# CHAPTER 3

# MAIN RESULTS

In this chapter, we divide into 4 sections. In section 3.1, we study the necessary conditions to be super vertex-magic graphs. In section 3.2 we study the conditions of paths and cycles to be super vertex-magic [7]. In section 3.3 we study the conditions of circulant graphs. In section 3.4 we study product of path $P_2$ to be super vertex-magic . In section 3.5 we collect the non super vertex-magic graphs.

## 3.1 On the Degrees of a Super Vertex-Magic Graphs

In this section we study the upper bounds and lower bounds of degrees of vertices in any super vertex-magic graphs.

**Lemma 3.1.1** *Let $G = (V, E)$ be a super vertex-magic graph of $n$ vertices and $e$ edges and $\lambda$ a super vertex-magic total labeling of $G$ in which the weight of each vertex is $h$. Then $h = e + \frac{n+1}{2} + \frac{e(e+1)}{n}$.*

**Proof.** Let $V = \{v_1, v_2, ..., v_n\}$ which implies that

$$nh = \sum_{x \in V} w_\lambda(x) = S_v + 2S_e,$$

where $S_v$ is the sum of all vertex labels and $S_e$ is the sum of all edge labels.

Therefore

$$\begin{aligned} nh &= ((e+1) + (e+2) + ... + (e+n)) + 2(1 + 2 + ... + e) \\ &= ne + \frac{n(n+1)}{2} + \frac{2e(e+1)}{2}. \end{aligned}$$

Therefore

$$h = e + \frac{n+1}{2} + \frac{e(e+1)}{n}.$$

□

Note: We also can see some results in the section 3.2 in [8].

**Lemma 3.1.2** *Every super vertex-magic graph has no isolated edges.*

**Proof.** Let $G$ be a super vertex-magic graph and $\lambda$ a super vertex-magic total labeling of $G$.

We prove by contradiction.

Suppose $s, t$ be the endpoints of an isolated edge in $G$, then

$$w_\lambda(s) = \lambda(s) + \lambda(st) = \lambda(t) + \lambda(st) = w_\lambda(t).$$

This implies that $\lambda(s) = \lambda(t)$,

which is a contradiction.

Therefore, $G$ has no isolated edges. □

**Theorem 3.1.3** *Every super vertex-magic graph has no isolated points.*

**Proof.** Let $G = (V, E)$ be a super vertex-magic graph with $V = \{v_0, v_1, ..., v_{n-1}\}$ $e$ edges and $\lambda$ a super vertex-magic total labeling of $G$.

From the definition of the super vertex-magic total labeling, $G$ cannot have more than one isolated vertex.

Suppose $G$ has exactly one isolated vertex, say $v_0$, W.L.O.G. assume that $\lambda(v_0)$ be $e + n$, $\lambda(v(i)) = e + n - i$ for $i = 1, 2, ..., n - 1$.

$$w_\lambda(v_i) = \lambda(v_i) + \sum_{x \in N(v_i)} \lambda(v_i x) = e + n$$

$$e + n - i + \sum_{x \in N(v_i)} \lambda(v_i x) = e + n = \sum_{x \in N(v_i)} \lambda(v_i x) = i$$

$d(v_i) = 1$ and $\lambda(v_i x) = i$.

There exists an isolated edge, contradiction.

Hence $G$ has no isolated points. □

**Corollary 3.1.4** *Let $G$ be a super vertex-magic graph of $n$ vertices $v_1, v_2, ..., v_n$ and $e$ edges. Then $e \geq \frac{n+1}{2}$.*

**Proof.** By Theorem 3.1.3, the graph $G$ has no isolated points, $d(v_i) \geq 1$

we have $n \leq \sum_{i=1}^{n} d(v_i)$.

Assume that $\sum_{i=1}^{n} d(v_i) = n$.

By Theorem 3.1.3, we have $d(v_i) = 1$ for all $i = 1, 2, ..., n$, thus $G$ has isolated edges. which is a contradiction with Lemma 3.1.2.

Then $n + 1 \leq \sum_{i=1}^{n} d(v_i)$.

By the Handshaking Lemma we have $\sum_{i=1}^{n} d(v_i) = 2e$.

We have $2e \geq n + 1$, therefore $e \geq \frac{n+1}{2}$. □

**Corollary 3.1.5** *Let $G$ be a super vertex-magic graph of $n$ vertices and $e$ edges*

*(i) If $G$ is a regular graph of even degree, then $n$ is odd.*

*(ii) If $G$ is a regular graph of odd degree, then $n$ is even.*

**Proof.** (i) Let $G$ be a regular graph of degree $2m$ for some positive integer $m$.

By the Handshaking Lemma we have

$$2mn = \sum_{v \in V(G)} d(v) = 2e.$$

Hence $e = mn$.

By Lemma 3.1.1 $h = e + \frac{e(e+1)}{n} + \frac{n+1}{2}$, we have

$$\begin{aligned} h &= mn + \frac{mn(mn+1)}{n} + \frac{n+1}{2} \\ &= mn + m(mn+1) + \frac{n+1}{2} \\ &= \frac{2mn + 2m^2n + 2m + n + 1}{2}. \end{aligned}$$

Therefore, $n$ is odd because $h$ must be an integer.

(ii) Let $G$ be a regular graph of degree $2m+1$ for some positive integer $m$.

By the Handshaking Lemma, we have

$$(2m+1)n = \sum_{i=1}^{n} d(v_i) = 2e.$$

Hence $\quad e = \frac{(2m+1)n}{2}$.

Therefore, $n$ is even. □

**Theorem 3.1.6** *Let $G$ be a super vertex magic graph of $n$ vertices and $e$ edges. If $n < -(\frac{2e-1}{2}) + \sqrt{3e^2 + e + \frac{1}{4}}$ then the minimum degree of $G$ is at least 2.*

**Proof.** By Lemma 3.1.2 G has no isolated points, the minimum degree is at least 1.

We prove by contradiction.

Assume that there exists a vertex $v$ with degree 1. Then the magic number

$$h = w_\lambda(v) \leq (e+n) + e = n + 2e.$$

By Lemma 3.1.1 $h = e + \frac{e(e+1)}{n} + \frac{n+1}{2}$. We have

$$e + \frac{e(e+1)}{n} + \frac{n+1}{2} \leq n + 2e \qquad ...(1)$$

$$\begin{aligned}
(1) \times 2n; 2ne + 2e(e+1) + n(n+1) &\leq 2n^2 + 4ne \\
2ne + 2e^2 + 2e + n^2 + n &\leq 2n^2 + 4ne \\
n^2 + 2ne - n - 2e^2 - 2e &\geq 0 \\
n^2 + (2e-1)n - (2e^2 + 2e) &\geq 0 \\
\left[n + (\frac{2e-1}{2})\right]^2 - (\frac{2e-1}{2})^2 - 2e^2 - 2e &\geq 0 \\
\left[n + (\frac{2e-1}{2})\right]^2 &\geq (\frac{2e-1}{2})^2 + 2e^2 + 2e \\
\left[n + (\frac{2e-1}{2})\right]^2 &\geq e^2 - e + \frac{1}{4} + 2e^2 + 2e \\
\left[n + (\frac{2e-1}{2})\right]^2 &\geq 3e^2 + e + \frac{1}{4}.
\end{aligned}$$

Then

$$n + (\frac{2e-1}{2}) \geq \sqrt{3e^2 + e + \frac{1}{4}}$$

$$n \geq -(\frac{2e-1}{2}) + \sqrt{3e^2 + e + \frac{1}{4}}.$$

Which is a contradiction with the hypothesis $n < -(\frac{2e-1}{2}) + \sqrt{3e^2 + e + \frac{1}{4}}$.

or

$$\left[n + (\frac{2e-1}{2})\right] \leq -\sqrt{3e^2 + e + \frac{1}{4}}$$

$$n \leq -(\frac{2e-1}{2}) - \sqrt{3e^2 + e + \frac{1}{4}}.$$

Which is a contradiction because $n$ is a positive integer.

Hence the minimum degree is at least two. □

**Theorem 3.1.7** *Let $G$ be a super vertex magic graph of $n$ vertices and $e$ edges with magic number $h$. Then the maximum degree $\Delta \leq \frac{-1}{2} + \sqrt{2(h-e) - \frac{7}{4}}$.*

**Proof.** Let us consider a vertex $v$ with maximum degree $\Delta$.

We have the magic number

$$h = w_\lambda(v) \geq e + 1 + (1 + 2 + ... + \Delta)$$

$$= e + 1 + \frac{\Delta(\Delta+1)}{2}$$

Since $h = e + \frac{e(e+1)}{n} + \frac{n+1}{2}$,

$$e + 1 + \frac{\Delta(\Delta+1)}{2} \leq e + \frac{e(e+1)}{n} + \frac{n+1}{2}$$

$$\frac{\Delta(\Delta+1)}{2} \leq \frac{e(e+1)}{n} + \frac{n-1}{2}$$

$$\Delta(\Delta+1) \leq \frac{2e(e+1)}{n} + n - 1$$

$$\Delta^2 + \Delta \leq \frac{2e(e+1)}{n} + n - 1$$

$$\Delta^2 + \Delta + \frac{1}{4} \leq \frac{2e(e+1)}{n} + n - \frac{3}{4}$$

$$\left(\Delta + \frac{1}{2}\right)^2 \leq \frac{2e(e+1)}{n} + n - \frac{3}{4}$$

$$\Delta \leq \frac{-1}{2} + \sqrt{\frac{2e(e+1)}{n} + n - \frac{3}{4}}$$

Therefore, $\Delta \leq \frac{-1}{2} + \sqrt{2(h-e) - \frac{7}{4}}$. □

**Theorem 3.1.8** *Let $G$ be a super vertex magic graph with $n$ vertices, $e$ edges and magic number $h$. Then the degree $d$ of any vertices of $G$ satisfy the following inequalities*

$$e+\frac{1}{2}-\sqrt{(e+1)^2-2(h-e-n)}\leq d\leq\frac{-1}{2}+\sqrt{2(h-e)-\frac{7}{4}}.$$

**Proof.** Let us consider a super vertex magic labeling $\lambda$ of $G$ in which magic number $h$.

Let $v$ be a vertex of degree $d$.

