Chapter 2

Basic Concepts and Preliminaries

The aim of this chapter is to give some definitions and properties of the distribution, the special function, the Fourier transform, partial differential equations and the elementary solutions of the partial differential operators which will be used in the later chapters.

2.1 Distribution

In this section, we shall use the standard notation $\mathcal{D}$ the space of testing functions, which consists of all real or complex functions with continuous derivative of all orders and with compact support. Every element in $\mathcal{D}$ is called a *testing function.*

Definition 2.1.1 A sequence of testing function $\varphi_i(x)_{i=1}^{\infty}$ is said to converge to $\varphi(x)$ in $\mathcal{D}$ if all $\varphi_i(x)$ are zero outside a certain region in $\mathbb{R}^n$ and if for every non-negative integers $m_1, m_2, \ldots, m_n$ the sequence $\{\frac{\partial^{m_1+m_2+\cdots+m_n}\varphi_i(x)}{\partial x_1^{m_1}\partial x_2^{m_2}\ldots\partial x_n^{m_n}}\}_{i=1}^{\infty}$ converges uniformly to $\frac{\partial^{m_1+m_2+\cdots+m_n}\varphi(x)}{\partial x_1^{m_1}\partial x_2^{m_2}\ldots\partial x_n^{m_n}}$ on $\mathbb{R}^n$.

Proposition 2.1.2 ([3]) *$\mathcal{D}$ is closed under convergence, that is, the limit of every sequence that converge in $\mathcal{D}$ is also in $\mathcal{D}$.*

Definition 2.1.3 A functional on a linear space E is mapping $f : E \longrightarrow \mathbb{C}$ which assigns to each member φ of E a certain complex number; the image of $\varphi \in E$ under f is usually written as $f(\varphi)$ or $\langle f, \varphi\rangle$.

One way to generate distributions is as follows. Let $f(x)$ be a locally integrable function, that is, a function that is integrable in the Lebesgue sense over every compact subset of $\mathbb{R}^n$. Corresponding to $f(x)$, we can define a distribution through the convergence integral

$$\langle f, \varphi\rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)dx. \tag{2.1}$$

Then by ([5]), $\langle f, \varphi \rangle$ is a distribution.

Definition 2.1.4 Distributions that can be generated through (2.1) from locally integrable functions shall be called regular, and all others will be called singular.

An important singular distribution is the so-called *Dirac-delta function* δ, which is defined by

$$\langle \delta, \varphi \rangle = \varphi(0). \tag{2.2}$$

It is to note that the Dirac-delta function is a singular distribution see ([5]) .

Proposition 2.1.5 ([25]) *Let x be an n-dimensional real variable and y an m-dimensional real variable. Also, let $\varphi(x,y)$ be a testing function in $\mathcal{D}$ define over $\mathbb{R}^{n+m}$. If $f(x)$ is a distribution defined over $\mathbb{R}^n$, then $\theta(y) = \langle f(x), \varphi(x,y) \rangle$ is a testing function of y in $\mathcal{D}$.*

Proposition 2.1.6 ([5]) *Let f be any distribution (in one dimension), then the functional g defined by*

$$\langle g, \varphi \rangle = \langle f, -\varphi' \rangle$$

is also a distribution.

Definition 2.1.7 The distribution g in proposition 2.1.6 is called the derivative of f and is denoted by f' or $\frac{df}{dx}$, that is,

$$\langle f', \varphi \rangle = \langle f, -\varphi' \rangle. \tag{2.3}$$

Similarly, in the case of several variable, the partial derivative of a distribution f with respect to each of the variables can be defined as

$$\langle \frac{\partial f}{\partial x_i}, \varphi \rangle = \langle f, -\frac{\partial \varphi}{\partial x_i} \rangle, \tag{2.4}$$

for $i = 1, \ldots, n$.

Proposition 2.1.8 ([5]) *Let f be a distribution in m dimensions and g be a distribution in n dimensions. Then the functional h defined by*

$$\langle h(x,y), \varphi(x,y) \rangle = \langle f(x), \langle g(y), \varphi(x,y) \rangle \rangle$$

is a distribution in $m + n$ dimensions.

