:*

(1) There exists A_α such that the map $\varphi : ran\ \alpha \to A_\alpha$ define by $a_i\varphi = b_i$ for all i is order preserving.

(2) If $B_\alpha \neq \emptyset$, then $lcm(A_\alpha(x)) \in I_n$ for all $x \in B_\alpha$.

Proof. Assume that $\alpha\beta\alpha = \alpha$ for some $\beta \in O(I_n)$. Then $x\alpha\beta\alpha = x\alpha$ for all $x \in I_n$, so $a_i\beta\alpha = a_i$ for all $i = 1, 2, ..., m$. Thus $b_i := a_i\beta \in a_i\alpha^{-1}$ for all $i = 1, 2, ..., m$.

(1) Let $A_\alpha = \{a_1\beta, a_2\beta, ..., a_m\beta\}$ and defined $\varphi : ran\ \alpha \to A_\alpha$ by $x\varphi = x\beta$ for all $x \in ran\ \alpha$. Then $\varphi = \beta_{|ran\ \alpha}$. Since $\beta_{|ran\ \alpha}$ is order preserving, φ is also order preserving.

(2) Assume that $B_\alpha \neq \emptyset$. Let $x \in B_\alpha$ and let $lcm(A_\alpha(x)) = d$. Let $b_i \in A_\alpha(x)$, then $a_i < x$. Since β is order preserving, $a_i\beta \leq x\beta$, i.e., $b_i \leq x\beta$. Thus $b_i | x\beta$ for all $b_i \in A_\alpha(x)$. So $x\beta$ is a common multiple of $A_\alpha(x)$. Since d is the least common multiple of $A_\alpha(x)$, $d \leq_{nat} x\beta \in I_n$. Therefore, $lcm(A_\alpha(x)) = d \in I_n$.

Conversely, assume that the conditions (1) and (2) hold.

Case 1: $B_\alpha = \emptyset$.

Case 1.1: $1 \notin ran\ \alpha$. Then we define $\beta : I_n \to I_n$ by

$$x\beta = \begin{cases} b_i & \text{if } x = a_i \ \ ; i = 1, 2, ..., m\ , \\ 1 & \text{if } x \in I_n \setminus \{a_1, a_2, ..., a_m\}. \end{cases}$$

To show that β is order preserving, let $x, y \in I_n$ be such that $x < y$.

If $x, y \in ran\ \alpha$, then $x\beta \leq y\beta$ by (1).

If $x, y \in I_n \setminus \{a_1, a_2, ..., a_m\}$, then $x\beta = 1 = y\beta$.

If $x \in I_n \setminus \{a_1, a_2, ..., a_m\}$ and $y \in \{a_1, a_2, ..., a_m\}$, then $x\beta = 1 \leq y\beta$.

If $x \in \{a_1, a_2, ..., a_m\}$ and $y \in I_n \setminus \{a_1, a_2, ..., a_m\}$, then $x = a_j = 1$ for some $j \in \{1, 2, ..., m\}$ (if $a_j \neq 1$, then $1 \neq a_j = x < y$ and thus $y \in B_\alpha$ which is a contradiction since $B_\alpha = \emptyset$). Thus this case can not be occurred since $1 \notin ran\ \alpha$.

Case 1.2: $1 \in ran\ \alpha$. Then let $a_1 = 1$ and we define $\beta : I_n \to I_n$ by

$$x\beta = \begin{cases} b_i & \text{if } x = a_i \ \ ; i = 2, 3, ..., m\ , \\ b_1 & \text{if } x \in I_n \setminus \{a_2, a_3, ..., a_m\}. \end{cases}$$

To show that β is order preserving, let $x, y \in I_n$ be such that $x < y$.

If $x, y \in \{a_2, a_3, ..., a_m\}$, then $x\beta \leq y\beta$ by (1).

If $x, y \in I_n \setminus \{a_2, a_3, ..., a_m\}$, then $x\beta = b_1 = y\beta$.

If $x \in I_n \setminus \{a_2, a_3, ..., a_m\}$ and $y \in \{a_2, a_3, ..., a_m\}$, then $x\beta = b_1 = a_1\beta$ and $y\beta = b_j$ for some $j \in \{2, 3, ..., m\}$. Since $a_1 = 1 \leq y \in ran\ \alpha$, $a_1\beta \leq y\beta$ and thus $x\beta \leq y\beta$.