Let $v_1, v_2, ..., v_{n-1}$ be the $n-1$ vertices other than $v$.

Let $e_1, e_2, ..., e_d$ be all edges which joint $v_0$ and other edges are $e'_1, e'_2, ..., e'_{e-d}$, we have

$$h=w_\lambda(v)=\lambda(v)+\sum_{i=1}^{d}\lambda(e_i) \qquad ...(2)$$

and

$$(n-1)h=\sum_{i=1}^{n-1}\lambda(v_i)+2\sum_{j=1}^{e-d}\lambda(e'_j)+\sum_{k=1}^{d}\lambda(e_k) \qquad ...(3)$$

From these two relations, we get

$$\begin{aligned}(n-1)(\lambda(v)+\sum_{i=1}^{d}\lambda(e_i)) &= \sum_{i=1}^{n-1}\lambda(v_i)+2\sum_{j=1}^{e-d}\lambda(e'_j)+\sum_{k=1}^{d}\lambda(e_k)\\ &= (1+2+...+(n+e))-\lambda(v_0)+\sum_{j=1}^{e-d}\lambda(e'_j)\end{aligned}$$

It follows that

$$n\lambda(v)+(n-1)\sum_{i=1}^{d}\lambda(e_i)-\sum_{j=1}^{e-d}\lambda(e'_j)=\frac{(n+e+1)(n+e)}{2} \qquad ...(4)$$

Let $\sum_{a}^{b} := (a+1)+(a+2)+...+b$

Among all the total labeling of $G$, the left-hand side of (4) is maximum when

$$\lambda(v)=n+e, \quad \sum_{i=1}^{d}\lambda(e_i)=\sum_{e-d}^{d}, \quad \sum_{j=1}^{e-d}\lambda(e'_j)=\sum_{0}^{e-d}$$

Substituting these values in (4) we get

$$n(n+e)+(n-1)\sum_{e-d}^{e}-\sum_{0}^{e-d}\geq\frac{(n+e+1)(n+e)}{2}$$

It follow that

$$n^2+ne+\Big[\frac{-nd^2+2ned+nd+d^2-2ed-d}{2}\Big]$$
$$-\Big[\frac{e^2-2ed+d^2+e-d}{2}\Big]\geq\frac{n^2+e^2+n+e+2ne}{2}$$
$$2n^2+2ne-nd^2+2ned+nd+d^2-2ed-d-e^2+2ed-d^2-e+d\geq n^2+e^2+n+e+2ne$$
$$-nd^2+2ned+nd\geq -n^2+2e^2+2e+n$$
$$-d^2+2ed+d\geq -n+\frac{2e^2}{n}+\frac{2e}{n}+1$$

Therefore

$$-d^2+2ed+d\geq 2(h-e-n)$$
$$d^2-2ed-d\leq -2(h-e-n)$$
$$d^2-2ed-d+e^2+e+\frac{1}{4}\leq e^2+e+\frac{1}{4}-2(h-e-n)$$
$$\Big(e-d+\frac{1}{2}\Big)^2\leq e^2+e+\frac{1}{4}-2(h-e-n)$$
$$e-d+\frac{1}{2}\leq\sqrt{e^2+e+\frac{1}{4}-2(h-e-n)}$$
$$d\geq e+\frac{1}{2}-\sqrt{e^2+e+\frac{1}{4}-2(h-e-n)}$$
$$d\geq e+\frac{1}{2}-\sqrt{\Big(e+\frac{1}{2}\Big)^2-2(h-e-n)}$$

By Theorem 3.1.7 the Theorem hold. □

## 3.2 Super Vertex-Magic of Paths and Cycles

In this section we study condition of paths and cycles to be a super vertex-magic graph.

**Theorem 3.2.1** ([7]) *A path $P_n$ is a super vertex-magic graph iff $n$ is odd.*

**Proof.** ($\Rightarrow$) Assume that path $P_n$ is a super vertex-magic graph with magic number $h$.

We have $e = n - 1$.

By Lemma 3.1.1, $h = e + \frac{n+1}{2} + \frac{e(e+1)}{n}$.

Then

$$\begin{aligned} h &= n - 1 + \frac{n+1}{2} + \frac{(n-1)n}{n} \\ &= n - 1 + \frac{n+1}{2} + n - 1 \\ &= 2n - 2 + \frac{n+1}{2} \\ &= \frac{5n-3}{2}. \end{aligned}$$

Hence $n$ is odd.

($\Leftarrow$) Assume that $n$ is odd.

Let $V = \{v_0, v_1, ..., v_{n-1}\}$ and define a labeling $\lambda$ by

$$\lambda(v_0) = 2n - 1$$
$$\lambda(v_i) = n + i - 1 \quad \text{for all} \quad i \neq 0$$

$$\lambda(v_i v_{i+1}) = \begin{cases} \frac{n-i-1}{2} & ; i = 0, 2, ..., n-3 \\ n - \frac{i+1}{2} & ; i = 1, 3, ..., n-2. \end{cases}$$

It is clear that $\lambda$ is well defined and 1-1 and $\lambda(v_i v_{i+1})$ is a positive integers for all $i = 0, 1, ..., n-1$.

To show that $\lambda$ is a super vertex-magic total labeling of $P_n$.

we will show that

(i) $\lambda(E(P_n)) = \{1, 2, ..., n-1\}$.

(ii) $\lambda(V(P_n)) = \{n, n-1, ..., 2n-1\}$.

(iii) $w_\lambda(v_i) = \frac{5n-3}{2}$ for all $i = 0, 1, 2, ..., n-1$.

Consider when $i = 1, 3, ..., n-2$,

$\lambda(v_i v_{i+1}) - 1 = n - \frac{i+1}{2} - 1 = \frac{2n-i-1}{2} \geq \frac{2n-(n-2)-1}{2} = \frac{n+1}{2} > 0$,

and when $i = 2, 4, ..., n-3$, $\lambda(v_i v_{i+1}) - 1 = \frac{n-i-1}{2} - 1 \geq \frac{n-(n-3)-1}{2} - 1 = 0$.

Therefore $\lambda(v_i v_{i+1}) \geq 1$.

Consider when $i = 1, 3, ..., n-2$,

$(n-1) - \lambda(v_i v_{i+1}) = n - 1 - (n - \frac{i+1}{2}) = \frac{i-1}{2} \leq \frac{(n-2)-1}{2} = \frac{n-3}{2} \geq 0$.

and when $i = 2, 4, ..., n-1$,

$(n-1) - \lambda(v_i v_{i+1}) = (n-1) - \frac{n-i-1}{2} = \frac{n-1+i}{2} \leq \frac{n-1+(n-1)}{2} = \frac{2n-2}{2} = n - 1 > 0$.

Therefore $\lambda(v_i v_{i+1}) \leq n - 1$.

Hence $\lambda(E(P_n)) \subseteq \{1, 2, ..., n-1\}$.

To show that $\{1, 2, ..., n-1\} \subseteq \lambda(E(P_n))$.

Let $i \in \{1, 2, ..., n-1\}$.

If $1 \leq i \leq \frac{n-1}{2}$, then $2 \leq 2i \leq n-1$.

Hence $2n - 3 \geq 2n - 2i - 1 \geq n$ and $2n - 2i - 1$ is odd.

$$\begin{aligned}\lambda(v_{2n-2i-1}v_{2n-2i}) &= n - \frac{2n-2i-1+1}{2} \\ &= \frac{2n-2n+2i}{2} \\ &= i.\end{aligned}$$

If $\frac{n+1}{2} \leq i \leq n-1$, then $n + 1 \leq 2i \leq 2n - 2$.

Hence $-2 \geq -2i + n - 1 \geq -n + 1$ and $-2i + n - 1$ is even.

$$\begin{aligned}\lambda(v_{-2i+n-1}v_{-2i+n}) &= \frac{n-(-2i+n-1)-1}{2} \\ &= i.\end{aligned}$$

Therefore $\{1, 2, ..., n-1\} \subseteq \lambda(E(P_n))$.

Altogether, we get $\lambda(E(P_n)) = \{1, 2, ..., n-1\}$.

(ii) From the defining of $\lambda(v_i)$,

$\lambda(v_{r+1}) = n + (r+1) - 1 = n + r$ for all $r = 0, 1, ..., n-2$. and $\lambda(v_0) = 2n - 1$.

Therefore the value $\lambda(v_1), \lambda(v_2), ..., \lambda(v_{n-1})$ are $n, n+1, n+2, ..., 2n-2$, respectively.

Hence $\lambda(V(P_n)) = \{n, n+1, n+2, ..., 2n-1\}$.

(iii) we will show that $w_\lambda(v_i) = \frac{5n-3}{2}$ for all $i = 0, 1, 2..., n-1$.

If $i = 0$, we have $\lambda(v_0) = 2n - 1$, $\lambda(v_0 v_1) = \frac{n-1}{2}$.

It follows that

$$\begin{aligned} w_\lambda(v_0) &= 2n - 1 + \frac{n-1}{2} \\ &= \frac{4n - 2 + n - 1}{2} \\ &= \frac{5n-3}{2}. \end{aligned}$$

If $i = n-1$, we have $\lambda(v_{n-1}) = 2n - 2$, $\lambda(v_{n-2}v_{n-1}) = n - \frac{n-1}{2}$.

It follows that

$$\begin{aligned} w_\lambda(v_{n-1}) &= 2n - 2 + n - \frac{n-1}{2} \\ &= 3n - 2 - \frac{n-1}{2} \\ &= \frac{6n - 4 - n + 1}{2} \\ &= \frac{5n-3}{2}. \end{aligned}$$

If $i \neq 0, n-1$, and $i$ is ood.

