CHAPTER 4

CONCLUSION

In this study, we found that:

1. Some properties of $(I_n, \leq)$

1.1 $(I_n, \leq)$ is a partially ordered set.

1.2 For each $m \in I_n$, m is a minimal element of $(I_n \setminus \{1\}, \leq)$ if and only if m is a prime number.

1.3 For each $x, y \in I_n$, if $x <_{nat} y$ and $x \in D_n$ then $x \nmid y$.

1.4 For each $M \in I_n$, M is a maximal element of $(I_n, \leq)$ if and only if $M \in D_n$.

1.5 The number of elements in D_n is $n - \left\lfloor \frac{n}{2} \right\rfloor$.

1.6 $(I_n, \leq)$ is a complete lower semilattice.

2. Regularity of $O(I_n)$

2.1 If $n \leq_{nat} 3$, then $O(I_n)$ is regular.

2.2 If $n \geq_{nat} 4$, then $O(I_n)$ is not regular.

2.3 $O(I_n)$ is regular if and only if $n \leq_{nat} 3$.

2.4 If X is isolated, then $O(X)$ is regular.

2.5 Let X be a partially ordered set such that $X = Y \cup Z$ where $|Y| \geq_{nat} 2$ and there exist $a, m \in Y$ with $a < m$; and Y, Z are disjoint partially ordered sets. Then $O(X)$ is not regular.

2.6 Let X be a proper partially ordered subset of I_4. Then $O(X)$ is regular if and only if X is one of the following forms :

(1) $\prod_1$ is a chain,

(2) $\prod_2 = \{a_1, a_2, a_3 : \ a_1 < a_2$ and $a_1 < a_3$ and $\{a_2, a_3\}$ is isolated $\}$,

(3) $\prod_3$ is isolated.

2.7 Let X be a proper partially ordered subset of I_5. Then $O(X)$ is regular if and only if X is one of the following forms :

(1) $\prod_1$ is a chain,

(2) $\prod_2 = \{a_1, a_2, a_3 : \ a_1 < a_2, \ a_1 < a_3$ and $\{a_2, a_3\}$ is isolated $\}$,

(3) $\prod_3 = \{a_1, a_2, a_3, a_4 : \ a_1 < a_i$ for all $i = 2, 3, 4$ and $\{a_2, a_3, a_4\}$ is isolated $\}$,

(4) $\prod_4$ is isolated.

2.8 Let $X = \{a_1, a_2, a_3, a_4 : \ a_1 < a_3, \ a_1 < a_4, \ a_2 < a_4;$ and $\{a_1, a_2\}$, $\{a_3, a_4\}$ are isolated $\}$ be a partially ordered set. Then any order preserving permutation of X equals to 1_X, the identity map on X.

2.9 Let $X = \{a_1, a_2, a_3, a_4 : \ a_1 < a_3, \ a_1 < a_4, \ a_2 < a_4;$ and $\{a_1, a_2\}, \{a_3, a_4\}$ are isolated $\}$ be a partially ordered set. Then $O(X)$ is regular.

2.10 Let $X = \{a_1, a_2, a_3, a_4, a_5 : \ a_1 < a_4, \ a_2 < a_4, \ a_2 < a_5$ and $a_3 < a_5$; and $\{a_1, a_2, a_3\}$, $\{a_1, a_5\}$ and $\{a_3, a_4\}$ are isolated $\}$ be a partially ordered set. Then $O(X)$ is not regular.

2.11 Let $X = \{a_1, a_2, a_3, a_4, a_5 : \ a_1 < a_3, \ a_1 < a_4, \ a_2 < a_4$ and $a_2 < a_5$; and $\{a_3, a_4, a_5\}$, $\{a_1, a_5\}$ and $\{a_2, a_3\}$ are isolated $\}$ be a partially ordered set. Then $O(X)$ is not regular.

2.12 Let $\alpha \in O(I_n)$. If there exist $x, y \in ran\ \alpha$ such that $x < y$ and $x\alpha^{-1}$ and $y\alpha^{-1}$ are disjoint partially ordered sets, then α is not regular.

2.13 Let $\alpha \in O(I_n)$.

(1) If α is regular, then for all $x, y \in ran\ \alpha, x < y$ implies $x\alpha^{-1}$ and $y\alpha^{-1}$ are not disjoint partially ordered sets.

(2) If $1 \in ran\ \alpha$, then $1\alpha = 1$.

2.14 Let $\alpha \in O(I_n)$ with $ran\ \alpha = \{a_1, a_2, ...a_m\}$. Then α is regular if and only if the following conditions hold:

(1) There exists A_α such that the map $\varphi : ran\ \alpha \to A_\alpha$ define by $a_i\varphi = b_i$ for all i is order preserving.

(2) If $B_\alpha \neq \emptyset$, then $lcm(A_\alpha(x)) \in I_n$ for all $x \in B_\alpha$.

3. Maximal Subgroup of $O(I_n)$

3.1 Let e be any idempotent of $O(I_n)$. For each $\alpha \in O(I_n)$, $\alpha e = \alpha = e\alpha$ if and only if $ran\ \alpha \subseteq ran\ e$ and $\pi_e \subseteq \pi_\alpha$.

3.2 Let $\alpha, \beta \in I_e$. If $\alpha\beta = e = \beta\alpha$, then $ran\ \alpha = ran\ e = ran\ \beta$ and

$\pi_\alpha = \pi_e = \pi_\beta$.

3.3 Let $\alpha, \beta \in I_e$ and σ, δ are the permutations of α and β respectively. If $\alpha\beta = e = \beta\alpha$, then $\sigma\delta = 1_M = \delta\sigma$.

3.4 For each $\alpha \in I_e$, $\alpha\beta = e = \beta\alpha$ for some $\beta \in I_e$ if and only if $\pi_\alpha = \pi_e = \pi_\beta$, $ran\ \alpha = ran\ e = ran\ \beta$ and $\sigma\delta = 1_M = \delta\sigma$ where σ, δ are the permutations of α and β respectively.

3.5 Let $H_e = \{\alpha \in I_e : \ \alpha\beta = e = \beta\alpha \text{ for some } \beta \in I_e\}$. Then H_e is a maximal subgroup of $O(I_n)$.