If $x \in \{a_2, a_3, ..., a_m\}$ and $y \in I_n \setminus \{a_2, a_3, ..., a_m\}$, then $x = a_j \neq 1$ for some $j \in \{2, 3, ..., m\}$, thus $y \in B_\alpha$ which is a contradiction since $B_\alpha = \emptyset$. So this case can not be occurred.

Case 2: $B_\alpha \neq \emptyset$. For each $x \in B_\alpha$, there exists $a_i \in ran\ \alpha \setminus \{1\}$ such that $a_i < x$. By assumption we have that $lcm(A_\alpha(x)) \in I_n$, say k_x.

Case 2.1: $1 \notin ran\ \alpha$. Then we define $\beta : I_n \to I_n$ by

$$x\beta = \begin{cases} b_i & \text{if } x = a_i \ \ ; i = 1, 2, ..., m\ , \\ k_x & \text{if } x \in B_\alpha, \\ 1 & \text{if } x \in I_n \setminus (\{a_1, a_2, ..., a_m\} \cup B_\alpha). \end{cases}$$

To show that β is order preserving, let $x, y \in I_n$ be such that $x < y$. We consider in three subcases.

Subcase 2.1.1: $x \in ran\ \alpha$ and $y \in I_n$. Then $x = a_j$ for some $j \in \{1, 2, ..., m\}$. If $y \in ran\ \alpha$, then $x\beta \leq y\beta$ by (1). If $y \in B_\alpha$, then $A_\alpha(y) = \{b_i \in a_i\alpha^{-1} :\ a_i < y\}$ and $y\beta = lcm(A_\alpha(y))$. Since $a_j = x < y$, $b_j \in A_\alpha(y)$ and so $x\beta = a_j\beta = b_j \leq lcm(A_\alpha(y)) = y\beta$. And, if $y \in I_n \setminus (ran\ \alpha \cup B_\alpha)$, then $x = a_j = 1$ (if $a_j \neq 1$, then $1 \neq a_j = x < y$ and thus $y \in B_\alpha$ which is a contradiction since $y \notin B_\alpha$). Thus this case can not be occurred since $1 \notin ran\ \alpha$.

Subcase 2.1.2: $x \in B_\alpha$ and $y \in I_n$. Then $y \in B_\alpha$. Thus $x\beta = lcm(A_\alpha(x))$ and $y\beta = lcm(A_\alpha(y))$. Since $x < y$, $A_\alpha(x) \subseteq A_\alpha(y)$ and thus $lcm(A_\alpha(x))$ divides $lcm(A_\alpha(y))$. Thus $x\beta | y\beta$ and $x\beta \leq y\beta$.

Subcase 2.1.3: $x \in I_n \setminus (ran\ \alpha \cup B_\alpha)$ and $y \in I_n$. In this subcase $x\beta = 1 \leq y\beta$.

Case 2.2: $1 \in ran\ \alpha$. Then we let $a_1 = 1$ and define $\beta : I_n \to I_n$ by

$$x\beta = \begin{cases} b_i & \text{if } x = a_i \ \ ; i = 2, 3, ..., m\ , \\ k_x & \text{if } x \in B_\alpha, \\ b_1 & \text{if } x \in I_n \setminus (\{a_2, a_3, ..., a_m\} \cup B_\alpha). \end{cases}$$

To show that β is order preserving, let $x, y \in I_n$ be such that $x < y$. We consider in three subcases.

Subcase 2.2.1: $x \in \{a_2, a_3, ..., a_m\}$ and $y \in I_n$. Then $x = a_j$ for some $j \in \{2, 3, ..., m\}$. If $y \in \{a_2, a_3, ..., a_m\}$, then $x\beta \leq y\beta$ by (1). If $y \in B_\alpha$, then $A_\alpha(y) = \{b_i \in a_i\alpha^{-1} : a_i < y\}$ and $y\beta = lcm(A_\alpha(y))$. Since $a_j = x < y$, $b_j \in A_\alpha(y)$ and so $x\beta = a_j\beta = b_j \leq lcm(A_\alpha(y)) = y\beta$. If $y \in I_n \setminus (\{a_2, a_3, ..., a_m\} \cup B_\alpha)$, then $x = a_j = 1$ (if $a_j \neq 1$, then $1 \neq a_j = x < y$ and thus $y \in B_\alpha$ which is a contradiction since $y \notin B_\alpha$). So $x\beta = b_1 = y\beta$.