Definition 2.1.9 The distribution h in proposition 2.1.8 is called the tensor (or direct) product of $f(x)$ and $g(y)$ and is denoted by $h(x,y) = f(x) \times g(y)$, that is,

$$\langle f(x) \times g(y), \varphi(x,y) \rangle = \langle f(x), \langle g(y), \varphi(x,y) \rangle \rangle. \tag{2.5}$$

Definition 2.1.10 The support of a distribution f is defined as the complement of the largest open set on which f is zero.

Proposition 2.1.11 ([5]) *Let f and g be distributions in n dimensions. Then the function h defined by*

$$\langle h, \varphi \rangle = \langle f(x) \times g(y), \varphi(x+y) \rangle \tag{2.6}$$

is a distribution provided that it satisfies either of the following conditions:

(1) *Either f or g has bounded support, or*

(2) *In one dimension the supports of both f and g are bounded on the same side (for instance, $f = 0$ for $x < a$, and $g = 0$ for $y < b$).*

Definition 2.1.12 The distribution h in proposition 2.1.11 is called the convolution of f and g and is denoted by $h = f * g$, that is,

$$\langle f * g, \varphi \rangle = \langle f(x) \times g(y), \varphi(x+y) \rangle. \tag{2.7}$$

Now we shall give some helpful properties of convolutions.

Proposition 2.1.13 ([5],[25]) *Let f, g and h be distributions.*

(1) *For δ is the Dirac-delta function, we have*

$$f * \delta = f. \tag{2.8}$$

(2) *If f and g satisfy at least one of the (1) and (2) of proposition 2.1.12, then*

$$f * g = g * f. \tag{2.9}$$

(3) *If $P(D)$ is linear partial differential operator with constant coefficients and f and g satisfy at least one of the (1) and (2) of proposition 2.1.12, then*

$$P(D)f * g = P(D)(f * g) = f * P(D)g. \tag{2.10}$$

Proposition 2.1.14 ([8]) $e^{\alpha t}\delta^{(k)} \equiv (D-\alpha)^k\delta$ *where* $D \equiv \frac{d}{dt}$ *and* $e^{\alpha t}\delta^{(k)} \equiv (D-\alpha)^k\delta$ *is a Tempered distribution of order k with support* 0.

2.2 The Special Functions and Fourier Transform

In this section, we shall present the definitions of the special function and Fourier transform given by Euler. In addition, we shall give some properties of the gamma function.

Definition 2.2.1 The *gamma function* is denote by Γ and is defined by

$$\Gamma(z) = \int_0^\infty e^t t^{z-1} dt, \tag{2.11}$$

where z is a complex number with $Rez > 0$.

A result that yields an immediate analytic continuation from the left haft plane is the following properties.

Proposition 2.2.2 ([1]) *Let z be a complex number. Then*

(1) $\Gamma(z) = \frac{\Gamma(z+1)}{z}$, $\quad z \neq 0, -1, -2, \ldots$

(2) $\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin\pi z}$, $\quad z \neq 0, \pm 1, \pm 2, \ldots.$

Proposition 2.2.3 ([1]) *Let z be a complex number. Then*

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z) , \quad z \neq 0, -1, -2, \ldots. \tag{2.12}$$

Definition 2.2.4 Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and write

$$v = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \qquad p + q = n.$$

Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } v > 0\}$ the interior of forward cone and $\overline{\Gamma}_+$ denote its closure.

For any complex number α, we define the function

$$R_\alpha^H(v) = \begin{cases} \frac{v^{(\alpha-n)/2}}{K_n(\alpha)} & \text{if } x \in \Gamma_+, \\ 0 & \text{if } x \notin \Gamma_+. \end{cases} \tag{2.13}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma\left(\frac{2+\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)}. \tag{2.14}$$

Let supp$R_\alpha^H(v) \subset \overline{\Gamma}_+$ where supp$R_\alpha^H(v)$ denotes the support of $R_\alpha^H(v)$. The function R_α^H is first introduced by Nozaki [20, p. 72] and is called the ultra-hyperbolic kernel of Marcel Riesz. Now $R_\alpha^H(v)$ ia an ordinary function if $\text{Re}(\alpha) \geq n$ and is a distribution of α if $\text{Re}(\alpha) < n$.

