CHAPTER IV

INTRACTABILITY

In the previous chapter we have shown that UCP and SRP are both in *P*. In this section we deal with the intractability of the third problem on the list. We show that APLP cannot be solved in polynomial time, unless *P* = *NP* [18].

4.1 (u,v)-Hamiltonian Path on a Grid

One known *NP*-complete problem is called the Hamiltonian Path Problem between two vertices on grid graphs. This problem can be formally stated as follows:

(u,v)-Hamiltonian Path on a Grid (UVHAMG)

INSTANCE: Graph $G = (V, E)$ on an integer grid and $u, v \in V$.

QUESTION: Does *G* contain a Hamiltonian path between vertices u and v?

This *NP*-completeness result regarding grid graphs is due to Itai, Papadimitriou, and Szwarcfiter [15].

4.2 Gadgets

A set of gadgets that will be used in the proof of APLP are presented as follows. There are six groups of gadgets that are created for representing sources and obstacles in the APLP according to points (vertices) and lines (edges) in UVHAMG in the two-dimensional grid. Five groups of gadgets represent all possible patterns of degree of points in UVHAMG. Each group of gadgets corresponds to points that have degrees one, two in pattern one, two in pattern two, three, and four. These types of gadgets are shown in Figures 4.1 to 4.5, respectively. In Figure 4.6 the last group of gadgets represents the pattern of sources and obstacles according to an edge in UVHAMG.

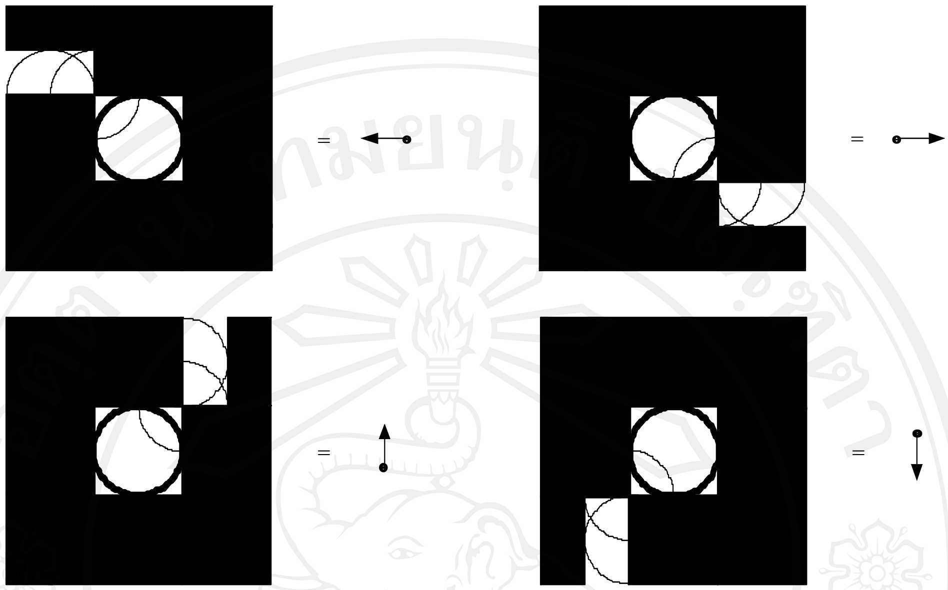

Figure 4.1 Gadgets in group 1 represents group of points that has degree one.

