CHAPTER 2

PRELIMINARIES

The aim of this chapter is to give some definitions, notations and dealing with some preliminaries and give some useful results that will be recalled in later chapters.

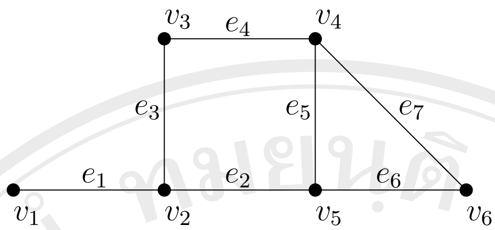
2.1 Basic Concepts on Graph

Definition 2.1.1 *A graph G consists of a finite nonempty set V of elements, called vertices, and a list E of unordered pairs of distinct vertices, the set V and E are the vertex set and edge set of G, respectively, so a graph G is a pair of two sets V and E for this reason, some write $G = (V, E)$. At times, it is useful to write $V(G)$ and $E(G)$ rather than V and E to emphasize that these are the vertex and edge sets of a particular graph G. If v and w are vertices of G, then an edge e of the form $\{v, w\}$or$\{w, v\}$, also e incidents with v (or w). If $e = \{v, w\}$ or $\{w, v\}$ is an edge of G, then v and w are joined by the edge e, the vertex v and the edge e (as well as w and e) are incident with each other the vertices v and w are endpoints of the edge e. We usually write $\{v, w\}$ or $\{v, w\}$ as vw wv.*
A graph H is a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

It is common to represent a graph by a diagram in the plane where the vertices are represented by points or small cycles and whose edges are indicated by the presence of a line segment or curve between the two points in the plane corresponding to the appropriate vertices.

Example 2.1.2 *Consider the graph $G = (V, E)$ with 6 vertices and 7 edges.*

G :



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$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\} \text{ and } E = \{e_1, e_2, e_3, e_4, e_5, e_6\}.$$

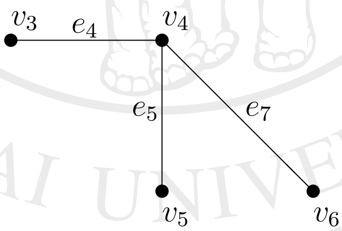
The vertices v_1 and v_2 are jointed by the edge e_1.

The edge e_4 is incident with v_3 (as well as v_4).

The vertex v_5 is not incident with e_3.

Example 2.1.3 *The graph H is a subgraph of G in the example 2.1.2.*

H :



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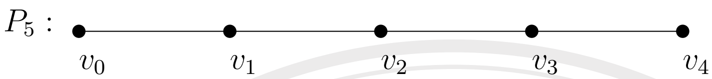
Definition 2.1.4 *Let x, y be a vertices of a graph G, The $x - y$ walk of G is meant a finite, alternating sequence $x = v_0, a_1, v_1, a_2, ..., v_{n-1}, a_n, v_n = y$ of vertices and edges, begining with the vertex x and ending with the vertex y such that $a_i = \{v_{i-1}v_i\}$ for all $i = 1, 2, ..., n$ the number of n is called length of walk.*

Definition 2.1.5 *A $x - y$ path in a graph G is a $x - y$ walk in which no vertices are repeated. A path P_n with n vertices, denoted by P_n is a graph where $V(P_n) = \{v_0, v_1, ..., v_{n-1}\}$ and $E(P_n) = \{\{v_iv_{i+1}\} | i = 0, 1, ..., n - 2\}$.*

Example 2.1.6 *The path P_5.*



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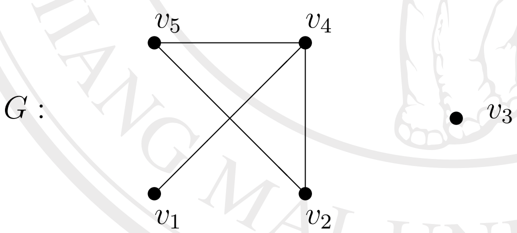
Definition 2.1.7 *A graph G is said to be connected if, for each pair of vertices x, y of G there exist a $x - y$ path. A maximal connected subgraph of a graph G is called a component.*

Definition 2.1.8 *Let G be a graph, and let v be a vertex of G. The degree of v, is denoted by $d(v)$ of edges incident with v. If $d(v) = 0$ then v is called isolated point.*

