Chapter 1

Introduction

Generalized functions have of late been commanding constantly expanding interest in several different branches of mathematics. In somewhat nonrigorous form, they have already long been used in essence by physicists and opened up a new area of mathematical research, which in turn provided an impetus in the development of a number of a number of mathematical disciplines, such as ordinary and partial differential equations, operational calculus, transformation theory, and functional analysis.

Modern developments in partial differential equations require a thorough grounding in the theory of distributions in more than one variable; for instance, in 3-space we can have sources concentrated not only at points but also on curves and surfaces, thereby giving rise to equivalent volume source densities which are clearly singular in varying degrees.

Euclidean n-space is denoted by $\mathbb{R}^n$ and a point in $\mathbb{R}^n$ is labeled $x = (x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are Cartesian coordinates with respect to a fixed frame of reference. We shall write

$$r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$$

for the distance between the point x and the origin. An element of volume $dx_1 \ldots dx_n$ will be abbreviated dx, so that the integral of a function f of position over a region R is written in either of the forms

$$\int_R f(x_1, \ldots, x_n) dx_1 \ldots dx_n \quad \text{or} \quad \int_R f(x) dx.$$

We shall also consider integrals over hypersurfaces of dimension $n-1$ in $\mathbb{R}^n$. Thus in $\mathbb{R}^3$ a hypersurface is an ordinary surface, whereas in $\mathbb{R}^2$ it is an ordinary curve. Typically a hypersurface is denoted by the letter Ω and an element of surface on Ω by dS. Occasionally a subscript on dS will be used to indicate the variable being integrated; for instance, if $f(x, \xi)$ is a function depending on two points in $\mathbb{R}^n$ and we are integrating f on Ω with respect to the variable x, we write

$$\int_{\Omega} f(x, \xi) dS_x.$$

It is well known that for the heat equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \Delta u(x, t) \tag{1.1}$$

with the initial condition

$$u(x, 0) = f(x)$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{1.2}$$

is the Laplace operator and $(x, t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, we obtain the solution

$$u(x, t) = \frac{1}{(4c^2 \pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x - y|^2}{4c^2 t}\right) f(y) dy$$

or the solution in the convolution form

$$u(x, t) = E(x, t) * f(x)$$

where

$$E(x, t) = \frac{1}{(4c^2 \pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4c^2 t}\right) \tag{1.3}$$

and the symbol $*$ designates as the convolution. The equation (1.3) is called *the heat kernel*, where $|x|^2 = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$ and $t > 0$, see ([7], pp. 208-209). Moreover, we obtain $E(x, t) \to \delta$ as $t \to 0$, where δ is the Dirac-delta distribution.

In [18], K. Nonlaopon and A. Kananthai extended (1.1) to the equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \Box u(x, t) \tag{1.4}$$

with the initial condition

$$u(x, 0) = f(x) \tag{1.5}$$

where $\Box$ is the ultra-hyperbolic operator, that is

$$\Box = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}, \tag{1.6}$$

with $p+q=n$ the dimension of Euclidean space $\mathbb{R}$. They obtain

$$u(x,t) = E(x,t) * f(x) \tag{1.7}$$

as a solution of (1.4) which satisfies (1.5) where $E(x,t)$ is the kernel of (1.4) and is defined by

$$E(x,t) = \frac{(i)^q}{(4c^2\pi t)^{n/2}} \exp\left[-\frac{\left(\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2\right)}{4c^2t}\right] \tag{1.8}$$

where $p+q=n$, $i=\sqrt{-1}$ and $\sum_{i=1}^{p} x_i^2 > \sum_{j=p+1}^{p+q} x_j^2$. Moreover, they obtain $E(x,t) \to \delta$ as $t \to 0$, where δ is the Dirac-delta distribution. In addition, they studied the ultra-hyperbolic heat kernel which is related to the spectrum.

In [20], they studied the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box^k u(x,t) \tag{1.9}$$

with the initial condition

$$u(x,0) = f(x), \quad \text{for} \quad x \in \mathbb{R}^n \tag{1.10}$$

where the operator $\Box^k$ is named *the ultra-hyperbolic operator* iterated k-times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k, \tag{1.11}$$

$p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is the given generalized function, k is a positive integer and c is a positive constant.