Subcase 2.2.2: $x \in B_\alpha$ and $y \in I_n$. Then $y \in B_\alpha$. Thus $x\beta = lcm(A_\alpha(x))$ and $y\beta = lcm(A_\alpha(y))$. Since $x < y$, $A_\alpha(x) \subseteq A_\alpha(y)$ and thus $lcm(A_\alpha(x))$ divides $lcm(A_\alpha(y))$. Thus $x\beta | y\beta$ and $x\beta \leq y\beta$.

Subcase 2.2.3: $x \in I_n \setminus (\{a_2, a_3, ..., a_m\} \cup B_\alpha)$ and $y \in I_n$. If $y \in \{a_2, a_3, ..., a_m\}$, then $x\beta = b_1 = a_1\beta$ and $y\beta = b_j$ for some $j \in \{2, 3, ..., m\}$. Since $a_1 = 1 < y \in \{a_2, a_3, ..., a_m\}$, $1\beta = a_1\beta \leq y\beta$ and thus $x\beta \leq y\beta$. If $y \in B_\alpha$, then $y\beta = lcm(A_\alpha(y))$ and thus $b_1 \in A_\alpha(y)$ since $a_1 = 1 < y$, so $x\beta = b_1 \leq lcm(A_\alpha(y)) = y\beta$. If $y \in I_n \setminus (\{a_2, a_3, ..., a_m\} \cup B_\alpha)$, then $x\beta = b_1 = y\beta$.

For each, $x \in I_n$, $x\alpha = a_i$ for some i and so $x\alpha\beta\alpha = (a_i\beta)\alpha = b_i\alpha = a_i = x\alpha$. Therefore, $x\alpha\beta\alpha = x\alpha$ for all $x \in I_n$ and hence $\alpha\beta\alpha = \alpha$. □

Example 3.2.18 For $n = 20$, let $\alpha \in O(I_{20})$ define by

$$\alpha = \begin{pmatrix} \{1,7,11,13,17,19\} & \{2,14\} & \{3,6,9,18\} & \{4,8,16\} & \{5,10,12,15,20\} \\ 1 & 3 & 6 & 9 & 18 \end{pmatrix}.$$

Then $ran\ \alpha = \{1,3,6,9,18\}$ and we choose $A_\alpha = \{1,2,6,4,12\}$. We see that there exists a bijection and order preserving from $ran\ \alpha$ to A_α and $B_\alpha = \{12,15\}$, $A_\alpha(12) = \{1,2,6\}$ and $A_\alpha(15) = \{1,2\}$. Then we define β by

$$\beta = \begin{pmatrix} 1 & 3 & 6 & 9 & 18 & 12 & 15 & \{2,4,5,7,8,10,11,13,14,16,17,19,20\} \\ 1 & 2 & 6 & 4 & 12 & 6 & 2 & 1 \end{pmatrix}.$$

Thus by Theorem 3.2.17, β is order preserving and $\alpha\beta\alpha = \alpha$.

□

Example 3.2.19 For $n = 16$, let $\alpha \in O(I_{16})$ define by

$$\alpha = \begin{pmatrix} \{1,7,13\} & \{2,3,6,9,11,14\} & \{4,8,12,16\} & \{5,10,15\} \\ 2 & 4 & 8 & 12 \end{pmatrix}.$$

Then $ran\ \alpha = \{2,4,8,12\}$ and we choose $A_\alpha = \{1,2,4,10\}$. We see that there exists a bijection and order preserving from A_α to $ran\ \alpha$ and $B_\alpha = \{6,10,14,16\}$ and $A_\alpha(6) = \{1\}$, $A_\alpha(10) = \{1\}$, $A_\alpha(14) = \{1\}$ and $A_\alpha(16) = \{1,2,4\}$. Then we define β by

$$\beta = \begin{pmatrix} 2 & 4 & 8 & 12 & 6 & 10 & 14 & 16 & \{1,3,5,7,9,11,13,15\} \\ 1 & 2 & 4 & 10 & 1 & 1 & 1 & 4 & 1 \end{pmatrix}.$$

Thus by Theorem 3.2.17, β is order preserving and $\alpha\beta\alpha = \alpha$.

