Chapter 6

Application of the Distributions $e^{\alpha t}\Box^k\delta$ to the Telegraph Equation

In this chapter, we introduce a new technique for solving the telegraph equation by using the distribution $e^{\alpha t}\Box^k\delta$.

6.1 Main Results

Consider the Telegraph equation

$$T_1^k u(x,t) = \delta(x,t) \tag{6.1}$$

and

$$T_2^k u(x,t) = \delta(x,t) \tag{6.2}$$

where T_1^k and T_2^k are defined by (1.33) and (1.34), respectively, $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

Changing the independent variables from x_1 to t, where t is the time and putting $p = 1$ and $q = n$, then the n-dimensional to be $n+1$-dimensional and choosing $\alpha_2 = \alpha_3 \ldots = \alpha_n = 0$ of the distribution $e^{\alpha t}\Box^k\delta$, we obtain

$$e^{\alpha t}\Box^k\delta = e^{\alpha_1 t}\left(\frac{\partial^2}{\partial t^2} - \triangle\right)^k \delta(x,t), \tag{6.3}$$

where $\triangle$ is the Laplace operator defined by (1.2).

Next, we recalling the Legendre's duplication of $\Gamma(z)$,

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\frac{1}{2})$$

then the function (2.13) may reduces to the function

$$M_\gamma(v) = \begin{cases} \frac{v^{(\gamma-n-1)/2}}{H_{n+1}(\gamma)} & \text{if } t \in \Gamma_+, \\ 0 & \text{if } t \notin \Gamma_+ \end{cases} \tag{6.4}$$

where $v = t^2 - x_1^2 - \cdots - x_n^2$, t is the time and

$$H_{n+1}(\gamma) = \pi^{(n-1)/2} 2^{\gamma-1} \Gamma\left(\frac{\gamma - n + 1}{2}\right) \Gamma(\frac{\gamma}{2}),$$

the function $M_\gamma(s)$ is called the hyperbolic kernel of Marcel Riesz.

By the property of $\delta(x,t) = \delta(x)\delta(t)$ and using the Proposition 2.1.14 of $e^{\alpha t}\delta^{(k)} = \left(\frac{\partial}{\partial t} - \alpha\right)^k \delta$ and $e^{\alpha_1 t}\delta = \delta$ we can express (6.3) for $k = 1$ as

$$\begin{aligned} e^{\alpha_1 t}\left(\frac{\partial^2}{\partial t^2} - \triangle\right)\delta(x,t) &= e^{\alpha_1 t}\frac{\partial^2}{\partial t^2}\,\delta - e^{\alpha_1 t}\triangle\delta \\ &= \left(\frac{\partial}{\partial t} - \alpha_1\right)^2 \delta - \triangle e^{\alpha_1 t}\delta \\ &= \left(\frac{\partial^2}{\partial t^2} - 2\alpha_1\frac{\partial}{\partial t} + \alpha_1^2 - \triangle\right)\delta. \end{aligned}$$

Choosing $\alpha_1 = -\beta$, it follows that

$$e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} - \triangle\right)\delta(x,t) = \left(\frac{\partial^2}{\partial t^2} + 2\beta\frac{\partial}{\partial t} + \beta^2 - \triangle\right)\delta(x,t) = T_1\delta(x,t). \tag{6.5}$$

Now, convolution both sides of (6.5) by the distribution $e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} - \triangle\right)\delta(x,t)$, k-times we obtain

$$e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} - \triangle\right)\delta(x,t) * \cdots * e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} - \triangle\right)\delta(x,t) = T_1\delta * \cdots * T_1\delta. \tag{6.6}$$

Then we have

$$e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} - \triangle\right)^k \delta(x,t) = T_1^k\delta. \tag{6.7}$$

Solving the Telegraph equation (6.1) after substituting (6.7) in to (6.1) and using the Lemma 2.4.9, thus

$$T_1^k u(x,t) = e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} - \triangle\right)^k \delta(x,t) * u(x,t) = \delta(x,t), \tag{6.8}$$

convolving both side by $e^{-\beta t}M_{2k}(v)$ of (6.8), we obtain

$$u(x,t) = e^{-\beta t}M_{2k}(v), \tag{6.9}$$

the elementary solution of the Telegraph equation of the (6.1).