We have $\lambda(v_i) = n + i - 1$, $\lambda(v_i v_{i+1}) = n - \frac{i+1}{2}$, $\lambda(v_{i-1} v_i) = \frac{n-i}{2}$.

It follows that

$$\begin{aligned} w_\lambda(v_i) &= n + i - 1 + n - \frac{i+1}{2} + \frac{n-i}{2} \\ &= 2n + i - 1 + \frac{n - 2i - 1}{2} \\ &= \frac{4n + 2i - 2 + n - 2i - 1}{2} \\ &= \frac{5n-3}{2}. \end{aligned}$$

If $i \neq 0, n-1$, and $i$ is even.

We have $\lambda(v_i) = n + i - 1$, $\lambda(v_i v_{i+1}) = \frac{n-i-1}{2}$, $\lambda(v_{i-1} v_i) = n - \frac{i}{2}$.

It follows that

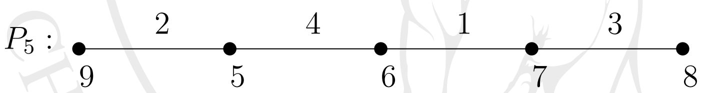
$$\begin{aligned} w_\lambda(v_1) &= n + i - 1 + \frac{n-i-1}{2} + n - \frac{i}{2} \\ &= 2n + i - 1 + \frac{n-2i-1}{2} \\ &= \frac{4n + 2i - 2 + n - 2i - 1}{2} \\ &= \frac{5n-3}{2}. \end{aligned}$$

Hence $w_\lambda(v_i) = \frac{5n-3}{2}$ for all $i = 0, 1, 2, ..., n-1$.

It implies that $\lambda$ is a super vertex-magic total labeling of $P_n$.

Hence $P_n$ is a super vertex-magic graph with magic number $\frac{5n-3}{2}$. □

**Example 3.2.2** *A path $P_5$ is a super vertex-magic graph with magic number* 11.




**Theorem 3.2.3** ([7]) *The Cycle $C_n$ is a super vertex-magic graph iff $n$ is odd.*

**Proof.** ($\Rightarrow$) Assume that $C_n$ is a super vertex-magic graph.

We have $e = n$.

By Lemma 3.1.1 $h = e + \frac{n+1}{2} + \frac{e(e+1)}{n}$,

Then

$$\begin{aligned} h &= n + \frac{n+1}{2} + \frac{n(n+1)}{n} \\ &= n + \frac{n+1}{2} + n + 1 \\ &= 2n + 1 + \frac{n+1}{2} \\ &= \frac{4n + 2 + n + 1}{2} \\ &= \frac{5n+3}{2}. \end{aligned}$$

Hence $n$ is odd.

($\Leftarrow$) Assume that $n$ is odd.

Let $V = \{v_0, v_1, ..., v_{n-1}\}$ and defind a labeling $\lambda$ by

$$\lambda(v_i) = n + 1 + i \quad ; i = 0, 1, ..., n-1$$

$$\lambda(v_{n-1}v_0) = \frac{n+1}{2}$$

$$\lambda(v_i v_{i+1}) = \begin{cases} n - \frac{i}{2} & ; i = 0, 2, ..., n-3 \\ \frac{n-i}{2} & ; i = 1, 3, ..., n-2 \end{cases}$$

It is clear that $\lambda$ is well defined and 1-1 and $\lambda(v_i v_{i+1})$ is a positive integers for all $i = 0, 1, ..., n-1$.

To show that $\lambda$ is a super vertex-magic total labeling of $C_n$.

we will show that

(i) $\lambda(E(C_n)) = \{1, 2, ..., n\}$

(ii) $\lambda(V(C_n)) = \{n+1, n+2, ..., 2n\}$

(iii) $w_\lambda(v_i) = \frac{5n+3}{2}$ for all $i = 0, 1, 2, ..., n-1$.

(i) From the defining of $\lambda$, we have

$\lambda(\{v_0v_1, v_2v_3, ..., v_{n-3}v_{n-2}\}) = \{n, n-1, ..., \frac{n+3}{2}\}$.

and $\lambda(\{v_1v_2, v_3v_4, ..., v_{n-1}v_0\}) = \{1, 2, ..., \frac{n-1}{2}\}$ and $\lambda(v_{n-1}v_0) = \frac{n+1}{2}$ .

Therefore $\lambda(E(C_n)) = \{1, 2, ..., n\}$.

(ii) From the defining of $\lambda$, we have

$\lambda(\{v_0, v_1, ..., v_{n-1}\}) = \{n+1, n+2, ..., 2n\}$.

Therefore $\lambda(V(C_n)) = \{n+1, n+2, ..., 2n\}$.

(iii) we will show that $w_\lambda(v_i) = \frac{5n+3}{2}$ for all $i = 0, 1, 2..., n-1$.

If $i = n-1$, we have $\lambda(v_{n-1}) = 2n$, $\lambda(v_{n-2}v_{n-1}) = 1$, $\lambda(v_{n-1}v_0) = \frac{n+1}{2}$.

$$\begin{aligned} w_\lambda(v_n) &= 2n + 1 + \frac{n+1}{2} \\ &= \frac{4n+2+n+1}{2} \\ &= \frac{5n+3}{2}. \end{aligned}$$

If $i = 0$, we have $\lambda(v_0) = n+1$, $\lambda(v_0v_1) = n$, $\lambda(v_{n-1}v_0) = \frac{n+1}{2}$.

$$\begin{aligned} w_\lambda(v_n) &= n+1+n+\frac{n+1}{2} \\ &= 2n+1+\frac{n+1}{2} \\ &= \frac{4n+2+n+1}{2} \\ &= \frac{5n+3}{2}. \end{aligned}$$

If $i$ is odd, we have $\lambda(v_i) = n+1+i$, $\lambda(v_iv_{i+1}) = \frac{n-i}{2}$, $\lambda(v_{i-1}v_i) = n - \frac{i-1}{2}$.

It follows that

$$\begin{aligned} w_\lambda(v_i) &= n+1+i-\frac{n-i}{2}+n-\frac{i-1}{2} \\ &= 2n+1+i+\frac{n-2i+1}{2} \\ &= \frac{4n+2+2i+n-2i+1}{2} \\ &= \frac{5n+3}{2}. \end{aligned}$$

If $i$ is even, we have $\lambda(v_i) = n+1+i$, $\lambda(v_iv_{i+1}) = n - \frac{i}{2}$, $\lambda(v_{i-1}v_i) = \frac{n-i+1}{2}$.

It follows that

$$\begin{aligned} w_\lambda(v_i) &= n+1+i+n-\frac{i}{2}+\frac{n-i+1}{2} \\ &= 2n+1+i+\frac{n-2i+1}{2} \\ &= \frac{4n+2+2i+n-2i+1}{2} \\ &= \frac{5n+3}{2}. \end{aligned}$$

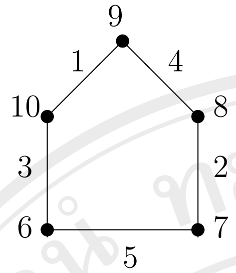
Hence $w_\lambda(v_i) = \frac{5n+3}{2}$ for all $i = 0, 1, 2, ..., n-1$.

It implies that $\lambda$ is a super vertex-magic total labeling of $C_n$.

Hence $C_n$ is a super vertex-magic graph with magic number $\frac{5n+3}{2}$. □

**Example 3.2.4** *The cycle $C_5$ is a super vertex-magic graph with magic number* 14.

$C_5$ :




## 3.3 Super Vertex-Magic of Circulant Graphs

The circulant graphs are important class of graphs, which can be used in the design of local area networks [2]. From [7], Balbuena, Barker, Das, Lin, Miller, Ryan, Slam, Sugeng and Tkac characterized the super vertex-magic graphs of the forms $C_n(1, m)$ and $C_n(1, 2, 3)$. In this section, we generalize those results into the circulant graphs of the forms $C_n(1, 2, s)$ and the disjoint union of k copies of $C_n(1, m)$ or $C_n(1, 2, s)$.

**Theorem 3.3.1** *For $n \geq 2m + 1$, The circulant graph $C_n(1, m)$ is a super vertex-magic graph with the magic number $h = \frac{13n+5}{2}$ iff $n$ is odd.*

**Proof.** ($\Rightarrow$) Let m be a positive integer greater than or equal to 2.

Assume that $C_n(1, m)$ is a super vertex-magic graph.

Since $e = 2n$, by Lemma 3.1.1,

$$\begin{aligned} h &= e + \frac{e(e+1)}{n} + \frac{n+1}{2} \\ &= 2n + \frac{2n(2n+1)}{n} + \frac{n+1}{2} \\ &= 2n + 4n + 2 + \frac{n+1}{2} \\ &= 6n + 2 + \frac{n+1}{2} \\ &= \frac{13n+5}{2}. \end{aligned}$$

Then $n$ is odd.

($\Leftarrow$) Assume that $n$ is odd.

Let $V = \{v_0, v_1, ..., v_{n-1}\}$ and define a labelling $\lambda$ by

$$\lambda(v_i) = \begin{cases} 3n - m + 1 + i, & i = 0, 1, ..., m-1 \\ 2n - m + 1 + i, & i = m, m+1, ..., n-1 \end{cases} \quad ...(5)$$

$$\lambda(v_i v_{i+1}) = \begin{cases} \frac{2n+2+i}{2}, & i = 0, 2, 4, ..., n-1 \\ \frac{3n+2+i}{2}, & i = 1, 3, 5, ..., n-2 \end{cases} \quad ...(6)$$

$$\lambda(v_i v_{i+m}) = n - i, \quad i = 0, 1, 2, ..., n-1.$$

It is clear that $\lambda$ is well defined and 1-1 and $\lambda(v_i v_{i+1})$ is a positive integers for all $i = 0, 1, ..., n-1$.