□

Example 3.2.20 For $n = 8$, let $\alpha \in O(I_8)$ define by

$$\alpha = \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6,7,8\} \\ 1 & 3 & 6 \end{pmatrix}.$$

Then $ran\ \alpha = \{1,3,6\}$ and we choose $A_\alpha = \{2,4,8\}$. We see that there exists a bijection and order preserving from $ran\ \alpha$ to A_α and $B_\alpha = \emptyset$. Then we define β by

$$\beta = \begin{pmatrix} 1 & 3 & 6 & \{2,4,5,7,8\} \\ 2 & 4 & 8 & 2 \end{pmatrix}.$$

Thus by Theorem 3.2.17, β is order preserving and $\alpha\beta\alpha = \alpha$.

□

Example 3.2.21 For $n = 7$, let $\alpha \in O(I_7)$ define by

$$\alpha = \begin{pmatrix} \{1,2,5\} & \{3,6\} & \{4,7\} \\ 2 & 4 & 6 \end{pmatrix}.$$

Then $ran\ \alpha = \{2,4,6\}$ and we choose $A_\alpha = \{1,3,4\}$. We see that there exists a bijection and order preserving from $ran\ \alpha$ to A_α and $B_\alpha = \emptyset$. Then we define β by

$$\beta = \begin{pmatrix} 2 & 4 & 6 & \{1,3,5,7\} \\ 1 & 3 & 4 & 1 \end{pmatrix}.$$

Thus by Theorem 3.2.17, β is order preserving and $\alpha\beta\alpha = \alpha$.

□

3.3 Maximal Subgroups of $O(I_n)$

In this section we will study a maximal subgroups of semigroup $O(I_n)$.

Definition 3.3.1 [2] An element e in a semigroup S is called *idempotent* if $e^2 = e$, and we set $E(S)$ to be the set of all idempotents in S.

Definition 3.3.2 [2] A subgroup M of a group G is said to be *maximal* in G if $M \neq G$ and for every subgroup H such that $M \subseteq H \subseteq G$ implies that $H = M$ or $H = G$.

Theorem 3.3.3 [2] *Let S be a semigroup and let e be any idempotent in S and*

$$G_e = \{x \in S : \ xe = x = ex, xy = e = yx \text{ for some } y \in S\}$$
$$= \{x \in S : \ x \in eS \cap Se \text{ and } e \in xS \cap Sx\}.$$

Then G_e is a maximal subgroup of S having e as an identity.

For each $\alpha \in O(I_n)$. The set $\pi_\alpha = \{(a, b) \in I_n \times I_n : \ a\alpha = b\alpha\}$ is an equivalent relation on I_n. We call π_α the partition of I_n corresponding to α.

Theorem 3.3.4 *Let e be any idempotent of $O(I_n)$. For each $\alpha \in O(I_n)$, $\alpha e = \alpha = e\alpha$ if and only if $ran\ \alpha \subseteq ran\ e$ and $\pi_e \subseteq \pi_\alpha$.*

Proof. Assume that $\alpha e = \alpha = e\alpha$. First, we prove that $ran\ \alpha \subseteq ran\ e$. Let $y \in ran\ \alpha$. Then there exists $x \in I_n$ such that $x\alpha = y$. Since $\alpha e = \alpha$, $y = x\alpha = x\alpha e = (x\alpha)e$. Thus $y \in ran\ e$. To prove $\pi_e \subseteq \pi_\alpha$, let $(a, b) \in \pi_e$. Then $ae = c = be$ for some $c \in I_n$. Thus $ae\alpha = c\alpha = be\alpha$ and so $a\alpha = d = b\alpha$ for some $d \in I_n$. Therefore, $(a, b) \in \pi_\alpha$.

