Chapter 5

Nonlinear Product of the Diamond and Klein-Gordon Operators Related to the Biharmonic Equation

In this chapter, we study the solution of the equation $\diamondsuit^k(\Box + m^2)^k u(x) = f(x, \triangle^{k-1}\Box^k(\Box + m^2)^k u(x))$ where $\diamondsuit^k(\Box + m^2)^k$ is the product of the Diamond and Klein-Gordon operators. The existence of the solution $u(x)$ of such equation depending on the conditions of f and $\triangle^{k-1}\Box^k(\Box + m^2)^k u(x)$. Moreover such solution $u(x)$ related to the biharmonic equation depending on the conditions of p, q and k.

5.1 Main Results

Theorem 5.1.1 *Consider the non-linear equation*

$$\diamondsuit^k(\Box + m^2)^k u(x) = f(x, \triangle^{k-1}\Box^k(\Box + m^2)^k u(x)), \qquad (5.1)$$

where $\diamondsuit^k$, $(\Box + m^2)^k$, $\Box^k$ *and* $\triangle^{k-1}$ *are defined by (1.16), (1.21), (1.11) and (1.2) respectively. Let* f *be defined and having continuous first derivative for all* $x \in \Omega \cup \partial\Omega$, Ω *is an open subset of* $\mathbb{R}^n$ *and* $\partial\Omega$ *denotes the boundary of* Ω *and* n *is even with* $n \geq 4$. *Suppose* f *is bounded function, that is*

$$|f(x, \triangle^{k-1}\Box^k(\Box + m^2)^k u(x))| \leq N \qquad (5.2)$$

for all $x \in \Omega$ *and the boundary condition*

$$\triangle^{k-1}\Box^k(\Box + m^2)^k u(x) = 0 \qquad (5.3)$$

for all $x \in \partial\Omega$. *Then we obtain*

$$u(x) = (-1)^{k-1} R^e_{2(k-1)}(s) * R^H_{2k}(v) * W^H_{2k}(v, m) * U(x), \qquad (5.4)$$

as a solution of (5.1) with the boundary condition

$$u(x) = R^H_{2k}(v) * W^H_{2k}(v,m) * (-1)^{k-2}\left(R^e_{2(k-2)}(s)\right)^{(l)}, \tag{5.5}$$

for all $x \in \partial\Omega$, $l = (n-4)/2$, $U(x)$ is a continuous function for $x \in \Omega \cup \partial\Omega$, $R^H_{2k}(v)$, $W^H_{2k}(v,m)$ and $R^e_{2(k-2)}(s)$ are given by (2.13), (1.22) and (2.15) respectively with $\alpha = 2k$ and $\beta = 2(k-2)$. Moreover $V(x) = (-1)^{k-1}R^e_{2(1-k)}(s) * u(x)$ is a solution of the equation $\Box^k(\Box + m^2)^k V(x) = U(x)$, where $\Box^k$ and $(\Box + m^2)^k$ are defined by (1.11) and (1.21) respectively and $u(x)$ is obtained from (5.4). Furthermore, if we put $k = 1$, $p = n$ and $q = 0$, then $V(x)$ is a solution of the inhomogeneous biharmonic equation $\Delta^2 V(x) + m^2\Delta V(x) = U(x)$.

Proof. We have

$$\Diamond^k(\Box + m^2)^k u(x) = \Delta(\Delta^{k-1}\Box^k(\Box + m^2)^k u(x)) = f(x, \Delta^{k-1}\Box^k(\Box + m^2)^k u(x)). \tag{5.6}$$

Since $u(x)$ has continuous derivative up to order $4k$ for $k = 1, 2, 3, \ldots$, thus we can assume