To show that $\lambda$ is a super vertex-magic total labelling of the graph $C_n(1, m)$. Since $e = 2n$, we will show that

(i) $\lambda(E(C_n(1, m))) = \{1, 2, ..., 2n\}$.

(ii) $\lambda(V(C_n(1, m))) = \{2n+1, 2n+2, ..., 3n\}$

(iii) $w_\lambda(v_i) = \frac{13n+5}{2}$ for all $i = 0, 1, ..., n-1$.

(i) To show that $\lambda(E(C_n(1, m))) = \{1, 2, ..., 2n\}$.

From (6), $\lambda(v_i v_{i+m}) = n - i, i = 0, 1, ..., n-1$, then $1 \leq \lambda(v_i v_{i+m}) \leq n$.
$\lambda(v_i v_{i+1}) = \frac{2n+2+i}{2}$, $i = 0, 2, ..., n-1$, then $n + 1 \leq \lambda(v_i v_{i+1}) \leq \frac{3n+1}{2}$. and
$\lambda(v_i v_{i+1}) = \frac{3n+2+i}{2}$, $i = 1, 3, ..., n-2$, then $\frac{3n+3}{2} \leq \lambda(v_i v_{i+1}) \leq 2n$.

It is also easy to see that $\lambda$ is a $1-1$ mapping.

Therefore $\lambda(E(C_n(1, m))) = \{1, 2, ..., 2n\}$.

(ii) To show that $\lambda(V(C_n(1, m))) = \{2n+1, 2n+2, ..., 3n\}$.

It is easy to see from (5) that

$\lambda(\{v_0, v_1, ..., v_{m-1}\}) = \{3n-m+1, 3n-m+2, ..., 3n\}$ and

$\lambda(\{v_m, v_{m+1}, ..., v_{n-1}\}) = \{2n+1, 2n+2, ..., 3n-m\}$.

Therefore $\lambda(V(C_n(1, m))) = \{2n+1, 2n+2, ..., 3n\}$.

(iii) we will show that $w_\lambda(v_i) = \frac{13n+5}{2}$ for all $i = 0, 1, ..., n-1$.

If $0 \leq i \leq m-1$, and $i$ is odd.

We have $\lambda(v_i) = 3n - m + 1 + i$, $\lambda(v_iv_{i+1}) = \frac{3n+2+i}{2}$, $\lambda(v_{i-1}v_i) = \frac{2n+1+i}{2}$,
$\lambda(v_iv_{i+m}) = n - i$, $\lambda(v_{i-m}v_i) = n - (n + i - m) = m - i$.

It follows that

$$\begin{aligned} w_\lambda(v_i) &= 3n - m + 1 + i + \frac{3n+2+i}{2} + \frac{2n+1+i}{2} + n - i + m - i \\ &= 4n + 1 - i + \frac{5n+3+2i}{2} \\ &= \frac{8n+2-2i+5n+3+2i}{2} \\ &= \frac{13n+5}{2}. \end{aligned}$$

If $0 \leq i \leq m-1$, and $i$ is even.

We have $\lambda(v_i) = 3n - m + 1 + i$, $\lambda(v_iv_{i+1}) = \frac{2n+2+i}{2}$, $\lambda(v_{i-1}v_i) = \frac{3n+1+i}{2}$,
$\lambda(v_iv_{i+m}) = n - i$, $\lambda(v_{i-m}v_i) = m - i$.

It follows that

$$\begin{aligned} w_\lambda(v_i) &= 3n - m + 1 + i + \frac{2n+2+i}{2} + \frac{3n+1+i}{2} + n - i + m - i \\ &= 4n + 1 - i + \frac{5n+3+2i}{2} \\ &= \frac{8n+2-2i+5n+3+2i}{2} \\ &= \frac{13n+5}{2}. \end{aligned}$$

If $m \leq i \leq n-1$, and $i$ is odd.

We have $\lambda(v_i) = 2n - m + 1 + i$, $\lambda(v_iv_{i+1}) = \frac{3n+2+i}{2}$, $\lambda(v_{i-1}v_i) = \frac{2n+1+i}{2}$,
$\lambda(v_iv_{i+m}) = n - i$, $\lambda(v_{i-m}v_i) = n - i + m$.

It follows that

$$\begin{aligned} w_\lambda(v_i) &= 2n - m + 1 + i + \frac{3n+2+i}{2} + \frac{2n+1+i}{2} + n - i + n - i + m \\ &= 4n + 1 - i + \frac{5n+3+2i}{2} \\ &= \frac{8n+2-2i+5n+3+2i}{2} \\ &= \frac{13n+5}{2}. \end{aligned}$$

If $m \leq i \leq n-1$, and $i$ is even.

We have $\lambda(v_i) = 2n - m + 1 + i$, $\lambda(v_iv_{i+1}) = \frac{2n+2+i}{2}$, $\lambda(v_{i-1}v_i) = \frac{3n+1+i}{2}$,

$\lambda(v_iv_{i+m}) = n - i$, $\lambda(v_{i-m}v_i) = n - i + m$.

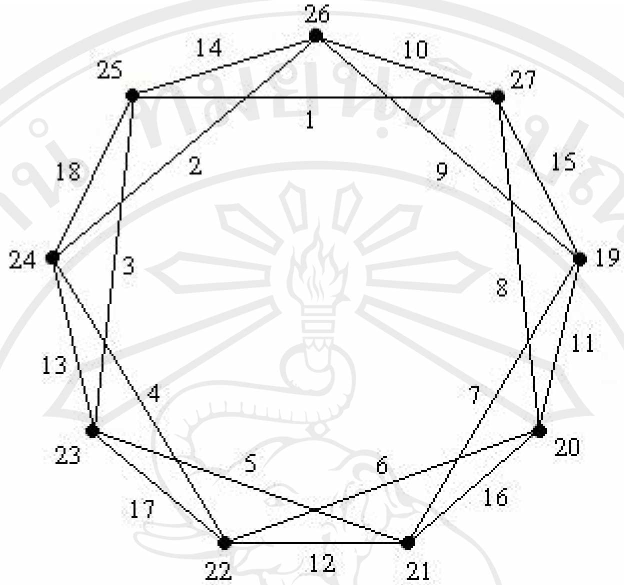
It follows that

$$\begin{aligned} w_\lambda(v_i) &= 2n - m + 1 + i + \frac{2n+2+i}{2} + \frac{3n+1+i}{2} + n - i + n - i + m \\ &= 4n + 1 - i + \frac{5n+3+2i}{2} \\ &= \frac{8n+2-2i+5n+3+2i}{2} \\ &= \frac{13n+5}{2}. \end{aligned}$$

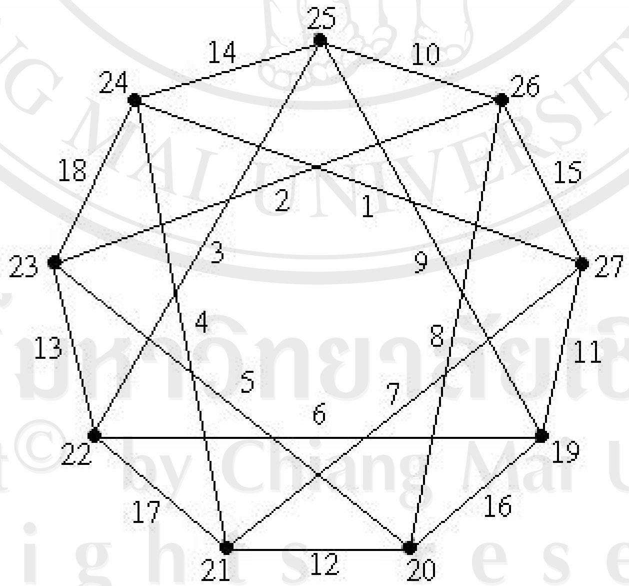
Hence $w_\lambda(v_i) = \frac{13n+5}{2}$ for all $i = 0, 1, ..., n-1$.

It implies that $\lambda$ is a super vertex-magic total labeling of $C_n(1, m)$. Hence $C_n(1, m)$ is a super vertex-magic graph with magic number $\frac{13n+5}{2}$. □

Example. The super vertex-magic graph C9(1,2) with the magic number 61.




Example. The super vertex-magic graph C9(1,3) with the magic number 61.



**Theorem 2.0.1** ([3]) *Let $k$ be a positive integer. If the graph $G$ is $r-$ regular graph that admits a super vertex-magic total labeling and $(k-1)(r+1)/2$ is an integer, then the graph $kG$ has a super vertex-magic total labeling.*

**Theorem 2.0.2** *For $n \geq 5$ and $n$ is odd and $m < \lfloor \frac{n}{2} \rfloor$. The graph $C_n(1, m)$ is a super vertex-magic graph iff $k$ is odd.*

**Proof.** ($\Rightarrow$) Assume that $kC_n(1, m)$ is a super vertex-magic graph.
We have $e = 2nk$.

By Lemma 3.1.1,

$$\begin{aligned} h &= e + \frac{e(e+1)}{n} + \frac{n+1}{2} \\ &= 2nk + \frac{2nk(2nk+1)}{nk} + \frac{nk+1}{2} \\ &= 2nk + 4nk + 2 + \frac{nk+1}{2}. \end{aligned}$$

Hence $k$ is odd.

($\Leftarrow$) Assume that $k$ is odd.
By Theorem 3.3.1 $C_n(1, m)$ is super when $n$ is $odd(n \geq 5)$ and $C_n(1, m)$ is 4-regular, $\frac{(k-1)(r+1)}{2} = \frac{(k-1)(4+1)}{2} = \frac{5(k-1)}{2}$ is an integer.
By Theorem 3.3.2, $kC_n(1, m)$ is a super vertex-magic graph. □

**Theorem 2.0.3** *For $n \geq 2s + 1$. A circulant graph $C_n(1, 2, s)$ is a super vertex-magic graph with the magic number $h = \frac{25n+7}{2}$ iff $n$ is odd.*

**Proof.** ($\Rightarrow$) Let $s$ be a positive integer greater than or equal to 2.
Assume that $C_n(1, 2, s)$ is a super vertex-magic graph.