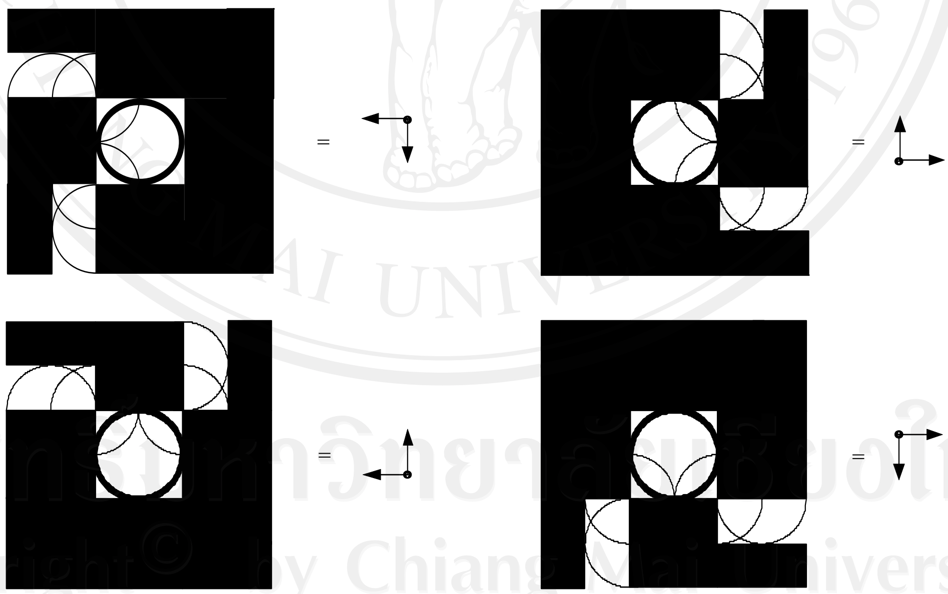

Figure 4.2 Gadgets in group 2 represents group of points that has degree two in pattern one.

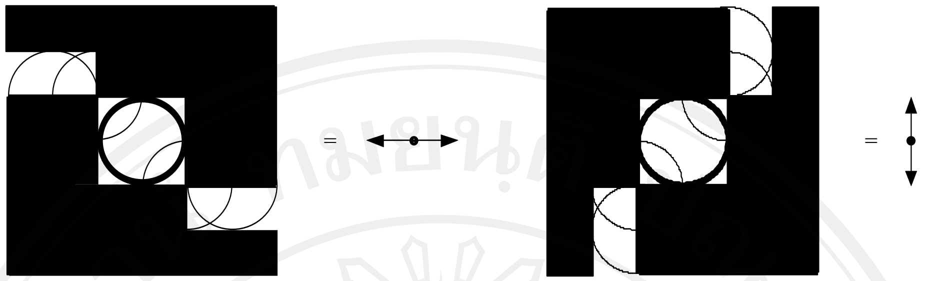

Figure 4.3 Gadgets in group 3 represents group of points that has degree two in pattern two.

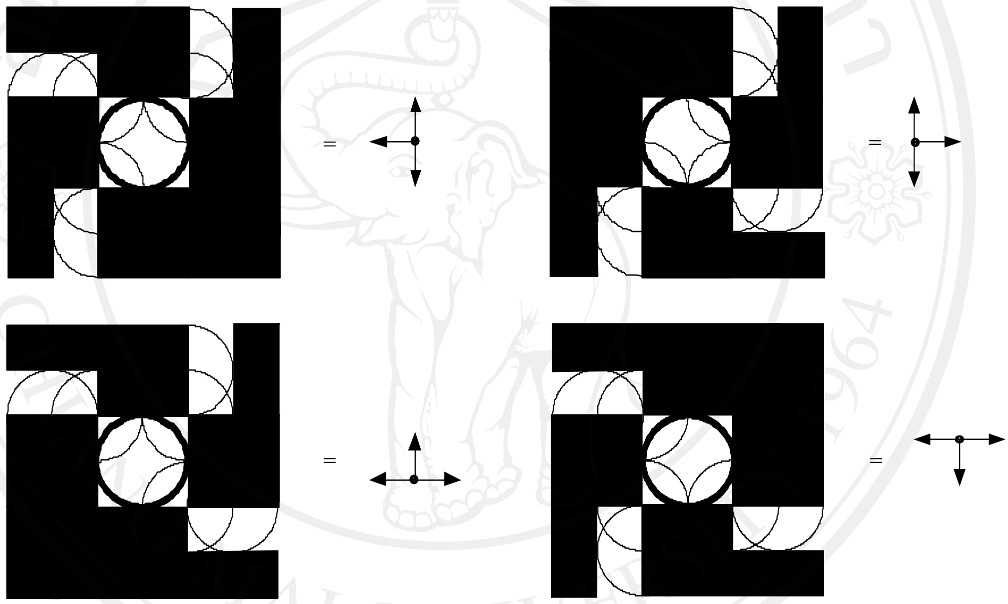

Figure 4.4 Gadgets group 4 represents group of points that has degree three.

