Chapter 3

Generalized Diamond Heat Kernel Related to the Spectrum

In this chapter, we find the solution of the generalized Diamond heat equation related to the spectrum and consider some properties of generalized Diamond heat kernel related to the spectrum.

3.1 Main Results

Theorem 3.1.1 *Given the equation*

$$\frac{\partial}{\partial t}u(x,t) = c^2 \diamondsuit^k u(x,t) \tag{3.1}$$

with the initial condition

$$u(x,0) = f(x) \tag{3.2}$$

where $\diamondsuit^k$ is the Diamond operator iterated k-times defined by

$$\diamondsuit^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k,$$

$p+q=n$ is the dimension of Euclidean space $\mathbb{R}^n$, k is a positive integer, $u(x,t)$ is an unknown function for $(x,t)=(x_1,x_2,\ldots,x_n,t)\in\mathbb{R}^n\times(0,\infty)$, $f(x)$ is the given generalized function, and c is a positive constant. Then we obtain

$$u(x,t) = E(x,t) * f(x)$$

as a solution of (3.1) which satisfies (3.2) where $E(x,t)$ is given by (2.44) and is called the generalized Diamond heat kernel of (3.1).

Proof. Taking the Fourier transform defined by (2.21) to both sides of (3.1), we obtain

$$\frac{\partial}{\partial t}\widehat{u}(\xi,t) = c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right]^k \widehat{u}(\xi,t).$$

Thus

$$\widehat{u}(\xi,t) = K(\xi)\exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k t\right] \tag{3.3}$$

where $K(\xi)$ is constant and $\widehat{u}(\xi,0) = K(\xi)$.

Now, by (3.2) we have

$$K(\xi) = \widehat{u}(\xi,0) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i(\xi,x)}f(x)\,dx \tag{3.4}$$

and by the inversion in (2.22), (3.3) and (3.4) we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{u}(\xi,t)\,d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\xi,x)}e^{-i(\xi,y)}f(y)\exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k t\right] dy\,d\xi. \end{aligned}$$

Thus

$$u(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\xi,x-y)}\exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k t\right] f(y)\,dy\,d\xi$$

or

$$u(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k t + i(\xi,x-y)\right] f(y)\,dy\,d\xi. \tag{3.5}$$

Set

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k t + i(\xi,x)\right] d\xi. \tag{3.6}$$

Since the integral of (3.6) is divergent, therefore we choose $\Omega \subset \mathbb{R}^n$ be the spectrum of $E(x,t)$ and by (2.44), we have

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k t + i(\xi,x)\right] d\xi$$
$$= \frac{1}{(2\pi)^n}\int_{\Omega} \exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k t + i(\xi,x)\right] d\xi. \tag{3.7}$$

$E(x,t)$ is called the diamond heat kernel of (3.1). Thus (3.5) can be written in the convolution form

$$u(x,t) = E(x,t) * f(x).$$

Since $E(x,t)$ exists, then

$$\begin{aligned}\lim_{t\to 0} E(x,t) &= \frac{1}{(2\pi)^n}\int_{\Omega} e^{i(\xi,x)}\, d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(\xi,x)}\, d\xi \\ &= \delta(x), \quad \text{for } x \in \mathbb{R}^n. \end{aligned} \tag{3.8}$$

See ([6], p. 396, Eq.(10.2.19(b))).

Thus for the solution $u(x,t) = E(x,t) * f(x)$ of (3.1), then

$$\begin{aligned}u(x,0) = \lim_{t\to 0} u(x,t) &= \lim_{t\to 0}\,(E(x,t) * f(x)) \\ &= \lim_{t\to 0} E(x,t) * f(x) \\ &= \delta * f(x) = f(x)\end{aligned}$$

which satisfies (3.2). □

Theorem 3.1.2 *The kernel $E(x,t)$ defined by (3.7) have the following properties:*

(1) $E(x,t) \in C^\infty$- the space of continuous function for $x \in \mathbb{R}^n$, $t > 0$ with infinitely differentiable.

(2) $\left(\frac{\partial}{\partial t} - c^2\Diamond^k\right) E(x,t) = 0$ for $t > 0$.

(3)

$$|E(x,t)| \le \frac{2^{2-n}}{\pi^{n/2}}\frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}, \quad \text{for } t > 0,$$

where $M(t)$ is a function of t in the spectrum Ω. Thus $E(x,t)$ is bounded for any fixed $t > 0$.

(4) $\lim_{t\to 0} E(x,t) = \delta$.

Proof.

(1) From (3.7), since

$$\frac{\partial^n}{\partial x^n} E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \frac{\partial^n}{\partial x^n}\exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k t + i(\xi,x)\right] d\xi.$$

Thus $E(x,t) \in \mathbb{C}^\infty$ for $x \in \mathbb{R}^n$, $t > 0$.