Definition 2.2.5 Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and write

$$s = x_1^2 + x_2^2 + \cdots + x_n^2.$$

For any complex number β, define the function

$$R_\beta^e(s) = 2^{-\beta}\pi^{-n/2}\Gamma\left(\frac{n-\beta}{2}\right)\frac{s^{(\beta-n)/2}}{\Gamma\left(\frac{\beta}{2}\right)} \qquad (2.15)$$

The function $R_\beta^e(s)$ is called the elliptic kernel of Marcel Riesz and is ordinary function if $\mathrm{Re}(\beta) \geq n$ and is a distribution of β if $\mathrm{Re}(\beta) < n$.

Definition 2.2.6 Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$ and we write

$$w = x_1^2 + x_2^2 + \cdots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2)$$

and

$$z = x_1^2 + x_2^2 + \cdots + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2), \quad p + q = n.$$

For any complex number γ and ν, we define

$$S_\gamma(w) = 2^{-\gamma}\pi^{-n/2}\Gamma\left(\frac{n-\gamma}{2}\right)\frac{w^{(\gamma-n)/2}}{\Gamma\left(\frac{\gamma}{2}\right)} \qquad (2.16)$$

and

$$T_\nu(z) = 2^{-\nu}\pi^{-n/2}\Gamma\left(\frac{n-\nu}{2}\right)\frac{z^{(\nu-n)/2}}{\Gamma\left(\frac{\nu}{2}\right)}. \qquad (2.17)$$

Definition 2.2.7 Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$ and the function $Y_{\alpha,\beta,\gamma,\nu}(v, s, w, z, m)$ is defined by

$$Y_{\alpha,\beta,\gamma,\nu}(v, s, w, z, m) = \sum_{r=0}^{\infty}\frac{(-1)^r\Gamma\left(\frac{\eta}{2}+r\right)}{r!\Gamma\left(\frac{\eta}{2}\right)}(m^2)^r K_{\alpha+2r,\beta+2r,\gamma+2r,\nu+2r}(v, s, w, z), \qquad (2.18)$$

and $K_{\alpha+2r,\beta+2r,\gamma+2r,\nu+2r}(v, s, w, z)$ is defined by

$$K_{\alpha+2r,\beta+2r,\gamma+2r,\nu+2r}(v, s, w, z) = (-1)^{\frac{\beta}{2}+r}R_{\alpha+2r}^H(v) * R_{\beta+2r}^e(s) * S_{\gamma+2r}(w) * T_{\nu+2r}(z) \qquad (2.19)$$

where η be any complex number and m is a nonnegative real number.

Definition 2.2.8 The spectrum of the kernel $E(x, t)$ of (1.17) is the bounded support of the Fourier transform $\widehat{E(\xi, t)}$ for any fixed $t > 0$.

Definition 2.2.9 Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$ and denote by

$$\Gamma_+ = \{\xi \in \mathbb{R}^n : \xi_1^2 + \xi_2^2 + \ldots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \ldots - \xi_{p+q}^2 > 0 \text{ and } \xi_1 > 0\}$$

the set of an interior of the forward cone, and $\overline{\Gamma}_+$ denotes the closure of Γ_+.

Let Ω be spectrum of $E(x,t)$ defined by definition 2.2.8 and $\Omega \subset \overline{\Gamma}_+$. Let $\widehat{E(\xi,t)}$ be the Fourier transform of $E(x,t)$ and define

$$\widehat{E(\xi,t)} = \begin{cases} \frac{1}{(2\pi)^{n/2}} \exp\left[c^2\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right)^k t\right] & \text{for } x \in \Gamma_+, \\ 0 & \text{for } \xi \notin \Gamma_+. \end{cases} \tag{2.20}$$

Lemma 2.2.10 ([3]) *The function $R^e_{-2k}(s)$ is the inverse of the convolution algebra of $R^e_{2k}(s)$, that is*

$$R^e_{-2k}(s) * R^e_{2k}(s) = R^e_{-2k+2k}(s) = R^e_0(s) = \delta.$$

Definition 2.2.11 Let f be a continuous function, the Fourier transform of f denoted by

$$\mathfrak{F}f = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi x} f(x)dx, \tag{2.21}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $\xi x = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$. Sometimes we write $\mathfrak{F}f \equiv \widehat{f}(\xi)$. By Eq. (2.21), we can define the inverse of the Fourier transform of $\widehat{f}(\xi)$ by

$$f(x) = \mathfrak{F}^{-1}\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi x} \widehat{f}(\xi)d\xi. \tag{2.22}$$