Example 2.1.9 *Consider the following graph G.*



$d(v_1) = 1$, $d(v_2) = 2$, $d(v_3) = 0$,
$d(v_4) = 3$, $d(v_5) = 2$

□

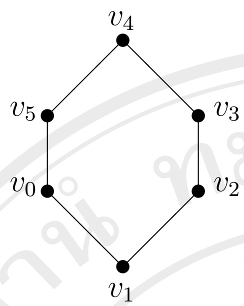
Definition 2.1.10 *We say a graph G is regular if all vertices of G have the same degree.*

Definition 2.1.11 *Let G be a graph. We called G have an isolated edge if a component of G is the path P_2.*

Definition 2.1.12 *A cycle of length n, denoted by C_n is a graph where $V(C_n) = \{v_0, v_1, ..., v_{n-1}\}$ and the edge set is $E(C_n) = \{\{v_i, v_{i+1}\} | i = 0, 1, ..., n-2\} \cup \{\{v_0, v_{n-1}\}\}$.*

Example 2.1.13 *The cycle of length 6, C_6.*

C_6 :

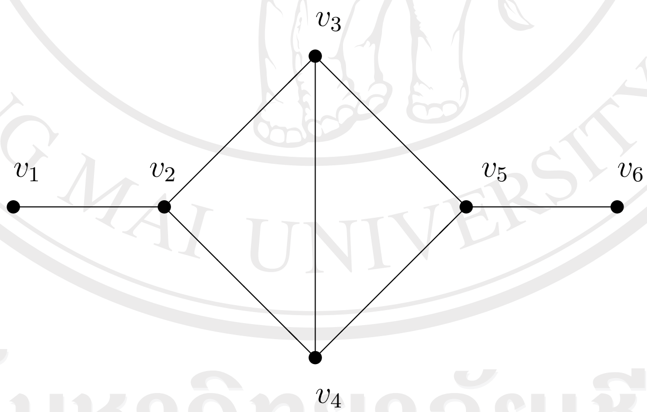


$\square$

Definition 2.1.14 *Let G be a graph. For a vertex $x \in V(G)$, we called $N(x) = \{y \in G | \{x, y\} \in E(G)\}$ the neighbourhood of x in G.*

Example 2.1.15 *The neighbourhood of v_i in the follwing graph G where i = 1, 2,...,6 are*

G :

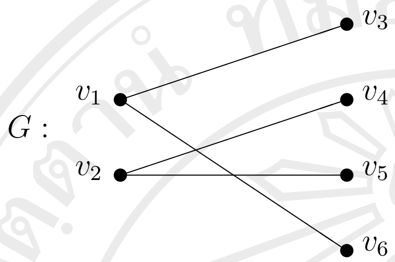


$N(v_1) = \{v_2\}$, $N(v_2) = \{v_1, v_3, v_4\}$, $N(v_3) = \{v_2, v_4, v_5\}$, $N(v_4) = \{v_2, v_3, v_5\}$, $N(v_5) = \{v_3, v_4, v_6\}$, $N(v_6) = \{v_5\}$. $\square$

Definition 2.1.16 *A graph $G = (V, E)$ is called bipartite graph if its vertex set can be partitioned into two parts V_1 and V_2 such that every edges has one endpoint in V_1 and another one in V_2.*

In addition, if $d(v_i) = r$ for all $v_i \in V_1$ and $d(w_i) = s$ for all $w_i \in V_2$ then the graph $G = (V, E)$ is called (r, s)-semiregular bipartite graph.

Example 2.1.17 *The following graph G is $(2,1)$-semiregular bipartite graph with $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_3, v_4, v_5, v_6\}$.*



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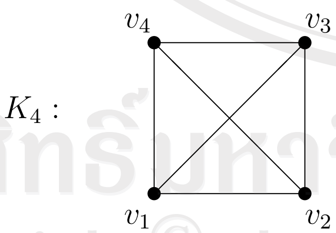
Lemma 2.1.18 ([5]) *(Handshaking Lemma)*

Let $G = (V, E)$ be a graph where $V = \{v_1, v_2, ..., v_n\}$. Then $\sum_{i=1}^{n} d(v_i) = 2|E|$.

