Chapter 4

The Green Function of the $(\oplus + m^2)^k$ Operator

In this chapter, we study the elementary solution or Green function of the operator $(\oplus + m^2)^k$ and such function can be related to the ultra-hyperbolic Klein-Gordon operator, the Helmholtz operator and the Diamond operator of the form $(\diamondsuit + m^2)^k$.

4.1 Main Results

Theorem 4.1.1 *Given the equation*

$$(\oplus + m^2)^k G(x) = \delta(x) \tag{4.1}$$

where $(\oplus+m^2)^k$ is the operator iterated k-times defined by (1.18), δ is the Dirac-delta distribution, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer. Then we obtain $G(x) = Y_{2k,2k,2k,2k}(v, s, w, z, m)$,

$$Y_{2k,2k,2k,2k}(v,s,w,z,m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r!\Gamma(k)} (m^2)^r K_{2k+2r,2k+2r,2k+2r,2k+2r}(v,s,w,z) \tag{4.2}$$

as an elementary solution of (4.1) where m is a nonnegative real number and $K_{2k+2r,2k+2r,2k+2r,2k+2r}(v,s,w,z)$ is defined by (2.19) with $\alpha = \beta = \gamma = \nu = \eta = 2k$.

Proof. At first, the following formula is valid [1, p. 3, formula (2)],

$$\Gamma\left(\frac{\eta}{2} + r\right) = \frac{\eta}{2}\left(\frac{\eta}{2} + 1\right) \cdots \left(\frac{\eta}{2} + r - 1\right) \Gamma\left(\frac{\eta}{2}\right).$$

Equivalently,

$$(-1)^r \frac{1}{r!}\Gamma\left(\frac{\eta}{2}+r\right) = \frac{(-1)^r \frac{\eta}{2}\left(\frac{\eta}{2}+1\right)\cdots\left(\frac{\eta}{2}+r-1\right)\Gamma\left(\frac{\eta}{2}\right)}{r!}$$
$$= \frac{\left(-\frac{\eta}{2}\right)\left(-\frac{\eta}{2}-1\right)\cdots\left[-\left(\frac{\eta}{2}+r-1\right)\right]}{r!}\Gamma\left(\frac{\eta}{2}\right).$$

Now we put, by definition,

$$(-1)^r \frac{1}{r!}\Gamma\left(\frac{\eta}{2}+r\right) = \binom{-\frac{\eta}{2}}{r}\Gamma\left(\frac{\eta}{2}\right).$$

Then, we obtain the function $Y_{\alpha,\beta,\gamma,\nu}(v,s,w,z,m)$ is defined by (2.18) become

$$Y_{\alpha,\beta,\gamma,\nu}(v,s,w,z,m) = \sum_{r=0}^{\infty}\binom{-\frac{\eta}{2}}{r}(m^2)^r K_{\alpha+2r,\beta+2r,\gamma+2r,\nu+2r}(v,s,w,z). \tag{4.3}$$

Putting $\eta = 2k$ and $\alpha = \beta = \gamma = \nu = 2k$ in (4.3), we have

$$\begin{aligned} Y_{2k,2k,2k,2k}(v,s,w,z,m) &= \sum_{r=0}^{\infty}\binom{-k}{r}(m^2)^r K_{2k+2r,2k+2r,2k+2r,2k+2r}(v,s,w,z) \\ &= \sum_{r=0}^{\infty}\binom{-k}{r}(m^2)^r(-1)^{k+r}R^H_{2k+2r}(v) \\ &\quad * R^e_{2k+2r}(s) * S_{2k+2r}(w) * T_{2k+2r}(z). \end{aligned} \tag{4.4}$$

Since, the operator $\diamondsuit, L_1, L_2, \square$ and $\triangle$ are defined by (1.14), (1.19), (1.20), (1.6) and (1.2) respectively, are linearly continuous and 1-1 mapping. Then all of them possess their own inverses. From Lemma 2.4.2, 2.4.3 and 2.4.4, we obtain

$$\begin{aligned} Y_{2k,2k,2k,2k}(v,s,w,z,m) &= \sum_{r=0}^{\infty}\binom{-k}{r}(m^2)^r \square^{-k-r}\delta * \triangle^{-k-r}\delta * L_1^{-k-r}\delta * L_2^{-k-r}\delta \\ &= \sum_{r=0}^{\infty}\binom{-k}{r}(m^2)^r \diamondsuit^{-k-r} L^{-k-r}\delta \\ &= \sum_{r=0}^{\infty}\binom{-k}{r}(m^2)^r \oplus^{-k-r}\delta \\ &= (\oplus + m^2)^{-k}\delta. \end{aligned} \tag{4.5}$$

By applying the operator $(\oplus + m^2)^k$ to both sides of (4.5), we obtain

$$(\oplus + m^2)^k Y_{2k,2k,2k,2k}(v,s,w,z,m) = (\oplus + m^2)^k . (\oplus + m^2)^{-k}\delta. \tag{4.6}$$