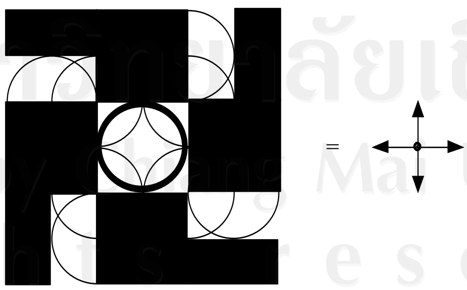

Figure 4.5 Gadgets group 5 represents group of points that has degree four.

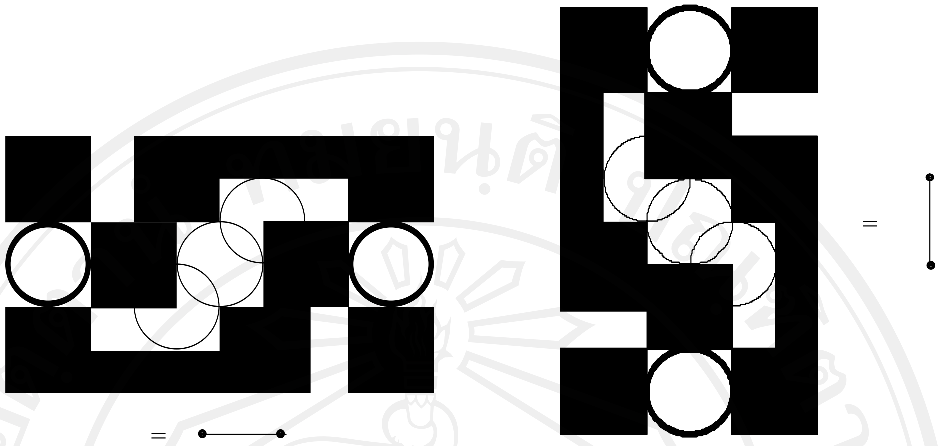

Figure 4.6 Gadgets group 6 represents pattern of sources and obstacles according to an edge in UVHAMG.

4.3 Time Complexity for the Access Point Location Problem

Theorem 4.1: The Access Point Location Problem (APLP) in a two-dimensional grid is *NP*-hard.

Proof: We make a reduction from UVHAMG. Let $S = V \cup S'$, where $S' = \{s'_1, s'_2, \ldots, s'_{m'}\}$ is a set of additional sources between a pair of sources in V and $m' = 7|E|$. Let $U = \{u_a = u, u_b = v\}$, $d = 2$, and $k = 8(|V| - 1) + 1$. There is no movement of both sources and users. The construction of obstacles will be described shortly. We assign users u_a and u_b the appropriate coordinates corresponding to the two sources u and v. Observe that we know coordinates of each source and therefore each coordinate of the additional sources is also known. The construction begins by laying gadget group 6 for each corresponding edge of the graph G from left to right and from top to bottom. At this point we have all the edges. We next want to appropriately lay gadgets group 1 to 5 over the sources that correspond to the vertices of the edges. Note that a gadget may need to be rotated correctly. Next remove all the sources and we have a mobility instance. It is not hard to see that this construction can be done in polynomial time.

We now argue that there is a (u, v)-Hamiltonian path on the grid if and only if users u_a and u_b can communicate throughout the duration of the model if $8(|V| - 1) + 1$ nonredundant sources of diameter 2 are placed appropriately in the grid and each source is accessed exactly once. Suppose there is a (*u, v*)-Hamiltonian path $(u, x_1, x_2, \ldots, x_{|v|-2}, v)$ on the grid. Observe that in the construction the number of sources in S' between a pair of sources in *V* is restricted by the obstacles to be *exactly* 7. Therefore, for the (*u, v*)-Hamiltonian path $(u, x_1, x_2, \ldots, x_{|v|-2}, v)$, we also have the corresponding communication path $(u_a, s'_1, \ldots, s'_7, x_1, s'_8, \ldots, s'_{14}, x_2, \ldots, x_{|v|-2}, s'_{f-6}, \ldots, s'_f, u_b)$ between users u_a and u_b, where $f = 7(|V|-1)$. Conversely, suppose we have the communication path $((u_a, s'_1, \ldots, s'_7, x_1, s'_8, \ldots, s'_{14}, x_2, \ldots, x_{|v|-2}, s'_{f-6}, \ldots, s'_f, u_b)$ between users u_a and u_b and this path has $8(|V|-1)+1$ nonredundant sources, where each source is accessed exactly once. Because $u_a = u$ and $u_b = v$ and by construction each edge is represented by 7 overlapping sources, the number of the corresponding vertices $|V|$ is precisely $(8(|V|-1)+1)-(7(|V|-1))$. By construction, the communication between users u_a and u_b must be made through these $|V|$ sources. Therefore, each corresponding vertex must be visited exactly once. By removing all s'_i in the communication path and replacing u_a, u_b by *u*, *v* respectively, we have the corresponding Hamiltonian path $(u, x_1, x_2, \ldots, x_{|v|-2}, u)$ between vertices *u* and *v*. This completes the proof.■

4.4 Example of Reducing UVHAMG into APLP

An example of an instance, graph G, for UVHAMG, is given in Figure 4.7.