Definition 2.1.19 *A complete graph is a graph in which every two distinct vertices are joined by exactly one edge. The complete graph with n vertices is denoted by K_n.*

Example 2.1.20 *The complete graph K_4 .*



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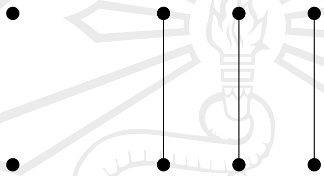
Definition 2.1.21 *We call G a complete bipartite graph if G is a bipartite graph and each vertex in the first part joint every vertices in another part.*

Definition 2.1.22 *Let G_1, G_2 be a graph where $V(G_1) \cap V(G_2) = \emptyset$ the disjoint union of G_1, G_2 is a graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph G consists of $n(\geq 2)$ disjoint copies of a graph H, Then we write $G = nH$.*

Example 2.1.23 *The disjoint union of 2 copies of K_1 and 3 copies of P_2, $2K_1 \cup 3P_2$ is show below.*



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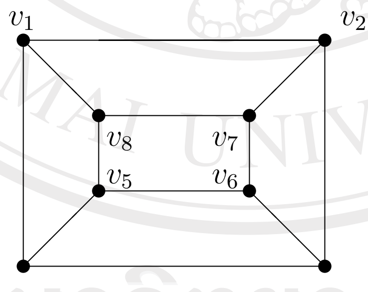
Definition 2.1.24 *Let $G_1 = (V_1, E_1)$, $G = (V_2, E_2)$ be graphs. The product of G_1 with G_2, denote by $G_1 \square G_2$, is a graph with $V(G_1 \square G_2) = V_1 \times V_2$ and $E(G_1 \square G_2) =$*
$$\Big\{\{(x,y),(x,y')\} | x \in G_1, \{y,y'\} \in E_2\Big\} \cup \Big\{\{(x,y),(x',y)\} | y \in G_2, \{x,x'\} \in E_1\Big\}$$

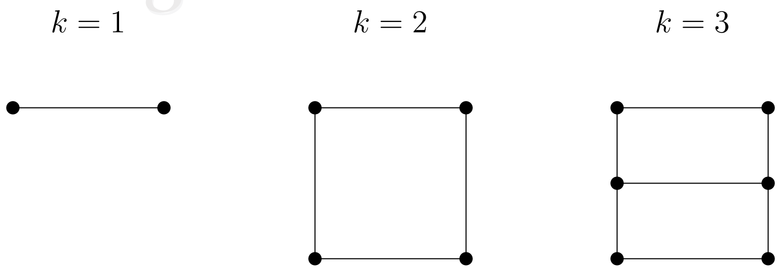
Example 2.1.25 *The graph $P_2 \square P_2 \square P_2$.*



$\square$

Definition 2.1.26 *The ladder graph L_k is a graph $K_2 \square P_k$.*

Example 2.1.27 *The ladder graph L_k where k=1,2,3.*

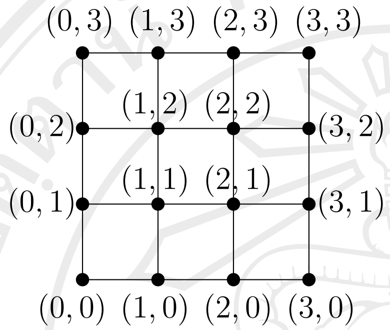


$\square$

Definition 2.1.28 *Let $m \geq 2$ be a positive integer. The graph G is called $m \times m$ square lattice if $G = P_m \square P_m$, where P_m is a path of m vertices.*

We denote vertices of $P_m \square P_m$ by (i, j) where $i, j = 0, 1, ..., m-1$

Example 2.1.29 *The 4×4 square lattice graph, $P_4 \square P_4$*



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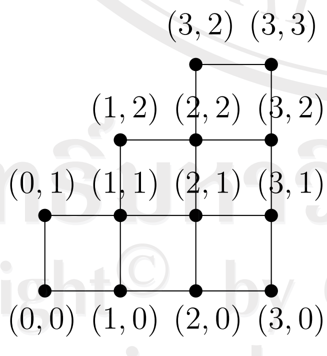
Definition 2.1.30 *The $1-$ ladder square lattice of $k-$ step is the graph obtained from $P_{k+1} \square P_{k+1}$ by deleting the set of vertices $\{(r, r_j) | r = 0, 1, ..., k-2, j = 2, 3, ..., k-r\}$ and all edges incident with them.*

Note that the number of vertices of the $1-$ ladder square lattice of $k-$ step is $\frac{k^2+5k+2}{2}$ and the number of edges is $k^2 + 3k$.