We have $e = 3n$.

By Lemma 3.1.1,

$$\begin{aligned} h &= e + \frac{e(e+1)}{n} + \frac{n+1}{2} \\ &= 3n + \frac{3n(3n+1)}{n} + \frac{n+1}{2} \\ &= 3n + 9n + 3 + \frac{n+1}{2} \\ &= 12n + 3 + \frac{n+1}{2} \\ &= \frac{25n+7}{2}. \end{aligned}$$

Then $n$ is odd.

($\Leftarrow$) Assume that $n$ is odd.

Let $V = \{v_0, v_1, ..., v_{n-1}\}$ and defined a labeling $\lambda$ by

$$\lambda(v_i) = \begin{cases} 4n - s + 1 + i, & i = 0, 1, ..., s-1 \\ 3n - s + 1 + i, & i = s, s+1, ..., n-1 \end{cases} \quad ...(7)$$

$$\lambda(v_iv_{i+1}) = \begin{cases} 2n+1, & i = 0 \\ \frac{5n+2-i}{2}, & i = 1, 3, 5, ..., n-2 \\ \frac{6n+2-i}{2}, & i = 2, 4, 6, ..., n-1 \end{cases} \quad ...(8)$$

$$\lambda(v_iv_{i+2}) = n + 1 + i, \quad i = 0, 1, 2, ..., n-1.$$

$$\lambda(v_iv_{i+s}) = n - i, \quad i = 0, 1, 2, ..., n-1.$$

It is clear that $\lambda$ is well defined and 1-1 and $\lambda(v_iv_{i+1})$ is a positive integers for all $i = 0, 1, ..., n-1$.

To show that $\lambda$ is a super vertex-magic total labeling of graph $C_n(1, 2, s)$.

Since $e = 3n$, we will show that

(i) $\lambda(E(C_n(1, 2, s))) = \{1, 2, ..., 3n\}$

(ii) $\lambda(V(C_n(1, 2, s))) = \{3n+1, 3n+2, ..., 4n\}$

(iii) $w_\lambda(v_i) = \frac{25n+7}{2}$ for all $i = 0, 1, ..., n-1$.

(i) To show that $\lambda(E(C_n(1, 2, s))) = \{1, 2, ..., 3n\}$.

From (8), $\lambda(v_0v_1) = 2n + 1$,

$\lambda(v_iv_{i+1}) = \frac{5n+2-i}{2}$, $i = 1, 3, 5, ..., n-2$, then $2n \leq \lambda(v_iv_{i+1}) \leq \frac{5n+1}{2}$,

$\lambda(v_iv_{i+1}) = \frac{6n+2-i}{2}$, $i = 2, 4, 6, ..., n-1$, then $\frac{5n+3}{2} \leq \lambda(v_iv_{i+1}) \leq 3n$,

$\lambda(v_iv_{i+2}) = n + 1 + i$, $i = 0, 1, 2, ..., n-1$, then $n + 1 \leq \lambda(v_iv_{i+2}) \leq 2n$,

$\lambda(v_iv_{i+s}) = n - i$, $i = 0, 1, 2, ..., n-1$, then $1 \leq \lambda(v_iv_{i+r}) \leq n$,

It is also easy to see that $\lambda$ is a $1-1$ mapping.

Therefore $\lambda(E(C_n(1, 2, s))) = \{1, 2, ..., 3n\}$.

(ii) To show that $\lambda(V(C_n(1, 2, s))) = \{3n+1, 3n+2, ..., 4n\}$.

It is easy to see from (7) that

$\lambda(\{v_0, v_1, ..., v_{s-1}\}) = \{4n-s+1, 4n-s+2, ..., 4n\}$ and

$\lambda(\{v_s, v_{s+1}, ..., v_{n-1}\}) = \{3n+1, 3n+2, ..., 4n-s\}$.

Then $\lambda(V(C_n(1, 2, s))) = \{3n+1, 3n+2, ..., 4n\}$.

(iii) we will show that $w_\lambda(v_i) = \frac{25n+7}{2}$ for all $i = 0, 1, ..., n-1$.

If $0 \le i \le s-1$, and $i$ is even.

We have $\lambda(v_i) = 4n - s + 1 + i$, $\lambda(v_iv_{i+1}) = \frac{6n+2-i}{2}$,$\lambda(v_{i-1}v_i) = \frac{5n+3-i}{2}$,

$\lambda(v_iv_{i+2}) = n + 1 + i$, $\lambda(v_{i-2}v_i) = n - 1 + i$,

$\lambda(v_iv_{i+s}) = n - i$, $\lambda(v_{i-s}v_i) = n - (n + i - s) = s - i$.

It follows that

$$\begin{aligned} w_\lambda(v_i) &= 4n - s + 1 + i + \frac{6n+2-i}{2} + \frac{5n+3-i}{2} + n + 1 + i + n - 1 + i + n - i + s - i \\ &= 7n + 1 + i + \frac{11n+5-2i}{2} \\ &= \frac{14n + 2 + 2i + 11n + 5 - 2i}{2} \\ &= \frac{25n+7}{2}. \end{aligned}$$

If $s \le i \le n-1$, and $i$ is odd.

We have $\lambda(v_i) = 3n - s + 1 + i$, $\lambda(v_iv_{i+1}) = \frac{5n+2-i}{2}$, $\lambda(v_{i-1}v_i) = \frac{6n+3-i}{2}$,

$\lambda(v_iv_{i+2}) = n + 1 + i$, $\lambda(v_{i-2}v_i) = n - 1 + i$,

$\lambda(v_iv_{i+s}) = n - i$, $\lambda(v_{i-s}v_i) = n - i + s$.

It follows that

$$\begin{aligned} w_\lambda(v_i) &= 3n - s + 1 + i + \frac{5n+2-i}{2} + \frac{6n+3-i}{2} + n + 1 + i + n - 1 + i + n - i + n - i + s \\ &= 7n + 1 + i + \frac{11n+5-2i}{2} \\ &= \frac{14n + 2 + 2i + 11n + 5 - 2i}{2} \\ &= \frac{25n+7}{2}. \end{aligned}$$

If $s \le i \le n-1$, and $i$ is even.

We have $\lambda(v_i) = 3n - s + 1 + i$, $\lambda(v_iv_{i+1}) = \frac{6n+2-i}{2}$, $\lambda(v_{i-1}v_i) = \frac{5n+3-i}{2}$,

$\lambda(v_iv_{i+2}) = n + 1 + i$, $\lambda(v_{i-2}v_i) = n - 1 + i$,

$\lambda(v_iv_{i+s}) = n - i$, $\lambda(v_{i-s}v_i) = n - i + s$.

It follows that

$$\begin{aligned} w_\lambda(v_i) &= 3n - s + 1 + i + \frac{6n+2-i}{2} + \frac{5n+3-i}{2} + n + 1 + i + n - 1 + i + n - i + n - i + s \\ &= 7n + 1 + i + \frac{11n+5-2i}{2} \\ &= \frac{14n+2+2i+11n+5-2i}{2} \\ &= \frac{25n+7}{2}. \end{aligned}$$

If $0 \leq i \leq s-1$, and $i$ is odd.

We have $\lambda(v_i) = 4n - s + 1 + i$, $\lambda(v_iv_{i+1}) = \frac{5n+2-i}{2}$, $\lambda(v_{i-1}v_i) = \frac{6n+3-i}{2}$,
$\lambda(v_iv_{i+2}) = n + 1 + i$, $\lambda(v_{i-2}v_i) = n - 1 + i$,
$\lambda(v_iv_{i+s}) = n - i$, $\lambda(v_{i-s}v_i) = n - (n + i - s) = s - i$.

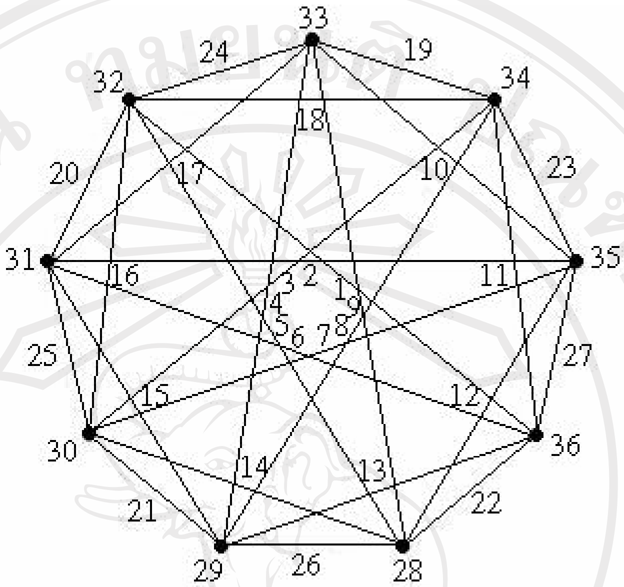
It follows that

$$\begin{aligned} w_\lambda(v_i) &= 4n - s + 1 + i + \frac{5n+2-i}{2} + \frac{6n+3-i}{2} + n + 1 + i + n - 1 + i + n - i + s - i \\ &= 7n + 1 + i + \frac{11n+5-2i}{2} \\ &= \frac{14n+2+2i+11n+5-2i}{2} \\ &= \frac{25n+7}{2}. \end{aligned}$$

Hence $w_\lambda(v_i) = \frac{25n+7}{2}$ for all $i = 0, 1, ..., n-1$.