Conversely, assume that $ran\ \alpha \subseteq ran\ e$ and $\pi_e \subseteq \pi_\alpha$. Let $x \in I_n$. Then $x\alpha = y$ for some $y \in I_n$. Thus $y \in ran\ \alpha \subseteq ran\ e$ and so $y \in ran\ e$. Then there exists $x' \in I_n$ such that $x'e = y$. Consider $x\alpha e = ye = x'ee = x'e = y = x\alpha$ for all $x \in I_n$ and so $\alpha e = \alpha$. Now, consider $xe = z$ for some $z \in I_n$. Since $z \in ran\ e$ and e is an idempotent, $ze = z$. But $xe = z = ze$, so we have that $(x, z) \in \pi_e \subseteq \pi_\alpha$. Then $(x, z) \in \pi_\alpha$ and so $x\alpha = z\alpha$. Since $z = xe$, $z\alpha = xe\alpha$. Thus $x\alpha = z\alpha = xe\alpha$ for all $x \in I_n$ and hence $e\alpha = \alpha$. Therefore, $\alpha e = \alpha = e\alpha$. $\square$

By the above Theorem we let

$$I_e = \{\alpha \in O(I_n) : \ \alpha e = \alpha = e\alpha\}$$
$$= \{\alpha \in O(I_n) : ran\ \alpha \subseteq ran\ e \text{ and } \pi_e \subseteq \pi_\alpha\}.$$

Lemma 3.3.5 *Let $\alpha, \beta \in I_e$. If $\alpha\beta = e = \beta\alpha$, then $ran\ \alpha = ran\ e = ran\ \beta$ and $\pi_\alpha = \pi_e = \pi_\beta$.*

Proof. Assume that $\alpha\beta = e = \beta\alpha$. First, we show that $ran\ \alpha = ran\ e = ran\ \beta$. Since $\alpha \in I_e$, $ran\ \alpha \subseteq ran\ e$. Let $z \in ran\ e$. Then there exists $x' \in I_n$ such that $x'e = z$. Since $e = \beta\alpha$, $x'e = x'\beta\alpha$. Thus $z = (x'\beta)\alpha$ and so $z \in ran\ \alpha$. Then $ran\ e \subseteq ran\ \alpha$. Therefore, $ran\ \alpha = ran\ e$. Since $\beta \in I_e$, $ran\ \beta \subseteq ran\ e$. Let $y \in ran\ e$. Then there exists $x'' \in I_n$ such that $x''e = y$. Since $\alpha\beta = e$, $x''\beta\alpha = x''e$. Thus $y = (x''\alpha)\beta$ and so $y \in ran\ \beta$. Then $ran\ e \subseteq ran\ \beta$ and hence $ran\ \beta = ran\ e$. Therefore, $ran\ \alpha = ran\ e = ran\ \beta$.

Next, we show that $\pi_\alpha = \pi_e = \pi_\beta$. Let $(a, b) \in \pi_\alpha$. Then $a\alpha = c = b\alpha$ for some $c \in I_n$. Since $\alpha\beta = e$, $a\alpha\beta = ae$ and $b\alpha\beta = be$. Thus $ae = a\alpha\beta = c\beta = b\alpha\beta = be$ and so $(a, b) \in \pi_e$. Hence $\pi_\alpha \subseteq \pi_e$. By Theorem 3.3.4 we have $\pi_e \subseteq \pi_\alpha$. Therefore, $\pi_\alpha = \pi_e$. Let $(a, b) \in \pi_\beta$. Then $a\beta = c = b\beta$ for some $c \in I_n$. Since $\beta\alpha = e$, $a\beta\alpha = ae$ and $b\beta\alpha = be$. Thus $ae = a\beta\alpha = c\alpha = b\beta\alpha = be$ and so $(a, b) \in \pi_e$. Hence $\pi_\beta \subseteq \pi_e$. By Theorem 3.3.4 we have $\pi_e \subseteq \pi_\beta$. Therefore, $\pi_e = \pi_\beta$ and hence $\pi_\alpha = \pi_e = \pi_\beta$. □

Let $M = \{a_1, a_2, ..., a_m\} = ran\ e$ where e is the idempotent of $O(I_n)$. Recall that the set of permutations of M is denoted by S_m. For convenience a permutation $\sigma \in S_m$ is usually represented as

$$\begin{pmatrix} a_1 & a_2 & \dots & a_m \\ a_1\sigma & a_2\sigma & \dots & a_m\sigma \end{pmatrix}.$$

In this notation the first column expresses the fact that σ maps a_1 to $a_1\sigma$; the second column, that σ maps a_2 to $a_2\sigma$ and the end column, that σ maps a_m to $a_m\sigma$.