If f is a distribution with compact supports by ([25]), Theorem 7.4-3, p. 187 Eq. (7) can be written as

$$\mathfrak{F}f = \frac{1}{(2\pi)^{n/2}} \langle f(x), e^{-i\xi x} \rangle. \tag{2.23}$$

2.3 Partial Differential Equations

Let L be an arbitrary linear differential operator of order p in the n independent variables $x_1, \ldots, x_n$ can be written

$$L = \sum_{|k| \le p} a_k(x) D^k, \tag{2.24}$$

where we shall assume that the functions $a_k(x)$ have partial derivatives of all orders. The formal adjoint of L is denoted by L^* defined from

$$L^*v = \sum_{|k|\leq p}(-1)^{|k|}D^k(a_k v).$$

Now let u and v be functions having continuous derivatives of order p in $\mathbb{R}^n$, that is, $D^k u$ and $D^k v$ are continuous for every multi-index k with $|k| \leq p$. Then it can be shown that

$$vLu - uL^*v = divJ(u,v), \tag{2.25}$$

where J is a vectorial bilinear form in u and v involving only derivatives of u and v of order $p-1$ or less. The integral form of (2.25) is known as *Green's theorem.*

Let $s(x)$ be a given continuous function in $\mathbb{R}^n$, and consider the partial differential equation

$$Lu = s \tag{2.26}$$

A function $u(x)$ is said to be a strict solution of (2.26) in an open region R in $\mathbb{R}^n$, if u has continuous derivatives of order p in R and if at every point of R we have $Lu = s$.

If s is a given distribution (and this, of course, includes the possibility that s is a continuous function), we may consider (2.26) as an equation for an unknown distribution u. We shall say that u is a generalized solution of (2.26) if it satisfies the equation in the distributional sense, that is, if $\langle Lu, \varphi\rangle = \langle s, \varphi\rangle$. Here the left side is defined for any distribution u from $\langle Lu, \varphi\rangle = \langle u, L^*\varphi\rangle$, so that u is a generalized solution of (2.26) if and only if

$$\langle u, L^*\varphi\rangle = \langle s, \varphi\rangle, \tag{2.27}$$

for every test function φ in $\mathcal{D}$.

Although (2.27) gives no hint for finding a solution u, it enables us in principle to determine if a distribution u actually satisfies (2.26). We only have to verify that, for each φ, the action of u on the test function $L^*\varphi$ is the same as the action of s on φ.

The definition just given for a generalized solution is global in character; that is, it applies to the whole of $\mathbb{R}^n$. We would also like to define a notion of generalized solution in an open region R. In the light of our earlier discussion of the values of a distribution, it is natural to use the following definition: u is a generalized solution of (2.26) in R if (2.27) holds for all test functions φ with support contained in R.

A partial differential equation is an equation containing a partial derivative which is to be taken of an unknown of more than one variable. A partial differential is called linear if it can be written in the form

$$\sum_{m_1+\cdots+m_n} a_{m_1,\ldots,m_n}(x_1,\ldots,x_n)\frac{\partial^{m_1+\cdots+m_n}u(x_1,\ldots,x_n)}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}} = f(x_1,\ldots,x_n),$$

where summation is taken over all nonnegative integer $m_1 + \cdots + m_n$, the a's and f are given functions, u is an unknown function, and m is a nonnegative integer. A partial differential equation is called nonlinear if it is not linear.

Now we shall given some examples of partial differential equations.

Example 2.3.1 The heat equation in n dimensions for $u = u(x_1, \ldots, x_n, t)$ is

$$\frac{\partial}{\partial t}u(x,t) = c^2 \triangle u(x,t) \tag{2.28}$$

with the initial condition

$$u(x,0) = f(x)$$

where $\triangle$ is the Laplace operator defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}, \tag{2.29}$$

and $(x,t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, we obtain the solution

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4c^2 t}\right) f(y)dy$$

or the solution in the convolution form

$$u(x,t) = E(x,t) * f(x)$$

where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4c^2 t}\right) \tag{2.30}$$

and the symbol $*$ designates as the convolution. The equation (2.30) is called *the heat kernel*, where $|x|^2 = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$ and $t > 0$, see ([7], pp. 208-209).