It implies that $\lambda$ is a super vertex-magic total labeling of $C_n(1, 2, s)$.
Hence $C_n(1, 2, s)$ is a super vertex-magic graph with magic number $\frac{25n+7}{2}$. □

Example. The super vertex-magic graph C9(1,2,4).



**Theorem 2.0.1** *For $n \geq 7$ and $n$ is odd. The graph $kC_n(1,2,s)$ is a super vertex-magic graph iff $k$ is odd.*

**Proof.** ($\Rightarrow$) Assume that $kC_n(1,2,s)$ is a super vertex-magic graph.

We have $e = 3nk$.

By Lemma 3.1.1,

$$\begin{aligned} h &= e + \frac{e(e+1)}{n} + \frac{n+1}{2} \\ &= 3nk + \frac{3nk(3nk+1)}{nk} + \frac{nk+1}{2} \\ &= 3nk + 9nk + 3 + \frac{nk+1}{2} \\ &= \frac{24nk + 6 + nk + 1}{2} \\ &= \frac{25nk+7}{2}. \end{aligned}$$

Hence $k$ is odd.

($\Leftarrow$) Assume that $k$ is odd.

By Theorem 3.3.4 and $C_n(1,2,s)$ is 6-regular and super vertex-magic with $n \geq 7$.

$\frac{(k-1)(r+1)}{2} = \frac{(k-1)(6+1)}{2} = \frac{7(k-1)}{2}$ is an integer.

By Theorem 3.3.2, $kC_n(1,2,s)$ is a super vertex-magic graph. □

## 2.1 Product of Paths $P_2$

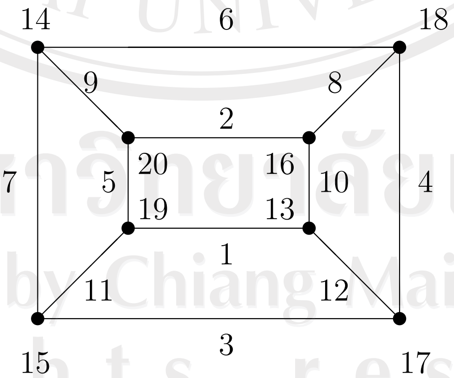
**Theorem 2.1.1** *The graph $k(P_2 \square P_2 \square P_2)$ is a super vertex-magic graph iff $k$ is a positive integer.*

**Proof.** $(\Rightarrow)$ Assume that $k(P_2 \square P_2 \square P_2)$ is a super vertex-magic graph.
We have $n = 8k$ and $e = 12k$.

By Lemma 3.1.1,

$$
\begin{aligned}
h &= e + \frac{e(e+1)}{n} + \frac{n+1}{2} \\
&= 12k + \frac{12k(12k+1)}{8k} + \frac{8k+1}{2} \\
&= 12k + \frac{3(12k+1)}{2} + \frac{8k+1}{2} \\
&= 12k + \frac{36k+3+8k+1}{2} \\
&= 12k + \frac{44k+4}{2} \\
&= 12k + 22k + 2 \\
&= 34k + 2.
\end{aligned}
$$

Therefore, $k$ is a positive integer.
$(\Leftarrow)$ assume that $k$ is a positive integer.




The above labeling on $P_2 \square P_2 \square P_2$ shows that the graph $P_2 \square P_2 \square P_2$ is super vertex-magic, 3- regular and $\frac{(k-1)(r+1)}{2} = \frac{(k-1)(3+1)}{2} = 2(k-1) = 2k-2$ is an integer for all positive integer $k$. By Theorem 3.3.2, $k(P_2 \square P_2 \square P_2)$ is a super vertex-magic graph. $\square$

## 2.2 Non Super Vertex-Magic Graphs

**Theorem 2.2.1** *Let $G$ be an $(r, s)$- semiregular bipartite graph of $n \geq 4$ vertices and $e$ edges , where $r > s$ and $e + 1 > n$ . Then $G$ is not a super vertex-magic graph.*

**Proof.** Let $k$ be the number of vertices of degree $r$ and $l$ the number of vertices of degree $s$. we have,

$$n = k + l, \qquad e = kr = ls.$$

Therefore,

$$\begin{aligned} kr &= (n-k)s \\ kr &= ns - ks \\ kr + ks &= ns \\ k(r+s) &= ns \\ k &= \frac{ns}{r+s} \end{aligned}$$

and

$$\begin{aligned} ls &= (n-l)r \\ ls &= nr - lr \\ lr + ls &= nr \\ l(r+s) &= nr \\ l &= \frac{nr}{r+s}. \end{aligned}$$

We prove by contradiction.

Let us assume that $G$ is a super vertex-magic graph.

Since $r > s$, which implies that $l > k$. The sum of the weights of the first $k$ vertices is at least the sum,

$$\begin{aligned} kh \geq \sum_{i=1}^{k}(e+i) + \sum_{i=1}^{e}(i) &= (e+1) + ... + (e+k) + 1 + 2 + ... + e \\ &= ke + \frac{k(k+1)}{2} + \frac{e(e+1)}{2} \qquad ...(9) \end{aligned}$$

On the an other part, the sum of the weights another part of $l$ vertices is at most,

$$lh \leq \sum_{i=k+1}^{n}(e+i) + \sum_{i=1}^{e} i = (e+k+1)+...+(e+n)+1+2+...+e$$
$$= \frac{(n+e)(n+e+1)}{2} - \frac{(e+k)(e+k+1)}{2} + \frac{e(e+1)}{2} \quad ...(10)$$

By (9) and (10) we have,

$$h \geq e + \frac{k+1}{2} + \frac{e(e+1)}{2k}$$
$$h \leq \frac{(n+e)(n+e+1)}{2l} - \frac{(e+k)(e+k+1)}{2l} + \frac{e(e+1)}{2l}.$$

It implies that,

$$e + \frac{k+1}{2} + \frac{e(e+1)}{2k} \leq \frac{(n+e)(n+e+1)}{2l} - \frac{(e+k)(e+k+1)}{2l} + \frac{e(e+1)}{2l}.$$

Hence

$$e + \frac{k+1}{2} + \frac{e(e+1)}{2k} - \frac{(n+e)(n+e+1)}{2l} + \frac{(e+k)(e+k+1)}{2l} - \frac{e(e+1)}{2l} \leq 0$$
$$e + \frac{e(e+1)}{2k} - \frac{e(e+1)}{2l} + \frac{k+1}{2} + \frac{(e+k)(e+k+1)-(n+e)(n+e+1)}{2l} \leq 0$$
$$e + e(e+1)\Big[\frac{1}{2k} - \frac{1}{2l}\Big] + \frac{k+1}{2} + \frac{(e+k)(e+k+1)-(n+e)(n+e+1)}{2l} \leq 0$$
$$e(e+1)\Big[\frac{1}{k} - \frac{1}{l}\Big] + 2e + k + 1 + \frac{(e+k)(e+k+1)-(n+e)(n+e+1)}{l} \leq 0. \quad ...(11)$$

Since

$(e+k)(e+k+1) = e^2 + 2ke + k^2 + e + k$

$(n+e)(n+e+1) = n^2+2ne+e^2+n+e$ ...(12)

Substituting $n = k + l$ in (12) then,

$$(n+e)(n+e+1) = (k+l)^2 + 2(k+l)e + e^2 + k + l + e$$
$$= k^2 + 2kl + l^2 + 2ke + 2el + e^2 + k + l + e$$
$$= k^2 + l^2 + e^2 + 2kl + 2ke + 2el + k + l + e.$$

It implies that

$(e+k)(e+k+1)-(n+e)(n+e+1)$
$= \Big[e^2+k^2+2ke+e+k\Big] - \Big[k^2+l^2+e^2+2kl+2ke+2el+k+l+e\Big]$
$= -l^2-2kl-2el-l.$

Thus

$$\frac{(e+k)(e+k+1)-(n+e)(n+e+1)}{l} = -l-2k-2e-1.$$

Hence (11) becomes

$$\begin{aligned}(e+1)\Big[\frac{e}{k}-\frac{e}{l}\Big]+2e+k+1-l-2k-2e-1 &\le 0\\ (e+1)(r-s)-l-k &\le 0\\ (e+1)(r-s)-(l+k) &\le 0\\ (e+1)(r-s)-n &\le 0.\end{aligned}$$

Since $r-s \ge 1$, then $n \ge e+1$.

which is a contradiction.

Hence $G$ is not a super vertex-magic graph. □

**Lemma 2.2.2** *The complete bipartite graph $K_{1,n}$ or the star where $n \ge 3$ is not a super vertex-magic graph.*

**Proof.** Let $K_{1,n}$ be the complete bipartite graph and $n \ge 3$.

Let $x$ be the central vertex of the star $K_{1,n}$ and $y$ be any remaining vertex.

Therefore $e = n$.

We prove by contradiction.

Assume that $K_{1,n}$ is a super vertex-magic graph.

It follow that

$$\begin{aligned}w_\lambda(x) &\ge n+1+(1+2+...+n)\\ &= n+1+\frac{n(n+1)}{2}.\\ w_\lambda(y) &\le (n+e)+n\\ &= 2n+e\\ &= 3n.\end{aligned}$$

Therefore

$$n+1+\frac{n(n+1)}{2} \leq 3n$$
$$\frac{2n+2+n^2+n}{2} \leq 3n$$
$$n^2+3n+2 \leq 6n$$
$$n^2-3n+2 \leq 0$$
$$(n-2)(n-1) \leq 0$$
$$1 \leq n \leq 2$$

Which is a contradiction because $n \geq 3$.
Hence $K_{1,n}$ is not a super vertex-magic graph. □

Note that the complete bipartite graph of 3 vertices(or $P_3$) is a super vertex-magic graph by Theorem 3.2.1. and the complete bipartite graph of 2 vertices(or $P_2$) is not a super vertex-magic graph by Theorem 3.1.2.