$$\Delta^{k-1}\Box^k(\Box + m^2)^k u(x) = U(x), \tag{5.7}$$

for all $x \in \Omega$. Then (5.6) can be written in the form

$$\Diamond^k(\Box + m^2)^k u(x) = \Delta U(x) = f(x, U(x)), \tag{5.8}$$

by (5.2),

$$|f(x, U(x))| \leq N, \tag{5.9}$$

for all $x \in \Omega$ and by (5.3) $U(x) = 0$ or

$$\Delta^{k-1}\Box^k(\Box + m^2)^k u(x) = 0, \tag{5.10}$$

for all $x \in \partial\Omega$, we obtain a unique solution $U(x)$ of (5.8) which satisfies (5.10) by Lemma 2.4.6. Now consider the equation (5.7), we have $\Delta^{k-1}(-1)^{k-1}R^e_{2(k-1)}(s) = \delta$, $\Box^k R^H_{2k}(v) = \delta$ and $(\Box + m^2)^k W^H_{2k}(v,m) = \delta$ where δ is the Dirac-delta distribution, that is $R^H_{2k}(v)$, $(-1)^{k-1}R^e_{2(k-1)}(s)$ and $W^H_{2k}(v,m)$ are the elementary solution of the operators $\Box^k$, Δ^{k-1} and $(\Box + m^2)^k$ respectively, see [21, p. 147], [3, p. 118] and [22, pp. 121-141]. The function $R^H_{2k}(v)$, $R^e_{2(k-1)}(s)$ and $W^H_{2k}(v,m)$ are defined by (2.13), (2.15) and (1.22), respectively, with $\alpha = 2k$, $\beta = 2(k-1)$. Thus convolving both sides of (5.7) by $(-1)^{k-1}R^e_{2(k-1)}(s) * R^H_{2k}(v) * W^H_{2k}(v,m)$ and by the properties of convolution, we obtain

$$u(x) = (-1)^{k-1}R^e_{2(k-1)}(s) * R^H_{2k}(v) * W^H_{2k}(v,m) * U(x), \tag{5.11}$$

as required.

Consider $\triangle^{k-1}\Box^k(\Box+m^2)^k u(x) = 0$ for $x \in \partial\Omega$. By Lemma 2.4.6 we have

$$\Box^k(\Box+m^2)^k u(x) = (-1)^{k-2}\left(R^e_{2(k-2)}(s)\right)^{(l)} \tag{5.12}$$

where$l = (n-4)/2$, $n \geq 4$ and n is even. Convolving both sides of above equation by $R^H_{2k}(v) * W^H_{2k}(v,m)$, we obtain

$$u(x) = R^H_{2k}(v) * W^H_{2k}(v,m) * (-1)^{k-2}\left(R^e_{2(k-2)}(s)\right)^{(l)} \tag{5.13}$$

for $x \in \partial\Omega$.

Lastly, convolving both sides of (5.11) by $(-1)^{k-1}R^e_{2(1-k)}(s)$ we obtain

$$(-1)^{k-1}R^e_{2(1-k)}(s) * u(x) = R^H_{2k}(v) * W^H_{2k}(v,m) * U(x) \tag{5.14}$$

by Lemma 2.2.10. Thus by Lemma 2.4.2, 2.4.5 we obtain $V(x) = (-1)^{k-1}R^e_{2(1-k)}(s) * u(x)$ as a solution of the equation $\Box^k(\Box+m^2)^k V(x) = U(x)$. If we put $k=1$, $p=n$ and $q=0$ then the operators $\Box^k$ and $(\Box+m^2)^k$ reduced to $\triangle$ and $\triangle+m^2$ respectively and then the solution $V(x)$ is the solution of the inhomogeneous biharmonic equation

$$\triangle^2 V(x) = g(x, \triangle V(x)), \tag{5.15}$$

where $g(x, \triangle V(x)) = U(x) - m^2\triangle V(x)$. □

5.2 Example

Example 5.2.1 Consider the biharmonic equation

$$(\triangle^2 + m^2\triangle)^k u(x) = f(x, (\triangle+m^2)u(x)) \tag{5.16}$$

where f is defined and has continuous and first derivatives for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ is the boundary of Ω. Now, we assume f is bounded on Ω. We assume $(\triangle+m^2)u(x) = U(x)$ and by the Theorem 4.1.1,5.1.1, we obtain

$$u(x) = Y_{-2r,2,-2r,-2r}(v,s,w,z,m) * U(x)$$

as a solution of (5.16) with the boundary condition

$$u(x) = Y_{-2r,2,-2r,-2r}(v,s,w,z,m) * -\left(R^e_{-2}(s)\right)^{(l)}$$

for all $x \in \partial\Omega$, $l = (n-4)/2$, $U(x)$ is a continuous function for $x \in \Omega \cup \partial\Omega$.