In [19], A. Kananthai studied the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Diamond\, u(x,t) \tag{1.12}$$

with the initial condition

$$u(x,0) = f(x), \quad \text{for} \quad x \in \mathbb{R}^n, \tag{1.13}$$

where the operator $\Diamond$ is first introduced by A. Kananthai ([9], pp. 27-37) and is named *the diamond operator* which is defined by

$$\Diamond = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2, \tag{1.14}$$

$p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown function for $(x,t)=(x_1,x_2,\ldots,x_n,t)\in\mathbb{R}^n\times(0,\infty)$, $f(x)$ is the given generalized function and c is a positive constant.

Now, the purpose of this work is to study the equation

$$\frac{\partial}{\partial t}u(x,t)=c^2\Diamond^k u(x,t) \tag{1.15}$$

with the initial condition

$$u(x,0)=f(x) \quad \text{for} \quad x\in\mathbb{R}^n,$$

where the operator $\Diamond^k$ is named *the diamond operator* iterated k-times defined by

$$\Diamond^k=\left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2-\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k, \tag{1.16}$$

$p+q=n$ is the dimension of space $\mathbb{R}^n$, $u(x,t)$ is an unknown function for $(x,t)=(x_1,x_2,\ldots,x_n,t)\in\mathbb{R}^n\times(0,\infty)$, $f(x)$ is the given generalized function, k is a nonnegative integer and c is a positive constant.

We obtain $u(x,t)=E(x,t)*f(x)$ as a solution of (1.15) where

$$E(x,t)=\frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k t+i(\xi,x)\right]d\xi. \tag{1.17}$$

and $\Omega\subset\mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t>0$. The function $E(x,t)$ is called *the diamond heat kernel* or the elementary solution of (1.15). All properties of $E(x,t)$ will be studied in details. Now, if we put $k=1$ in (1.17) then we obtain the kernel of (1.12) which is satisfies (1.13).

The operator $\oplus^k$ has been studied first by A. Kananthai, S. Suantai and V. Longani [15] and is defined by

$$\begin{aligned}\oplus^k &=\left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2-\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k\times\left[\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}+i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k\\ &\quad\times\left[\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}-i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k\\ &=\left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^4-\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k\end{aligned} \tag{1.18}$$

where $p+q=n$ is the dimension of $\mathbb{R}^n$, $i=\sqrt{-1}$ and k is a nonnegative integer. The Diamond operator $\Diamond$ is defined by (1.14) and the operators L_1 and L_2 are defined by

$$L_1 = \sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \tag{1.19}$$

and

$$L_2 = \sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}. \tag{1.20}$$

Thus (1.18) can be written by

$$\oplus^k = \Diamond^k L_1^k L_2^k.$$

Otherwise, the operator $\Diamond$ can also be expressed in the form $\Diamond = \Box\triangle = \triangle\Box$ where $\Box$ is the ultra-hyperbolic operator defined by (1.16) and $\triangle$ is the Laplacian defined (1.2)

In 1988, S.E. Trione [23] was study the elementary solution of the ultra-hyperbolic Klein-Gordon operator iterate $k-$times is defined by

$$(\Box + m^2)^k = \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} + m^2\right]^k, \tag{1.21}$$

and we obtain the elementary solution $W_{2k}^H(v,m)$ defined by

$$W_{2k}^H(v,m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r!\Gamma(k)} (m^2)^r R_{2k+2r}^H(v), \tag{1.22}$$

where $R_{\alpha+2r}^H(v)$ is defined by (2.13). See, also ([24] pp. 154-156). After that, A. Kanathai [13], obtained the function

$$G(x) = \left[W_{2k}^H(v,m) * W_{2k}^e(s,m)\right] * \left(S^{*k}(x)\right)^{-1} \tag{1.23}$$

is a Green function for the operator

$$(\Diamond + m^2)^k = \left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right)^k, \tag{1.24}$$

where $W_{2k}^e(s,m)$ is defined by

$$W_{2k}^e(s,m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r!\Gamma(k)} (m^2)^r (-1)^{k+r} R_{2k+2r}^e(s) \tag{1.25}$$

and $R^e_{2k+2r}(s)$ is defined by (2.15) with $\beta = 2k + 2r$, m is a nonnegative real number and

$$S(x) = \delta - m^2 \left(W_2^H(v,m) * W_2^e(s,m)\right) * \left(R^H_{-2}(v) + R^e_{-2}(s)\right). \qquad (1.26)$$

$S^{*k}(x)$ denotes the convolution of S itself k-times, $\left(S^{*k}(x)\right)^{*-1}$ is the inverse of $S^{*k}(x)$ in the convolution algebra.