We next consider the distribution $e^{\alpha t}\Box^k\delta$ with $k = 1$, $\alpha_1 = -\beta$, $p = n+1$, $q = 0$ and $x_1 = t$. then we obtain

$$\begin{aligned} e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} + \triangle\right)\delta(x,t) &= e^{-\beta t}\frac{\partial^2}{\partial t^2}\,\delta + e^{-\beta t}\triangle\delta \\ &= \left(\frac{\partial}{\partial t} + \beta\right)^2\delta + \triangle e^{-\beta t}\delta \\ &= \left(\frac{\partial^2}{\partial t^2} + 2\beta\frac{\partial}{\partial t} + \beta^2 + \triangle\right)\delta. \end{aligned}$$

Similarly, convolving the distribution $e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} + \triangle\right)\delta(x,t)$, k-times we obtain the operator T_2^k as

$$e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} + \triangle\right)^k\delta(x,t) = T_2^k\delta, \tag{6.10}$$

where T_2^k is defined by (1.34), $(x,t) \in \mathbb{R}^n \times (0,\infty)$. By Legendre's duplication of $\Gamma(z)$ with $p = n+1$ and $q = 0$, the hyperbolic kernel reduce to the elliptic kernel of Marcel Riesz denoted by $R_\lambda(s) = s^{(\lambda-n-1)/2}/W_{n+1}(\lambda)$, where $s^2 = t^2 + x_1^2 + x_2^2 + \cdots + x_n^2$ and $W_{n+1}(\lambda) = \pi^{n+1/2}2^\lambda\Gamma(\lambda/2)/\Gamma((n+1-\lambda)/2)$. Solving the Telegraph equation (6.2) after substituting (6.10) in to (6.2) and using the Lemma 2.4.2, see also [11], we have

$$T_2^k u(x,t) = e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} + \triangle\right)^k\delta(x,t) * u(x,t) = \delta(x,t), \tag{6.11}$$

convolving both side by $e^{-\beta t}(-1)^k R_{2k}(s)$ of (6.11), we obtain

$$u(x,t) = e^{-\beta t}(-1)^k R_{2k}(s), \tag{6.12}$$

the elementary solution of the Telegraph equation of the (6.2) as required.

6.2 Example

In this section, we want to show an example about application of the distributions to the telegraph equation.

Example 6.2.1 We now to fine the elementary solution of the equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + 2\beta\frac{\partial}{\partial t}u(x,t) + \beta^2 u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = 0 \tag{6.13}$$

with the initial conditions

$$u(x,0)=0, \qquad \frac{\partial}{\partial t}u(x,0)=\delta(x) \tag{6.14}$$

where $\delta(x)$ is the Dirac-delta function.

Now, we consider the equation

$$\frac{\partial^2}{\partial t^2}u(x,t)+2\beta\frac{\partial}{\partial t}u(x,t)+\beta^2 u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t)=\delta(x)\delta(t). \tag{6.15}$$

Taking the Fourier transform to both side of the equation (6.15)

$$\frac{\partial^2}{\partial t^2}\hat{u}(\xi,t)+2\beta\frac{\partial}{\partial t}\hat{u}(\xi,t)+\beta^2\hat{u}(\xi,t)+\xi^2\hat{u}(\xi,t)=\delta(t). \tag{6.16}$$