(2) By computing directly, we obtain

$$\left(\frac{\partial}{\partial t} - c^2\Diamond^k\right) E(x,t) = 0.$$

(3) We have

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k t + i(\xi,x)\right] d\xi.$$

$$|E(x,t)| \le \frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k t\right] d\xi.$$

By changing to bipolar coordinates

$\xi_1 = r\omega_1, \xi_2 = r\omega_2, \ldots, \xi_p = r\omega_p$ and $\xi_{p+1} = s\omega_{p+1}, \xi_{p+2} = s\omega_{p+2}, \ldots, \xi_{p+q} = s\omega_{p+q}$ where $\sum_{i=1}^{p}\omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q}\omega_j^2 = 1$. Thus

$$|E(x,t)| \le \frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2\left(r^4 - s^4\right)^k t\right] r^{p-1}s^{q-1}\,dr\,ds\,d\Omega_p\,d\Omega_q$$

where $d\xi = r^{p-1}s^{q-1}\,dr\,ds\,d\Omega_p\,d\Omega_q$, $d\Omega_p$ and Ω_q are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Since $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ and we suppose $0 \le r \le R$ and $0 \le s \le T$ where R and T are constants. Thus we obtain

$$\begin{aligned}|E(x,t)| &\le \frac{\Omega_p\,\Omega_q}{(2\pi)^n}\int_0^R\int_0^T \exp\left[c^2\left(r^4 - s^4\right)^k t\right] r^{p-1}s^{q-1}\,ds\,dr\\ &= \frac{\Omega_p\,\Omega_q}{(2\pi)^n}M(t) \quad \text{for any fixed } t > 0 \text{ in the spectrum } \Omega\\ &= \frac{2^{2-n}}{\pi^{n/2}}\frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}\end{aligned}$$

where

$$M(t) = \int_0^R \int_0^T \exp\left[c^2\left(r^4 - s^4\right)^k t\right] r^{p-1} s^{q-1}\, ds\, dr$$

is a function of t, $\Omega_p = \frac{2\pi^{p/2}}{\Gamma(\frac{p}{2})}$ and $\Omega_q = \frac{2\pi^{p/2}}{\Gamma(\frac{q}{2})}$. Thus, for any fixed $t > 0$, $E(x,t)$ is bounded.

(4) Obvious by (3.8). □

3.2 Example

In this section, we want to show an example the generalized Diamond heat kernel in $\mathbb{R}^2$.

Example 3.2.1 Consider the equation

$$\frac{\partial}{\partial t} u(x_1, x_2, t) = c^2 \left[\left(\frac{\partial^2}{\partial x_1^2}\right)^2 - \left(\frac{\partial^2}{\partial x_2^2}\right)^2\right]^3 u(x_1, x_2, t) \tag{3.9}$$

with the initial condition

$$u(x_1, x_2, 0) = f(x_1, x_2).$$

Taking the Fourier transform both sides of (3.9), we obtain

$$\frac{\partial}{\partial t}\hat{u}(\xi_1, \xi_2, t) = c^2\left[\xi_1^4 - \xi_2^4\right]^3 \hat{u}(\xi_1, \xi_2, t). \tag{3.10}$$

Thus

$$\hat{u}(\xi_1, \xi_2, t) = \hat{f}(\xi_1, \xi_2) \exp\left[c^2\left[\xi_1^4 - \xi_2^4\right]^3 t\right] \tag{3.11}$$

where $\hat{f}(\xi_1, \xi_2)$ is a constant and

$$\hat{u}(\xi_1, \xi_2, 0) = \hat{f}(\xi_1, \xi_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi_1 x_1, \xi_2 x_2)} f(x_1, x_2)\, dx_1 dx_2$$

From relation (2.22) its inverse transform is

$$u(x_1, x_2, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\xi_1 x_1, \xi_2 x_2)} \hat{u}(\xi_1, \xi_2, t)\, d\xi_1 d\xi_2 \tag{3.12}$$

or $u(x_1, x_2, t)$ equal to

$$\frac{1}{4\pi^2}\int\int\int\int e^{i(\xi_1 x_1, \xi_2 x_2)} e^{-i(\xi_1 y_1, \xi_2 y_2)} f(y_1, y_2) \exp\left[c^2\left\{\xi_1^4 - \xi_2^4\right]^3 t\right] dy_1 dy_2 d\xi_1 d\xi_2.$$

Let $\Gamma_+ = \{\xi_1^2 - \xi_2^2 > 0 \text{ and } \xi_1 > 0\}$ be the set of an interior of the forward cone and let Ω be the spectrum of $E(x_1, x_2, t)$ defined by definition 2.2.8 and $\Omega \subset \overline{\Gamma}_+$. Choose $(a, b) \subset \mathbb{R}(\xi_1)$, $(c, d) \subset \mathbb{R}(\xi_2)$ and $(a, b) \times (c, d) \subset \Omega$, we obtain

$$E(x_1, x_2, t) = \frac{1}{4\pi^2} \int_a^b \int_c^d \exp\left[c^2 \left[\xi_1^4 - \xi_2^4\right]^3 t + i(\xi_1 x_1, \xi_2 x_2)\right] d\xi_1 d\xi_2. \quad (3.13)$$

Then, we obtain the solution

$$\begin{aligned} u(x_1, x_2, t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x_1 - y_1, x_2 - y_2, t) f(y_1, y_2) dy_1 dy_2 \\ &= E(x_1, x_2, t) * f(x_1, x_2, t), \end{aligned}$$

where $E(x_1, x_2, t)$ is called the diamond heat kernel of (3.9).