For each $\alpha, \beta \in I_e$ with $\alpha\beta = e = \beta\alpha$, by Lemma 3.3.5 we have $ran\ \alpha =$

$ran\ e = ran\ \beta$ and $\pi_\alpha = \pi_e = \pi_\beta$ and so we write

$$e = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ a_1 & a_2 & \dots & a_m \end{pmatrix},$$

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ a_1\sigma & a_2\sigma & \dots & a_m\sigma \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ a_1\delta & a_2\delta & \dots & a_m\delta \end{pmatrix}$$

where $\sigma, \delta \in S_m$. We call σ and δ the *permutations of* α *and* β respectively.

Lemma 3.3.6 *Let* $\alpha, \beta \in I_e$ *and* σ, δ *are the permutations of* α *and* β *respectively. If* $\alpha\beta = e = \beta\alpha$*, then* $\sigma\delta = 1_M = \delta\sigma$.

Proof. Assume that $\alpha\beta = e = \beta\alpha$. Let

$$e = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ a_1 & a_2 & \dots & a_m \end{pmatrix},$$

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ a_1\sigma & a_2\sigma & \dots & a_m\sigma \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ a_1\delta & a_2\delta & \dots & a_m\delta \end{pmatrix}$$

where $\sigma, \delta \in S_m$ and $a_i \in A_i$ for all i since e is an idempotent. Then $x\alpha\beta = xe = x\beta\alpha$ for all $x \in I_n$. Let $x \in I_n$. Then $x \in A_i$ for some i. Since $x\alpha\beta = xe$, $(a_i\sigma)\beta = a_i$. Then $(a_i\sigma)\beta = a_j\beta($ for some a_j $) = a_j\delta = a_i\sigma\delta = a_i = a_i 1_M$ for all $a_i \in M$. Thus $\sigma\delta = 1_M$. Since $x\beta\alpha = xe$, $(a_i\delta)\alpha = a_i$. Then $(a_i\delta)\alpha = a_j\alpha($ for some a_j $) = a_j\sigma = a_i\delta\sigma = a_i = a_i 1_M$ for all $a_i \in M$. Thus $\delta\sigma = 1_M$. Therefore, $\sigma\delta = 1_M = \delta\sigma$. □

Theorem 3.3.7 *For each $\alpha \in I_e$, $\alpha\beta = e = \beta\alpha$ for some $\beta \in I_e$ if and only if $\pi_\alpha = \pi_e = \pi_\beta$, $ran\ \alpha = ran\ e = ran\ \beta$ and $\sigma\delta = 1_M = \delta\sigma$ where σ, δ are the permutations of α and β respectively.*

Proof. Assume that $\alpha\beta = e = \beta\alpha$ for some $\beta \in I_e$. By Lemma 3.3.5 and Lemma 3.3.6 we have $\pi_\alpha = \pi_e = \pi_\beta$ and $ran\ \alpha = ran\ e = ran\ \beta$ and $\sigma\delta = 1_M = \delta\sigma$.

Coversely, assume that $\pi_\alpha = \pi_e = \pi_\beta$ and $ran\ \alpha = ran\ e = ran\ \beta$ and $\sigma\delta = 1_M = \delta\sigma$. Since $ran\ e = ran\ \beta$ and $\pi_e = \pi_\beta$, by Theorem 3.3.4 we have that $\beta e = \beta = e\beta$. Thus $\beta \in I_e$. Let

$$e = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ a_1 & a_2 & \dots & a_m \end{pmatrix},$$

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ a_1\sigma & a_2\sigma & \dots & a_m\sigma \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ a_1\delta & a_2\delta & \dots & a_m\delta \end{pmatrix}$$

where $\sigma, \delta \in S_m$ and $a_i \in A_i$ for all i since e is an idempotent. Let $x \in I_n$. Then $x \in A_i$ for some i and we have that $xe = a_i$. Suppose that $a_i\sigma = a_j$ and $a_i\delta = a_k$ for some j, k . So $x\alpha\beta = (a_i\sigma)\beta = a_j\beta = a_j\delta = a_i\sigma\delta = a_i 1_M = a_i = xe$ and $x\beta\alpha = (a_i\delta)\alpha = a_k\alpha = a_k\sigma = a_i\delta\sigma = a_i 1_M = a_i = xe$ for all $x \in I_n$. Therefore, $\alpha\beta = e = \beta\alpha$. □