**Theorem 2.2.3** *Every complete bipartite graph of $n \geq 4$ vertices is not a super vertex-magic graph.*

**Proof.** Let $K_{x,y}$ be a complete bipartite graph of $n \geq 4$ vertices and $e$ edges.

We have $n = x + y$.

Case I: $x \neq y$,

If $x = 1$ or $y = 1$ , by Lemma 3.5.2. $K_{x,y}$ is not a super vertex-magic graph.
If $x, y \geq 2$, we have $xy \geq x + y$.

It implied that $xy + 1 > x + y$.

Therefore $e + 1 > n$.

By Theorem 3.5.1 $K_{x,y}$ is not a super vertex-magic graph.

Case II: $x = y$,

We prove by contradiction.

Assume that $K_{x,x}$ is a super vertex-magic graph.

By Lemma 3.1.1 $h = e + \frac{e(e+1)}{n} + \frac{n+1}{2}$.

Therefore

$$\begin{aligned} h &= x^2 + \frac{x^2(r^2+1)}{2x} + \frac{2x+1}{2} \\ &= x^2 + \frac{x(x^2+1)}{2} + \frac{2x+1}{2}. \end{aligned}$$

If $x$ is odd. we see that $x^2+1$ is even.

Therefore $\frac{x(x^2+1)}{2}$ is a positive integer.

It implies that $h$ is not an integer.

Which is a contradiction.

If $x$ is even. we see that $\frac{x(x^2+1)}{2}$ is an positive integer.

It implied that $h$ is not an integer.

Which is a contradiction.

From Case I and Case II.

Hence $K_{x,y}$ is not a super vertex-magic graph. □

**Theorem 2.2.4** *Let $m > 1$ be a positive integer and $m \neq 4$. Every $m \times m$ square lattice graph is not a super vertex-magic graph.*

**Proof.** Let $G$ be a $m \times m$ square lattice graph of $n$ vertices and $e$ edges

We have $n = m^2$ and $e = 2m^2 - 2m$.

We prove by contradiction.

Assume that $G$ is a super vertex-magic graph.

By Lemma 3.1.1,

$$\begin{aligned} h &= e + \frac{e(e+1)}{n} + \frac{n+1}{2} \\ &= 2m^2 - 2m + \frac{(2m^2-2m)(2m^2-2m+1)}{m^2} + \frac{m^2+1}{2} \\ &= 2m^2 - 2m + \frac{4m^4 - 8m^3 + 4m^2 + 2m^2 - 2m}{m^2} + \frac{m^2+1}{2} \\ &= 2m^2 - 2m + 4m^2 - 8m + 6 - \frac{2}{m} + \frac{m^2+1}{2} \\ &= 6m^2 - 10m + 6 - \frac{2}{m} + \frac{m^2+1}{2} \\ &= 6m^2 - 10m + 6 + \frac{m^3+m-4}{2m}. \end{aligned}$$

Case I: $m$ is odd, therefore $\frac{m^3+m-4}{2m}$ is an integer.

It implies that $m$=1.

Case II: $m$ is even, we have $m = 2t$ for some $t \in \mathbb{Z}^+$.

We see that $\frac{m^3+m-4}{2m} = \frac{8t^3+2t-4}{4t} = 2t^2 + \frac{1}{2} - \frac{1}{t}$.

Therefore $t = 2$ and implies that $m = 4$.

Form Case I and Case II, if $m \neq 1, 4$ then $h$ is not an integer.

Which is a contradiction.

Hence $G$ is not a super vertex-magic graph. □

**Theorem 2.2.5** *Every ladder graph $L_k = P_2 \Box P_k$ is not a super vertex-magic graph for all positive integer $k$.*

**Proof.** Let $k$ be a positive integer.

Let $L_k$ be a ladder graph of $n$ vertices and $e$ edges

We have $n = 2k$ and $e = 3k - 2$.

We prove by contradiction.

Assume that $L_k$ is a super vertex-magic graph.

By Lemma 3.1.1, $h = e + \frac{e(e+1)}{2} + \frac{n+1}{2}$.

Therefore

$$\begin{aligned} h &= 3k - 2 + \frac{(3k-2)(3k-1)}{2k} + \frac{2k+1}{2} \\ &= 3k - 2 + \frac{9k^2 - 9k + 2}{2k} + \frac{2k+1}{2} \\ &= 3k - 2 + \frac{18k^2 - 18k + 4 + 4k^2 + 2k}{4k} \\ &= 3k - 2 + \frac{22k^2 - 16k + 4}{4k} \\ &= 3k - 2 + \frac{11k}{2} - 4 + \frac{1}{k} \\ &= 3k - 6 + \frac{11k}{2} + \frac{1}{k} \\ &= 3k - 6 + \frac{11k^2 + 2}{2k}. \end{aligned}$$

Case I: $k$ is even, then $k = 2t$ for some $t \in \mathbb{Z}^+$.

Therefore $\frac{11k^2+2}{2k} = \frac{44t^2+2}{4t} = 11t + \frac{1}{2}$ is not an integer.

It implied that $h$ is not an integer.

Case II: $k$ is odd, then $\frac{11k^2+2}{2k}$ is not an integer.

It implied that $h$ is not an integer.

From Case I and Case II, $h$ is not an integer.

Which is a contradiction.

Hence ladder graph $L_k$ is not a super vertex-magic graph. $\square$

**Theorem 2.2.6** *The graph $P_m \square P_{m+1}$ is not a super vertex-magic graph for all positive integer $m$.*

**Proof.** Let $m$ be a positive integer and $G$ is a graph $P_m \square P_{m+1}$.

We have $n = m^2 + m$ and $e = 2m^2 - 1$.

We prove by contradiction.

Assume that $G$ is a super vertex-magic graph.

By Lemma 3.1.1 $h = e + \frac{e(e+1)}{n} + \frac{n+1}{2}$.

Therefore

$$\begin{aligned} h &= 2m^2 - 1 + \frac{(2m^2-1)(2m^2)}{m^2+m} + \frac{m^2+m+1}{2} \\ &= 2m^2 - 1 + \frac{4m^4 - 2m^2}{m^2+m} + \frac{m^2+m+1}{2} \\ &= 2m^2 - 1 + \frac{4m^3 - 2m}{m+1} + \frac{m^2+m+1}{2} \\ &= 2m^2 - 1 + 4m^2 - 4m + 2 - \frac{2}{m+1} + \frac{m^2+m+1}{2}. \end{aligned}$$

Case I: $m$ is even, $m = 2t$ for some $t \in \mathbb{Z}^+$, then

$$\begin{aligned} \frac{-2}{m+1} + \frac{m^2+m+1}{2} &= \frac{-2}{2t+1} + \frac{4t^2+2t+1}{2} \\ &= \frac{-2}{2t+1} + 2t^2 + t + \frac{1}{2} \\ &= 2t^2 + t + \frac{1}{2} - \frac{2}{2t+1} \\ &= 2t^2 + t + \frac{2t+1-4}{4t+2} \\ &= 2t^2 + t + \frac{2t-3}{4t+2}. \end{aligned}$$

Therefore $\frac{2t-3}{4t+2}$ is an integer. It implies that

$$
\begin{aligned}
2t-3 &\geq 4t+2\\
-5 &\geq 2t\\
\frac{-5}{2} &\geq t.
\end{aligned}
$$

Which is a contradiction.

Case II: $m$ is odd, $m = 2t+1$ for some $t \in \mathbb{Z}^+ \cup \{0\}$, then

$$
\begin{aligned}
\frac{-2}{m+1} + \frac{m^2+m+1}{2} &= \frac{-2}{2t+2} + \frac{4t^2+4t+1+2t+1+1}{2}\\
&= \frac{-1}{t+1} + \frac{4t^2+6t+3}{2}\\
&= \frac{-1}{t+1} + 2t^2+3t+1+\frac{1}{2}\\
&= 2t^2+3t+1+\frac{1}{2}-\frac{1}{t+1}\\
&= 2t^2+3t+1+\frac{t+1-2}{2t+2}\\
&= 2t^2+3t+1+\frac{t-1}{2t+2}.
\end{aligned}
$$

Therefore $\frac{t-1}{2t+2}$ is an integer. It implies that

$$
\begin{aligned}
t-1 &\geq 2t+2\\
-3 &\geq t.
\end{aligned}
$$

Which is a contradiction.

From Case I and Case II.

Hence $P_m \square P_{m+1}$ is not a super vertex-magic graph. $\square$

**Theorem 2.2.7** *Every 1- ladder square lattice of k- step is not a super vertex-magic graph.*

**Proof.** Let $G$ be a 1- ladder square lattice graph of $n$ vertices, $e$ edges and $k$ step. We have $n = \frac{k^2+5k+2}{2}, e = k^2 + 3k$.

We prove by contradiction.

Assume that $G$ is a super vertex-magic graph.

By Lemma 3.1.1,

$$\begin{aligned}
h &= e + \frac{e(e+1)}{n} + \frac{n+1}{2} \\
&= k^2 + 3k + \frac{2(k^2+3k)(k^2+3k+1)}{k^2+5k+2} + \frac{k^2+5k+4}{4} \\
&= k^2 + 3k + \frac{8(k^2+3k)(k^2+3k+1) + (k^2+5k+2)(k^2+5k+4)}{4(k^2+5k+2)} \\
&= k^2 + 3k + \frac{8(k^4+6k^3+10k^2+3k) + (k^4+10k^3+31k^2+30k+8)}{4k^2+20k+8} \\
&= k^2 + 3k + \frac{8k^4+48k^3+80k^2+24k+k^4+10k^3+31k^2+30k+8}{4k^2+20k+8} \\
&= k^2 + 3k + \frac{9k^4+58k^3+111k^2+54k+8}{4k^2+20k+8} \\
&= k^2 + 3k + \frac{9k^2}{4} + \frac{13k}{4} + 7 - \frac{112k+48}{4k^2+20k+8} \\
&= k^2 + 3k + \frac{9k^2}{4} + \frac{13k}{4} + 7 - \frac{28k+12}{k^2+5k+2} \\
&= k^2 + 3k + 7 + \frac{k(9k+13)}{4} - \frac{28k+12}{k^2+5k+2}.
\end{aligned}$$

Case I: $k = 4m$ for some $m \in \mathbb{Z}^+$ , then $\frac{k(9k+13)}{4}$ is an integer.