Example 2.3.2 The Schödinger equation in quantum mechanics is the operator equation corresponding to the non-relativistic expression for the energy,

$$E = \frac{p^2}{2m}, \tag{2.31}$$

under the substitution $E \to i\frac{\partial}{\partial t}$, $p \to -i\nabla$. Our starting point is the relativistic energy-momentum equation,

$$E^2 = p^2 + m^2. \tag{2.32}$$

We may try to quantise the theory by replacing observable by the corresponding hermitian operators, which gives

$$-\frac{\partial^2}{\partial t^2}\phi(x,t) = -\triangle\phi(x,t) + m^2\phi(x,t), \tag{2.33}$$

where $\phi(x,t)$ is the wavefunction. This can be rewritten as

$$\left(\frac{\partial^2}{\partial t^2} - \triangle\right)\phi(x,t) + m^2\phi(x,t) = 0 \Rightarrow \left(\Box + m^2\right)\phi(x,t) = 0, \tag{2.34}$$

which is the *Klein-Gordon equation* for a free particle.

Example 2.3.3 In the study of the boundary value problems in elasticity one encounters the biharmonic equation $\triangle^2 u = 0$. Indeed, many plane problems of elasticity, when studied with the help of analytic functions, reduce to the solution of the two-dimensional biharmonic equation. Similarly, the discussion of the theory of elastic plates and shells leads to the three-dimensional biharmonic equation.

Example 2.3.4 In a long electrical cable or a telephone wire both the current and voltage depend upon position along the wire as well as the time. It is possible to show, using basic laws of electrical circuit theory, that the electrical current $i(x,t)$ satisfies the PDEs

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (RC + GL)\frac{\partial i}{\partial t} + RGi \tag{2.35}$$

where the R, L, C and G are, for unit length of cable, respectively the resistance, inductance, capacitance and leakage conductance. The voltage $v(x,t)$ also satisfies (2.35). Renaming some constants we get the *telegraph equation*

$$u_{tt} + (\alpha+\beta)u_t + \alpha\beta u = c^2 u_{xx} \tag{2.36}$$

where

$$c^2 = \frac{1}{LC}, \qquad \alpha = \frac{G}{C}, \qquad \beta = \frac{R}{L}.$$

2.4 Elementary Solutions

We shall mainly be interested in the equations where in the coefficients are constants. The theory of partial differential equations stems from the intensive and extensive study of a few basic equations of mathematical physics, and the coefficients in all of these are constants. Such equations arise in the study of gravitation, electromagnetism, perfect fluids, elasticity, heat transfer, and quantum mechanics. Of great importance in the study of these equations are their elementary solutions. Recall that a elementary solution $E(x)$ is a generalized function that satisfies the equation

$$LE(x) = \delta(x). \tag{2.37}$$

This solution is not unique, because we can add to it any solution of the homogeneous equation. This understood, in the sequel we shall select the elementary solution from among the particular solutions according to its behavior at infinity or other appropriate criteria. In the study of these solutions the following interesting concept is helpful. It is called Hadamard's method of descent:

Given the solution of a partial differential equation in $\mathbb{R}^{n+1}$, we can find its solution in $\mathbb{R}^n$ or in a still lower dimension. In doing so, we descend from the higher-dimensional problem to a lower-dimensional one. For instance, the solution of the initial value problem for the wave equation in two dimensions can be obtained from that in three dimensions. Specifically, let us consider a linear partial differential equation

$$L\left(D, \frac{\partial}{\partial x_{n+1}}\right) u = f(x) \otimes \delta(x_{n+1}), \tag{2.38}$$

in the space $\mathbb{R}^{n+1}$ of variable (x, x_{n+1}), where $x = (x_1, \ldots, x_n)$, D is $\partial/\partial x_j$, $j = 1, \ldots, n$, $f \in \mathcal{D}'(\mathbb{R}^n)$,

$$L\left(D, \frac{\partial}{\partial x_{n+1}}\right) u = \sum_{q=1}^{p} \frac{\partial^q}{\partial x_{n+1}} L_q(D) + L_0(D), \tag{2.39}$$

and $L_q(D)$ are partial differential operators involving the variables $x_1, \ldots, n$.