Example 2.1.31 *The $1-$ ladder square lattice of $3-$ step.*

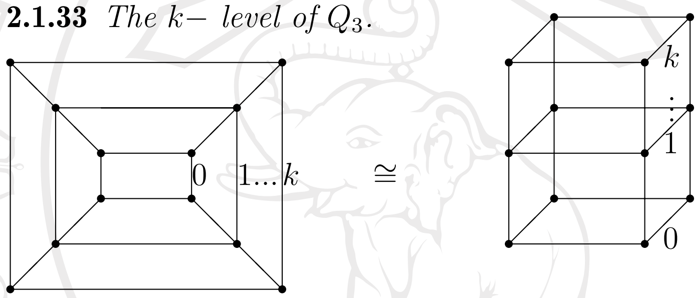


$\square$

Definition 2.1.32 *The graph G is called the $k-$ level of Q_3 is a graph where* $V(G) = \{(1,0,j),(0,0,j),(0,1,j),(1,1,j)|j = 0,1,...,k-1\}$ *and*
$E(G) = \{\{(0,0,j),(0,1,j)\},\{(0,1,j),(1,1,j)\},\{(1,1,j),(1,0,j)\},\{(1,0,j),(0,0,j)\}|j = 0,1,...,k-1\} \cup \{\{(0,0,j),(0,0,j+1)\},\{(0,1,j),(0,1,j+1)\},\{(1,1,j),(1,1,j+1)\},\{(1,0,j),(1,0,j+1)\}|j = 0,1,...,k-2\}$.

Note that the number of vertices of the $k-$ level of Q_3 is $4k$ and the number of edges is $8k-4$.

Example 2.1.33 *The $k-$ level of Q_3.*



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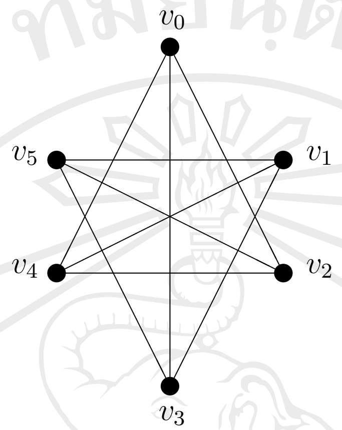
Definition 2.1.34 *Let $1 \leq a_1 < a_2 < ... < a_k \leq \lfloor n/2 \rfloor$, where n and $a_i (i = 1,2,...,k)$ are positive integers. A circulant graph $C_n(a_1,a_2,...,a_k)$ is a regular graph whose set of vertices is $V = \{v_0,v_1,...,v_{n-1}\}$ and whose set of edges is*

$$E = \{\{v_i, v_{i+a_j}\}(mod\ n) : i = 0,1,...,n-1, j = 1,2,...,k\}.$$

Note that if $a_k < \lfloor n/2 \rfloor$ then $C_n(a_1,a_2,...,a_k)$ is a $2k$-regular graph.

Example 2.1.35 *The graph* $C_6(2,3)$.



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2.2 Basic Concepts on Labelling.

Definition 2.2.1 *Let* $G=(V,E)$ *be a graph, where* $|V|=n$ *and* $|E|=e$ *and* λ *be a* $1-1$ *mapping from* $V\cup E$ *to* $\{1,2,...,n+e\}$. *For each* $x\in V$, *define the weight of the vertex* x *by* $w_\lambda(x)$ *where*

$$w_\lambda(x) := \lambda(x) + \sum_{y\in N(x)} \lambda(xy)$$

Then λ *is called vertex-magic total labeling of* G *if there exists* $h\in\mathbb{N}$ *such that* $w_\lambda(v_i)=h$ *for all* $v_i\in V$ *and* h *is called magic number for* λ.

Definition 2.2.2 *Let* $G=(V,E)$ *be a graph where* $|V|=n$ *and* $|E|=e$ *and* λ *a labeling of* G. *Then* λ *is called super vertex-magic total labeling of* G *if* $\lambda(V)=\{e+1,e+2,...,e+n\}$ *and* G *is called a super vertex-magic graph.*

Example 2.2.3 *A super vertex-magic graph with magic number* 9.