Now consider the equation

$$\frac{\partial^2}{\partial t^2}\hat{u}(\xi,t)+2\beta\frac{\partial}{\partial t}\hat{u}(\xi,t)+\beta^2\hat{u}(\xi,t)+\xi^2\hat{u}(\xi,t)=0 \tag{6.17}$$

with the initial conditions

$$\hat{u}(\xi,0)=0, \qquad \frac{\partial}{\partial t}\hat{u}(\xi,0)=1.$$

Then we obtain $\hat{\phi}(\xi,t)$ as a solution of equation (6.17) where

$$\hat{\phi}(\xi,t)=\frac{e^{-\beta t}}{2\xi i}\left(\exp[i\xi t]-\exp[-i\xi t]\right).$$

Thus equation (6.16) has solution

$$\hat{u}(\xi,t)=H(t)\hat{\phi}(\xi,t)$$

where $H(t)$ a Heaviside function. Since the right hand side of the equation is tempered distribution, but they are not locally integrable. Thus we can not find the inverse Fourier transform $u(x,t)$. So, we define

$$\hat{\phi}^{\epsilon}(\xi,t)=e^{-\epsilon\xi}\hat{\phi}(\xi,t)$$

and

$$\hat{\phi}^{\epsilon}(\xi,t)\rightarrow\hat{\phi}(\xi,t) \qquad \text{uniformly as} \qquad \epsilon\rightarrow 0.$$

Thus

$$\hat{\phi}^{\epsilon}(\xi,t)=\frac{e^{-\beta t}}{2i}\left[\frac{e^{-(\epsilon-it)\xi}-e^{-(\epsilon+it)\xi}}{\xi}\right]$$

is locally integrable and then we can find the inverse Fourier transform $\phi(x,t)$.

Now

$$\hat{\phi}^{\epsilon}(\xi,t) = \frac{e^{-\beta t}}{2i}\int_{\epsilon-it}^{\epsilon+it} e^{-s\xi}\, ds.$$

Applying the inverse Fourier transform, we obtain

$$\begin{aligned}\phi^{\epsilon}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\hat{\phi}^{\epsilon}(\xi,t)\, d\xi \\ &= \frac{e^{-\beta t}}{4\pi i}\int_{\epsilon-it}^{\epsilon+it}\int_{-\infty}^{\infty} e^{i\xi x}e^{-s\xi}\, ds d\xi.\end{aligned}$$

By directly computation, we have

$$\int_{-\infty}^{\infty} e^{i\xi x}e^{-s\xi}\, d\xi = \frac{2s}{s^2+x^2}.$$

Thus

$$\begin{aligned}\phi^{\epsilon}(x,t) &= \frac{e^{-\beta t}}{4\pi i}\int_{\epsilon-it}^{\epsilon+it}\frac{2s}{s^2+x^2}\, ds \\ &= \frac{e^{-\beta t}}{4\pi i}\left(\ln[(\epsilon+it)^2+x^2] - \ln[(\epsilon-it)^2+x^2]\right).\end{aligned}$$

Let $\epsilon \to 0$, we obtain

$$\phi(x,t) = \frac{e^{-\beta t}}{4\pi i}\left(\ln[(x^2-t^2)+0i] - \ln[(x^2-t^2)-0i]\right).$$

Since, we have

$$\ln[(x^2-t^2)+0i] - \ln[(x^2-t^2)-0i] = 2\pi i H(t^2-x^2).$$

Thus, we obtain

$$\phi(x,t) = \frac{e^{-\beta t}}{2}H(t^2-x^2).$$

Thus equation (6.13) has

$$E(x,t) = H(t)\phi(x,t) = \frac{e^{-\beta t}}{2}H(t^2-x^2)H(t)$$

as elementary solution.

Now, we using the distribution $e^{\alpha t}\square^k\delta$ for solve this problem. Recalling the solution of the telegraph equation (6.9) of the telegraph operator (6.1) with $k = 1$, we have

$$u(x,t) = e^{-\beta t}M_2(v).$$

Putting $v = t^2 - x^2$, $t > 0$ and $n = 1$, we obtain

$$u(x,t) = \frac{e^{-\beta t}}{2}$$

as elementary solution.