When we say that the generalized function $g \in \mathcal{D}'(\mathbb{R}^{n+1})$ allows the continuation over functions of the form $\varphi(x)1(x_{n+1})$ where $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we mean the following: Given an arbitrary sequence of functions $\psi_m(x_{n+1})$, $m = 1, 2, \ldots$, belonging to $\mathcal{D}(\mathbb{R}^1)$, where $\mathbb{R}^1$ is the space with variables X_{n+1} and converging to 1 in $\mathbb{R}^1$[i.e.$1(x_{n+1})$], then there is the limit

$$\lim_{m \to \infty} \langle g, \varphi(x)\psi_m(x_{n+1})\rangle = \langle g, \varphi(x)1(x_{n+1})\rangle = \langle g_0, \varphi\rangle \tag{2.40}$$

$\varphi \in \mathcal{D}(\mathbb{R}^n)$. In view of the completeness of $\mathcal{D}'$, we find that $g_0 \in \mathcal{D}'(\mathbb{R}^n)$.

Specifically, for $g(x)$ such that $g(x) = f(x) \otimes \delta(x_{n+1})$, the inhomogeneous term in (2.38), we have

$$\begin{aligned}
\langle g_0, \varphi\rangle &= \lim_{m \to \infty} \langle g(x), \varphi(x)\psi_m(x_{n+1})\rangle \\
&= \lim_{m \to \infty} \langle f(x) \otimes \delta(x_{n+1}), \varphi(x)\psi_m(x_{n+1})\rangle \\
&= \lim_{m \to \infty} \langle f(x), \delta(x_{n+1})\varphi(x)\psi_m(x_{n+1})\rangle \\
&= \lim_{m \to \infty} \langle f(x), \varphi(x)\psi_m(0)\rangle \\
&= \langle f(x), \varphi(x)\rangle, \qquad\qquad \varphi \in \mathcal{D}.
\end{aligned}$$

Accordingly, the method of descent can be stated as follows: If the solution $u \in \mathcal{D}'(\mathbb{R}^{n+1})$ of (2.37) allows the continuation (2.40), then the distribution $u_0 \in \mathcal{D}'(\mathbb{R}^n)$ is the solution of the equation

$$L_0(D)u_0 = f(x). \tag{2.41}$$

For instance, if the locally integrable function $E(x,t)$ is the elementary solution of the operator $L(D, \partial/\partial t)$, then the distribution

$$E_0(x) = \int_{-\infty}^{\infty} E(x,t)\, dt, \tag{2.42}$$

is the elementary solution of the operator L_0. Indeed, in view of the Lebesgue theorem on the passage of the limit under the integral sign, we have

$$\begin{aligned}
\lim_{m\to\infty} \langle E(x,t), \varphi(x)\psi_m(t)\rangle &= \lim_{m\to\infty} \int E(x,t)\varphi(x)\psi_m(t)dxdt \\
&= \int E(x,t)\varphi(x)dxdt \\
&= \int \varphi(x)\left(\int_{-\infty}^{\infty} E(x,t)dt\right) dx \\
&= \langle E_0(x), \varphi(x)\rangle,
\end{aligned}$$

where E_0 is defined in (2.42) and $\varphi \in \mathcal{D}$. Moreover, this limit does not depend on the sequence $\psi_m(t)$. Here $E_0(x)$ is the elementary solution of the operator L_0, as required.

Proposition 2.4.1 *Let* L *be the operator defined by*

$$\mathrm{L} = \frac{\partial}{\partial \mathrm{t}} - \mathrm{c}^2 \Diamond^{\mathrm{k}} \tag{2.43}$$

where $\Diamond^k$ is the Diamond operator iterated k-times defined by

$$\Diamond^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k,$$

$p+q=n$ is the dimension of $\mathbb{R}^n$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $t \in (0,\infty)$, k is a positive integer and c is a positive constant. Then we obtain

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right)^k t + i(\xi, x)\right] d\xi \tag{2.44}$$

as a elementary solution of (2.43) in the spectrum $\Omega \subset \mathbb{R}^n$ for $t > 0$.