Theorem 3.3.8 *Let $H_e = \{\alpha \in I_e :\ \alpha\beta = e = \beta\alpha$ for some $\beta \in I_e\}$. Then H_e is a maximal subgroup of $O(I_n)$.*

Proof. We will prove that $G_e = \{\alpha \in O(I_n) :\ \alpha e = \alpha = e\alpha$ and $\alpha\beta = e = \beta\alpha$ for some $\beta \in O(I_n)\} = H_e$. Let $\alpha \in H_e$. Then $\alpha \in I_e \subseteq O(I_n)$ and $\alpha\beta = e = \beta\alpha$ for some $\beta \in I_e$. Since $\alpha \in I_e$, $\alpha e = \alpha = e\alpha$ and so $\alpha \in G_e$. Thus $H_e \subseteq G_e$. Let $\alpha \in G_e$. Then $\alpha e = \alpha = e\alpha$ and $\alpha\beta = e = \beta\alpha$ for some $\beta \in O(I_n)$. Thus $\alpha \in I_e$. Since G_e is a group having e as its identity, β is an inverse element of α. Then

$\beta \in G_e$ and so $\beta \in I_e$. Thus $G_e \subseteq H_e$ and so $G_e = H_e$. Since G_e is a maximal subgroup, H_e is also a maximal subgroup. $\square$

Example 3.3.9 For $n = 8$, we let e be an idempotent of $O(I_8)$ defined by

$$e = \begin{pmatrix} \{1\} & \{2,3,5\} & \{4,8\} & \{6,7\} \\ 1 & 2 & 4 & 6 \end{pmatrix}.$$

Then by Lemma 3.3.4 we have that

$$I_e = \{\alpha \in O(I_8) : \, ran\ \alpha \subseteq \{1,2,4,6\} \text{ and } \pi_e \subseteq \pi_\alpha\}.$$

Thus

$$\beta = \begin{pmatrix} \{1\} & \{2,3,5\} & \{4,6,7,8\} \\ 1 & 2 & 4 \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} \{1,2,3,5\} & \{4,6,7,8\} \\ 2 & 4 \end{pmatrix}$$

are elements of I_e. Thus for each $\alpha \in O(I_8)$,

$\alpha \in H_e \quad \Rightarrow \quad \alpha \in I_e$ and $\alpha\beta = e = \beta\alpha$ for some $\beta \in I_e$

$\Rightarrow \quad \alpha \in I_e$, $\pi_\alpha = \pi_e = \pi_\beta$, $ran\ \alpha = ran\ e = ran\ \beta$ and $\sigma\delta = 1_M = \delta\sigma$ for some $\beta \in I_e$

$$\Rightarrow \quad \alpha = \begin{pmatrix} \{1\} & \{2,3,5\} & \{4,8\} & \{6,7\} \\ 1 & a & b & c \end{pmatrix} \text{ where } \{a,b,c\} = \{2,4,6\}$$

$$\Rightarrow \quad \alpha = \begin{pmatrix} \{1\} & \{2,3,5\} & \{4,8\} & \{6,7\} \\ 1 & 2 & b & c \end{pmatrix} \text{ since } 2 < 4 \text{ and } 2 < 6$$

$$\Rightarrow \quad \alpha = \begin{pmatrix} \{1\} & \{2,3,5\} & \{4,8\} & \{6,7\} \\ 1 & 2 & 4 & 6 \end{pmatrix} \text{ or}$$

$$\alpha = \begin{pmatrix} \{1\} & \{2,3,5\} & \{4,8\} & \{6,7\} \\ 1 & 2 & 6 & 4 \end{pmatrix}.$$

Then we see that $\beta, \gamma \notin H_e$. That is $I_e \neq H_e$ in general. For each $\alpha \in H_e$, we choose $\beta = \alpha$ and then $\alpha\beta = e = \beta\alpha$. Thus $H_e = \{e, \alpha\}$ is a maximal subgroup of $O(I_8)$. □

Example 3.3.10 For $n = 8$, we let e be an idempotent of $O(I_8)$ defined by

$$e = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 3 & 8 & 5 & 7 \end{pmatrix}.$$