It implies that $\frac{28k+12}{k^2+5k+2}$ is an integer, we have

$$28k + 12 \geq k^2 + 5k + 2$$

$$0 \geq k^2 - 23k - 10$$

$$\frac{23 - \sqrt{(23)^2 - 4(1)(-10)}}{2} \leq k \leq \frac{23 + \sqrt{(23)^2 - 4(1)(-10)}}{2}$$

$$\frac{23 - \sqrt{569}}{2} \leq k \leq \frac{23 + \sqrt{569}}{2}$$

$$\frac{23 - 23.85}{2} \leq k \leq \frac{23 + 23.85}{2}$$

$$0 \leq k \leq 23.$$

Consider

If $k = 4$, $n = 19$ and $e = 28$, we have $h = 38 + \frac{812}{19} \approx 80.74$

If $k = 8$, $n = 53$ and $e = 88$, we have $h = 105 + \frac{1991}{26} \approx 116.58$

If $k = 12$, $n = 103$ and $e = 180$, we have $h = 232 + \frac{32{,}580}{103} \approx 548.31$

If $k = 16$, $n = 169$ and $e = 304$, we have $h = 389 + \frac{92{,}720}{169} \approx 937.64$

If $k = 20$, $n = 251$ and $e = 460$, we have $h = 586 + \frac{212{,}060}{251} \approx 1,430.86$

We see that if $0 \leq k \leq 23$ and $k = 4m$ for some $m \in \mathbb{Z}^+$, then $h$ is not integer.

Which is a contradiction.

Hence $G$ is not a super vertex-magic graph.

Case II: $k = 4t + 1$ for some $t \in \mathbb{Z}^+ \cup \{0\}$, we have

$$
\begin{aligned}
\frac{9k^2}{4} + \frac{13k}{4} - \frac{28k+12}{k^2+5k+2} &= \frac{9(4t+1)^2}{4} + \frac{13(4t+1)}{4} - \frac{28(4t+1)+12}{(4t+1)^2+5(4t+1)+2} \\
&= \frac{9(16t^2+8t+1)}{4} + \frac{52t+13}{4} - \frac{112t+40}{16t^2+8t+1+20t+5+2} \\
&= \frac{144t^2+72t+9}{4} + \frac{52t+13}{4} - \frac{112t+40}{16t^2+28t+8} \\
&= \frac{144t^2+72t+9}{4} + \frac{52t+13}{4} - \frac{28t+10}{4t^2+7t+2} \\
&= \frac{144t^2+124t+22}{4} - \frac{28t+10}{4t^2+7t+2} \\
&= \frac{72t^2+62t+11}{2} - \frac{28t+10}{4t^2+7t+2} \\
&= 36t^2+31t+5+\frac{1}{2} - \frac{28t+10}{4t^2+7t+2} \\
&= 36t^2+31t+5+\frac{4t^2+7t+2-56t-20}{8t^2+14t+4} \\
&= 36t^2+31t+5+\frac{4t^2-49t-18}{8t^2+14t+4}.
\end{aligned}
$$

Therefore $\frac{4t^2-49t-18}{8t^2+14t+4}$ is an integer, we have

$$\begin{aligned}
4t^2-49t-18 &\geq 8t^2+14t+4\\
0 &\geq 4t^2+63t+22\\
\frac{-63-\sqrt{63^2-4(4)(22)}}{8} \leq t &\leq \frac{-63+\sqrt{63^2-4(4)(22)}}{8}\\
\frac{-63-\sqrt{3,617}}{8} \leq t &\leq \frac{-63+\sqrt{3,617}}{8}\\
\frac{-63-60.14}{8} \leq t &\leq \frac{-63+60.14}{8}\\
-15 \leq t &< 0.
\end{aligned}$$

Which is a contradiction.

Hence if $k = 4t+1$ for some $t \in \mathbb{Z}^+ \cup \{0\}$ then $G$ is not a super vertex-magic graph.

Case III: $k = 4t+2$ for some $t \in \mathbb{Z}^+ \cup \{0\}$, we have

$$\begin{aligned}
\frac{9k^2}{4}+\frac{13k}{4}-\frac{28k+12}{k^2+5k+2} &= \frac{9(4t+2)^2}{4}+\frac{13(4t+2)}{4}-\frac{28(4t+2)+12}{(4t+2)^2+5(4t+2)+2}\\
&= \frac{9(16t^2+16t+4)}{4}+\frac{52t+2}{4}-\frac{112t+68}{16t^2+16t+4+20t+10+2}\\
&= 36t^2+36t+9+\frac{26t+1}{2}-\frac{112t+68}{16t^2+36t+16}\\
&= 36t^2+36t+9+\frac{26t+1}{2}-\frac{28t+17}{4t^2+9t+4}\\
&= 36t^2+36t+9+13t+\frac{1}{2}-\frac{28t+17}{4t^2+9t+4}\\
&= 36t^2+36t+9+13t+\frac{4t^2+9t+4-56t-34}{8t^2+18t+8}\\
&= 36t^2+36t+9+13t+\frac{4t^2-47t-30}{8t^2+18t+8}.
\end{aligned}$$

Therefore $\frac{4t^2-47t-30}{8t^2+18t+8}$ is an integer, we have

$$\begin{aligned}
4t^2 - 47t - 30 &\geq 8t^2 + 18t + 18 \\
0 &\geq 4t^2 + 65t + 48 \\
\frac{-65 - \sqrt{65^2 - 4(4)(48)}}{8} \leq t &\leq \frac{-65 + \sqrt{65^2 - 4(4)(48)}}{8} \\
\frac{-65 - \sqrt{3457}}{8} \leq t &\leq \frac{-65 + \sqrt{3457}}{8} \\
\frac{-65 - 58.80}{8} \leq t &\leq \frac{-65 + 58.80}{8} \\
-15 \leq t &< 0.
\end{aligned}$$

Which is a contradiction.

Hence if $k = 4t + 2$ for some $t \in \mathbb{Z}^+ \cup \{0\}$ then $G$ is not a super vertex-magic graph.

Case IV: $k = 4t + 3$ for some $t \in \mathbb{Z}^+ \cup \{0\}$, we have

$$\begin{aligned}
\frac{9k + 13}{4} &= \frac{9(4t + 3) + 13}{4} \\
&= \frac{36t + 27 + 13}{4} \\
&= \frac{36t + 40}{4} \\
&= 9t + 10.
\end{aligned}$$

we see that $\frac{9k+13}{4}$ is an integer.

Therefore $\frac{28k+12}{k^2+5k+2}$ is an integer.

We see that $0 \leq k \leq 23$.

Consider

If $k = 3$, $n = 13$ and $e = 18$, we have $h = 25 + \frac{342}{13} \approx 51.31$

If $k = 7$, $n = 43$ and $e = 71$, we have $h = 93 + \frac{5{,}112}{43} \approx 211.88$

If $k = 11$, $n = 89$ and $e = 154$, we have $h = 199 + \frac{23{,}870}{89} \approx 467.20$

If $k = 15$, $n = 151$ and $e = 270$, we have $h = 346 + \frac{73{,}170}{151} \approx 830.57$

If $k = 19$, $n = 229$ and $e = 418$, we have $h = 553 + \frac{175{,}142}{229} \approx 1,297.81$

If $k = 23$, $n = 323$ and $e = 598$, we have $h = 760 + \frac{358{,}202}{323} \approx 1,868.98$

We see that if $0 \leq k \leq 23$ and $k = 4t + 3$ $\exists t \in \mathbb{Z}^+ \cup \{0\}$, then $h$ is not integer.

Which is a contradiction.

Hence $G$ is not a super vertex-magic graph.

From Case I, Case II, Case III and Case IV.

Hence G is not a super vertex magic graph. □

**Theorem 2.2.8** *Let $k$ be a positive integer where $k \neq 2, 6$. A $k$- level of $Q_3$ is not a super vertex-magic graph.*

**Proof.** Let $G$ be a $k$- level of $P_2$ of $n$ vertices and $e$ edges .

We have $n = 4k$ and $e = 8k - 4$.

We prove by contradiction.

Assume that $G$ is a super vertex-magic graph.

By Lemma 3.1.1,

$$\begin{aligned} h &= e + \frac{e(e+1)}{n} + \frac{n+1}{2} \\ &= 8k - 4 + \frac{(8k-4)(8k-3)}{4k} + \frac{4k+1}{2} \\ &= 8k - 4 + \frac{1}{4k}(64k^2 - 56k + 12) + \frac{4k+1}{2} \\ &= 8k - 4 + 16k - 14 + \frac{3}{k} + 2k + \frac{1}{2} \\ &= 26k - 18 + \frac{3}{k} + \frac{1}{2}. \end{aligned}$$

We see that $\frac{3}{k} + \frac{1}{2} = \frac{6+k}{2k}$ is a positive integer, then

$$6 + k = 2kt \quad for\ some\ t \in \mathbb{Z}^+$$

$$k(2t-1) = 6$$

$$k = \frac{6}{2t-1}$$

$$\therefore t = 1, 2.$$

It implies that $k = 2$ or $k = 6$.

We see that if $k \neq 2, 6$ then $h$ is not an integer, which is a contradiction.

Hence $G$ is not a super vertex-magic graph. □

Note that if $k$=2 by Theorem 3.4.1, $G$ is super vertex-magic graph. We still cannot show whether the 6− level of $Q_3$ is super vertex-magic or not.