Proof. Let $LE(x,t) = \delta(x,t)$ where $E(x,t)$ is the kernel or the elementary solution of operator L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t}E(x,t) - c^2\diamondsuit^k E(x,t) = \delta(x)\delta(t).$$

Take the Fourier transform defined by (2.21) to both sides of the equation, we obtain

$$\frac{\partial}{\partial t}\widehat{E(\xi,t)} - c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k \widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\delta(t).$$

Thus

$$\widehat{E(\xi,t)} = \frac{H(t)}{(2\pi)^{n/2}}\exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k t\right]$$

where $H(t)$ is the Heaviside function. Since $H(t) = 1$ for $t > 0$. Therefore,

$$\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k t\right]$$

which has been already defined by (2.20). Thus

$$\begin{aligned} E(x,t) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{E(\xi,t)}\,d\xi \\ &= \frac{1}{(2\pi)^{n/2}}\int_{\Omega} e^{i(\xi,x)}\widehat{E(\xi,t)}\,d\xi \end{aligned}$$

where Ω is the spectrum of $E(x,t)$. Thus from (2.20)

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\Omega}\exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k t + i(\xi,x)\right]d\xi$$

for $t > 0$. □

Lemma 2.4.2 ([9],[22]) *The functions $R_{2k}^H(v)$ and $(-1)^k R_{2k}^e(s)$ are the elementary solutions of the operator $\Box^k$ and $\triangle^k$, respectively, where $\Box^k$ and $\triangle^k$ are the operators iterated k-times defined by (1.6) and (1.2), respectively $R_{2k}^H(v)$ defined by (2.13) with $\alpha = 2k$ and $R_{2k}^e(s)$ defined by (2.15) with $\beta = 2k$.*

Lemma 2.4.3 ([9]) *The convolution $R_{2k}^H(v) * (-1)^k R_{2k}^e(s)$ is an elementary solution of the operator $\diamondsuit^k$ iterated k-times and is defined by (1.16).*

Lemma 2.4.4 *The functions $S_{2k}(w) * T_{2k}(z)$ is the elementary solutions of the operator $L^k = L_1^k L_2^k = L_2^k L_1^k$ denoted by,*

$$L^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \tag{2.45}$$

where $S_{2k}(w)$ and $T_{2k}(z)$ are defined by (2.16) and (2.17), respectively, with $\gamma = \nu = 2k$. The operator L_1^k and L_2^k are defined by (1.19) and (1.20), respectively.

Proof. We need to show that

$$L^k (S_{2k}(w) * T_{2k}(z)) = L_1^k \left((-1)^k(-i)^{q/2} S_{2k}(w)\right) * L_2^k \left((-1)^k(i)^{q/2} T_{2k}(z)\right) = \delta.$$

To prove this, see ([15], p. 223). □

Lemma 2.4.5 ([23]) *Given the equation*

$$(\Box + m^2)^k G(x) = \delta(x) \tag{2.46}$$

where $(\Box + m^2)^k$ is the ultra-hyperbolic Klein-Gordon operator iterated k times defined by (1.21) then

$$G(x) = W_{2k}^H(v, m)$$

is an elementary solution or Green function of (2.46) where $W_{2k}^H(v, m)$ is defined by (1.22) with $\alpha = 2k$.

Lemma 2.4.6 ([14],[16]) *Given the equation*

$$\triangle^k u(x) = 0, \tag{2.47}$$

we obtain $u(x) = (-1)^{k-1} \left(R_{2(k-1)}^e(s) \right)^{(l)}$ as a solution of (2.47) where l is a nonnegative integer with $l = (n-4)/2$, $n \geq 4$ and n is even and $\left(R_{2(k-1)}^e(s) \right)^{(l)}$ is a function defined by (2.15) with l derivatives and $\beta = 2(k-1)$.

Lemma 2.4.7 ([2]) *Given the equation*

$$\triangle u(x) = f(x, u(x)) \tag{2.48}$$

where f is defined and has continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ is the boundary of Ω. Assume that f is bounded on Ω that is $|f(x, u(x))| \leq N$ for all $x \in \Omega$. Then we obtain a continuous function $u(x)$ as a unique solution of (2.48) with the boundary condition $u(x) = 0$ for $x \in \partial\Omega$.

Lemma 2.4.8 ([10]) *Let*

$$(e^{\alpha t} \Box^k \delta) * u(t) = \delta \tag{2.49}$$

be the differential equation, where $u(t)$ is any tempered distribution. Then $u(t) = e^{\alpha t} R_{2k}^H(v)$ is a unique elementary solution of (2.49), where $R_{2k}^H(v)$ is defined by (2.13) with $\alpha = 2k$.