Thus for each $\alpha \in O(I_8)$,

$\alpha \in H_e \quad \Rightarrow \quad \alpha \in I_e$ and $\alpha\beta = e = \beta\alpha$ for some $\beta \in I_e$

$\Rightarrow \quad \alpha \in I_e$, $\pi_\alpha = \pi_e = \pi_\beta$, $ran\ \alpha = ran\ e = ran\ \beta$ and $\sigma\delta = 1_M = \delta\sigma$ for some $\beta \in I_e$

$\Rightarrow \quad \alpha = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & a & b & c & d \end{pmatrix}$ where

$\{a, b, c, d\} = \{3, 5, 7, 8\}$.

Since $\{3,6\}, \{4,8\}, \{5\}, \{7\}$ are pairwise disjoint partially ordered sets, so they are $4! = 24$ permutations of $\{3,5,7,8\}$. Thus all elements of H_e are

$$\alpha_1 = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 3 & 5 & 7 & 8 \end{pmatrix},$$

$$\alpha_2 = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 3 & 5 & 8 & 7 \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 3 & 7 & 5 & 8 \end{pmatrix},$$

$$\alpha_4 = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 3 & 7 & 8 & 5 \end{pmatrix},$$

$$\alpha_5 = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 3 & 8 & 7 & 5 \end{pmatrix},$$

$$\alpha_6 = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 3 & 8 & 5 & 7 \end{pmatrix},$$

$$\alpha_7 = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 5 & 3 & 7 & 8 \end{pmatrix},$$

$$\alpha_8 = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 5 & 3 & 8 & 7 \end{pmatrix},$$

$$\alpha_9 = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 5 & 7 & 3 & 8 \end{pmatrix},$$

$$\alpha_{10} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 5 & 7 & 8 & 3 \end{pmatrix},$$

$$\alpha_{11} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 5 & 8 & 7 & 3 \end{pmatrix},$$

$$\alpha_{12} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 5 & 8 & 3 & 7 \end{pmatrix},$$

$$\alpha_{13} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 7 & 3 & 5 & 8 \end{pmatrix},$$

$$\alpha_{14} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 7 & 3 & 8 & 5 \end{pmatrix},$$

$$\alpha_{15} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 7 & 5 & 3 & 8 \end{pmatrix},$$

$$\alpha_{16} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 7 & 5 & 8 & 3 \end{pmatrix},$$

$$\alpha_{17} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 7 & 8 & 5 & 3 \end{pmatrix},$$

$$\alpha_{18} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 7 & 8 & 3 & 5 \end{pmatrix},$$

$$\alpha_{19} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 8 & 5 & 7 & 3 \end{pmatrix},$$

$$\alpha_{20} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 8 & 5 & 3 & 7 \end{pmatrix},$$

$$\alpha_{21} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 8 & 3 & 5 & 7 \end{pmatrix},$$

$$\alpha_{22} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 8 & 3 & 7 & 5 \end{pmatrix},$$

$$\alpha_{23} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 8 & 7 & 5 & 3 \end{pmatrix},$$

$$\alpha_{24} = \begin{pmatrix} \{1,2\} & \{3,6\} & \{4,8\} & \{5\} & \{7\} \\ 1 & 8 & 7 & 3 & 5 \end{pmatrix}.$$

□

Example 3.3.11 For $n = 7$, we let e be the identity map of $O(I_7)$. Then

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}.$$

Thus for each $\alpha \in O(I_7)$,

$\alpha \in H_e \quad \Rightarrow \quad \alpha \in I_e$ and $\alpha\beta = e = \beta\alpha$ for some $\beta \in I_e$

$\Rightarrow \quad \alpha \in I_e$, $\pi_\alpha = \pi_e = \pi_\beta$, $ran\ \alpha = ran\ e = ran\ \beta$ and $\sigma\delta = 1_M = \delta\sigma$ for some $\beta \in I_e$

$\Rightarrow \quad \alpha$ is an order preserving permutation of I_7 and there exists an order preserving permutation β of I_7 such that $\alpha\beta = e = \beta\alpha$.

Therefore, α can be

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

or

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 7 & 6 & 5 \end{pmatrix}$$

and hence $H_e = \{e, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 7 & 6 & 5 \end{pmatrix}\}$.

□