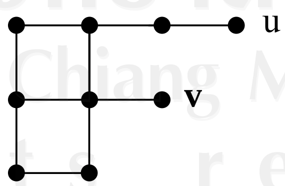

Figure 4.7 Example of an instance, graph $G = (V, E)$ on an integer grid, for UVHAMG.

Now we construct an instance for APLP from the instance in Figure 4.7. First, we lay a gadget from group 6 for each corresponding edge of the graph G from left to right and from top to bottom, let a thick circle lay at the same position, and assign users u_a and u_b the appropriate coordinates corresponding to the two sources u and v. The result is shown in Figure 4.8.

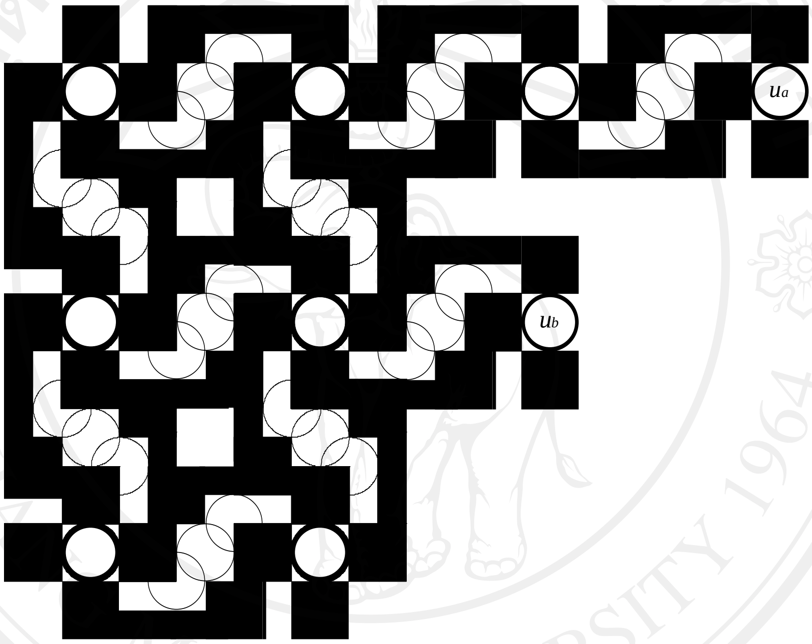

Figure 4.8 The result of laying gadget 6 and assigning u_a and u_b corresponding to the graph G in Figure 4.7.

Next, we appropriately lay gadgets group 1 to 5 over the sources that correspond to the vertices of the edges and let thick circle lay at the same position. The results are shown in Figure 4.9.

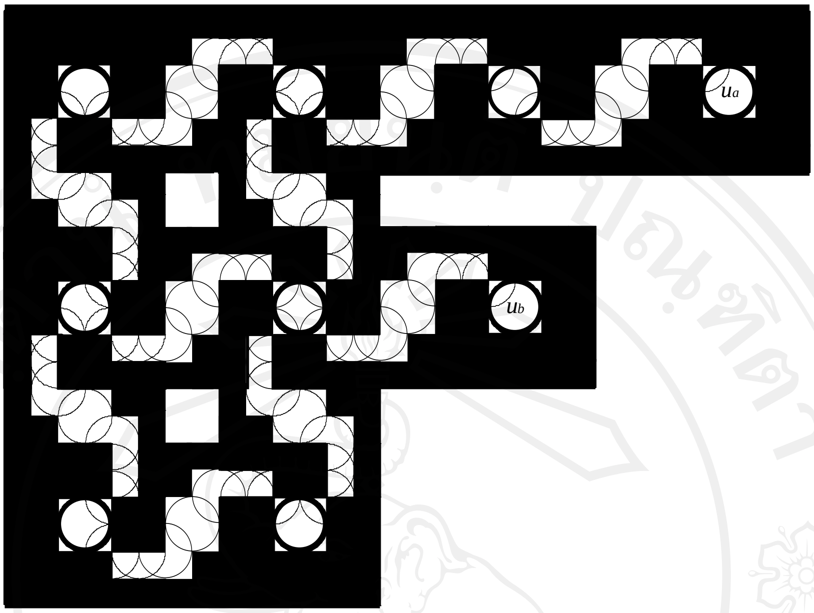

Figure 4.9 Example of an intermediate constructed instance.