Thus

$$(\oplus + m^2)^k Y_{2k,2k,2k,2k}(v,s,w,z,m) = \delta.$$

Moreover, from (4.6) by putting $\beta = \gamma = \nu = -2r$, we obtain

$$\begin{aligned} K_{\alpha+2r,0,0,0}(v,s,w,z) &= R^H_{\alpha+2r}(v) * R^e_0(s) * S_0(w) * T_0(z) \\ &= R^H_{\alpha+2r}(v) * \delta * \delta * \delta \\ &= R^H_{\alpha+2r}(v). \end{aligned}$$

Then (4.3) become

$$Y_{\alpha,-2r,-2r,-2r}(v,s,w,z,m) = \sum_{r=0}^{\infty} \binom{-\frac{\eta}{2}}{r} (m^2)^r R^H_{\alpha+2r}(v). \tag{4.7}$$

Now, putting $\alpha = \eta = 2k$ to obtain

$$\begin{aligned} Y_{2k,-2r,-2r,-2r}(v,s,w,z,m) &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r R^H_{2k+2r}(v) \\ &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \Box^{-k-r} \delta \\ &= (\Box + m^2)^{-k} \delta. \end{aligned} \tag{4.8}$$

By applying the operator $(\Box + m^2)^k$ to both sides of (4.8), we obtain

$$(\Box + m^2)^k Y_{2k,-2r,-2r,-2r}(v,s,w,z,m) = (\Box + m^2)^k (\Box + m^2)^{-k} \delta = \delta. \tag{4.9}$$

Then $Y_{2k,-2r,-2r,-2r}(v,s,w,z,m) = W^H_{2k}(v,m)$ as an elementary solution of the ultra-hyperbolic Klein-Gordon operator iterated k-times defined by (1.22). In particular, putting $\alpha = \gamma = \nu = -2r$ and $\beta = \eta = 2k$ of (4.3), we obtain

$$\begin{aligned} Y_{-2r,2k,-2r,-2r}(v,s,w,z,m) &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R^e_{2k+2r}(s) \\ &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \triangle^{-k-r} \delta \\ &= (\triangle + m^2)^{-k} \delta. \end{aligned} \tag{4.10}$$

By applying the operator $(\triangle + m^2)^k$ to both sides of (4.10), we obtain $Y_{-2r,2k,-2r,-2r}(v,s,w,z,m)$ is an elementary solution of Helmholtz operator k-times. Similarly,

$$Y_{2k,2k,-2r,-2r} = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R^H_{\alpha+2r}(v) * R^e_{2k+2r}(s)$$

is the Green function of the operator $(\Diamond + m^2)^k$. $\Box$

Theorem 4.1.2 *Given the equation*

$$(\oplus + m^2)^k U(x) = f(x) \tag{4.11}$$

where f is a given generalized function and $U(x)$ is an unknown function; we obtain the solution

$$U(x) = Y_{2k,2k,2k,2k}(v, s, w, z, m) * f(x) \tag{4.12}$$

where $Y_{2k,2k,2k,2k}(v, s, w, z, m)$ is defined by (4.4).

Proof. Convolving both sides of (4.11) by $Y_{2k,2k,2k,2k}(v, s, w, z, m)$ and applying the Theorem 3.1, we obtain (4.12) as required. □

4.2 Example

In this section, we want to show an example of the operator $(\oplus + m^2)^k U(x) = f(x)$.

Example 4.2.1 Consider the equation

$$(m + \triangle^2)^k (m - \triangle^2)^k u(x) = f(x) \tag{4.13}$$

where $\triangle^2$ is the biharmonic operator defined by (1.30), $x \in \mathbb{R}^n$, $f(x)$ is a given generalized function and $u(x)$ is an unknown function. For solving the product of biharmonic operators, we can rewrite the equation (4.13) as

$$(m^2 - \triangle^4)^k u(x) = f(x) \tag{4.14}$$

and we know that the operator in the equation (4.14) is the operator $(\oplus + m^2)^k$ with $p = 0$ and $q = n$. By the definition 2.2.7, we obtain the function $Y_{\alpha,\beta,\gamma,\nu}(v, s, w, z, m)$ where $v = -x_1^2 - x_2^2 - \cdots - x_n^2$, $s = x_1^2 + x_2^2 + \cdots + x_n^2$, $w = -i(x_1^2 + x_2^2 + \cdots + x_n^2)$ and $z = i(x_1^2 + x_2^2 + \cdots + x_n^2)$, $i = \sqrt{-1}$ and the function $K_n(\alpha)$ in (4.15) be reduce to

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma\left(\frac{2+\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha}{2}\right)\Gamma\left(\frac{-\alpha}{2}\right)}. \tag{4.15}$$

Convolving both sides of (4.14) by the new elementary solution $Y_{2k,2k,2k,2k}(v, s, w, z, m)$ then we obtain the solution.