Now, the purpose of this work is to find the elementary solution or Green function of the operator $(\oplus + m^2)^k$, that is

$$(\oplus + m^2)^k U(x) = \delta(x), \qquad (1.27)$$

where $U(x)$ is the Green function, δ is the Dirac-delta distribution, k is a nonnegative integer and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. After that, we find the solution of the equation $(\oplus + m^2)^k U(x) = f(x)$ where f is a given generalized function and $U(x)$ is an unknown function.

Now consider the non-linear equation

$$\Diamond^k u(x) = f(x, \triangle^{k-1}\Box^k u(x)), \qquad (1.28)$$

where $\Diamond^k$ is defined by (1.16), $\triangle$ is the Laplacian operator, $\Box$ is the ultra-hyperbolic operator, $p + q = n$, $x \in \mathbb{R}^n$ and $u(x)$ is am unknown. We know that if f is continuous first derivative for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω, n is even with $n \geq 4$ and if f is bounded on Ω, then $u(x) = (-1)^{k-1} R^e_{2(k-1)}(s) * R^H_{2k}(v) * W(x)$ is a solution of (1.28) with the boundary condition

$$u(x) = R^H_{2k}(v) * (-1)^{k-2} (R^e_{2(k-2)}(r))^{(m)}$$

for $x \in \partial\Omega$ and $m = (n-4)/2$, [16].

In this work, we study the non-linear equation of the form

$$\Diamond^k (\Box + m^2)^k u(x) = f(x, \triangle^{k-1}\Box^k (\Box + m^2)^k u(x)), \qquad (1.29)$$

with f defined and having continuous first derivative for all $x \in \Omega \cup \partial\Omega$, where Ω is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω, f is bounded on Ω that is $|f| \leq N$, N is constant and the ultra-hyperbolic Klein-Gordon operator is defined by (1.21). We can find the solution $u(x)$ of (1.29) and is unique under the boundary condition $\triangle^{k-1}\Box^k(\Box + m^2)^k u(x) = 0$ for $x \in \partial\Omega$. By [[2], p. 369] there exists a unique solution $U(x)$ of the equation $\triangle U(x) = f(x, U(x))$ for all $x \in \Omega$ with the boundary condition $U(x) = 0$ for all $x \in \partial\Omega$ where $U(x) = \triangle^{k-1}\Box^k(\Box + m^2)^k u(x)$. Then if we put $k = 1$, $p = n$ and $q = 0$ in

$\Box^k(\Box+m^2)^k V(x) = U(x)$, we found that $V(x) = (-1)^{k-1} R^e_{2(1-k)}(s) * u(x)$ is a solution of the inhomogeneous biharmonic equation,

$$\triangle^2 V(x) + m^2 \triangle V(x) = U(x)$$

where $\triangle^2$ denotes the biharmonic operator is defined by

$$\triangle^2 = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^2. \tag{1.30}$$

The telegraph equation plays an important role in transmission and propagation of electrical signals, vibrational systems and mechanical systems. Apart from these situations, heat diffusion and wave propagation equations are particular cases of the telegraph equation. It is well known that separation of variables technique is useful for obtaining exact series solutions and the Fourier transform is used for the n-dimensional cases.

The distributions $e^{\alpha t}\Box^k\delta$ have studied by Kananthai [10] where $\Box^k$ is the ultra-hyperbolic operator defined by (1.11) and $p+q=n$ the dimension of the space $\mathbb{R}^n$ and α is a constant, $\alpha = (\alpha_1, \alpha_2 \ldots, \alpha_n)$ and $\alpha t = \alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_n t_n$.

In this work, we applied the distribution $e^{\alpha t}\Box^k\delta$ for finding the elementary solution of the n-dimensional telegraph equations of the type

$$\left(\frac{\partial^2}{\partial t^2} + 2\beta\frac{\partial}{\partial t} + \beta^2 - \triangle\right)^k u(x,t) = T_1^k u(x,t) = \delta(x,t) \tag{1.31}$$

and

$$\left(\frac{\partial^2}{\partial t^2} + 2\beta\frac{\partial}{\partial t} + \beta^2 + \triangle\right)^k u(x,t) = T_2^k u(x,t) = \delta(x,t) \tag{1.32}$$

where $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is the Laplacian operator $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, t is the time and δ is the Dirac delta function and T_1^k, T_2^k are the operator iterated k-times defined by

$$T_1^k = \left(\frac{\partial^2}{\partial t^2} + 2\beta\frac{\partial}{\partial t} + \beta^2 - \triangle\right)^k, \tag{1.33}$$

$$T_2^k = \left(\frac{\partial^2}{\partial t^2} + 2\beta\frac{\partial}{\partial t} + \beta^2 + \triangle\right)^k. \tag{1.34}$$