After removal of all the sources except u_a and u_b, we have a mobility instance, as shown in Figure 4.10.

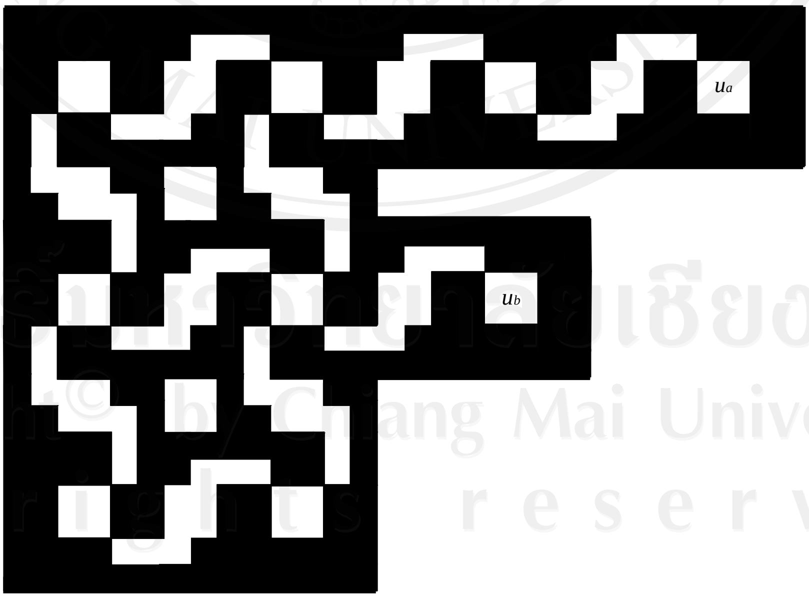

Figure 4.10 Example of a mobility instance.

We argue that there is a (u, v)-Hamiltonian path on the grid if and only if users u_a and u_b can communicate throughout the duration of the model if $k = 8(|V| - 1) + 1$ nonredundant sources of diameter $d = 2$ are placed appropriately in the grid and each source is accessed exactly once. Such a path is shown in Figure 4.11.

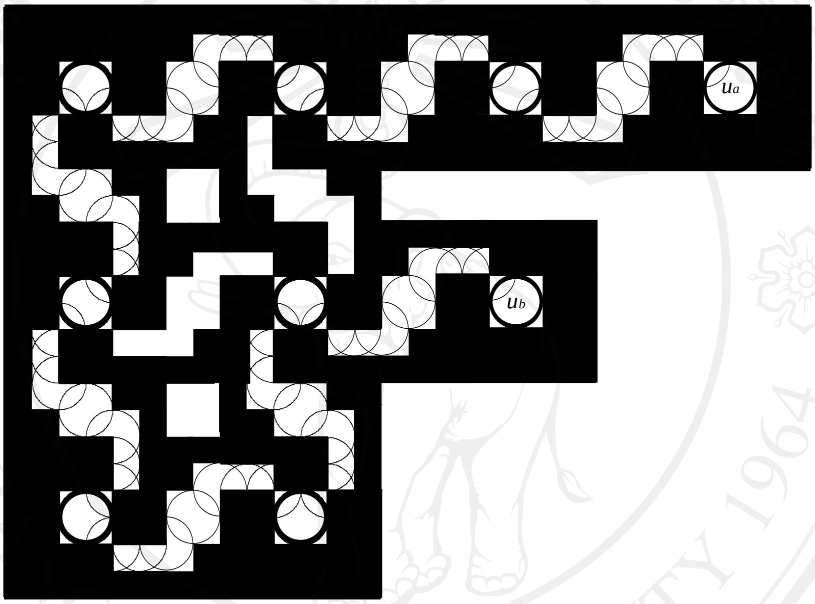

Figure 4.11 A YES instance of APLP.