CHAPTER 2

PRELIMINARIES

This chapter is essentially introductory in nature. Its remain purpose is to present some basic concepts from the theory of delay differential equations and to sketch some preliminary results which will be used throughout the thesis.

Definition of Oscillation

Before we define oscillation of solutions, let us consider some simple examples.

Example 1: The equation

$$y'' + y = 0$$

Has the periodic solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$.

Example 2: Consider the equation

$$y''(t) + \frac{1}{t} y'(t) + 4t^2 y(t) = 0$$

whose solution is $y(t) = \sin t^2$. This solution is not periodic but has an oscillatory properties.

Example 3: Consider the equation

$$y''(t) - y'(-t) = 0$$

This equation has an oscillatory solution $y_1(t) = \sin t$ and a nonoscillatory solution $y_2(t) = e^t + e^{-t}$.

Let us now restrict our discussion to those solutions y of equation

$$y''(t) + a(t) y(t - \tau(t)) = 0$$

which exist on some $[T_0, \infty)$ and satisfied $\sup\{|y(t)| : t > T\} > 0$ for every $T \geq T_0$. In other word, $|y(t)| \neq 0$ on any infinite interval . Such a solution sometimes is said to be a regular solution. We usually assume that $a(t) \geq 0$ and in doing so we mean to imply that $a(t) \neq 0$ on any infinite interval $[T_0, \infty)$.

There are various possibilities of defining oscillation of solutions of ODEs(with or without delay) . In this section, we give a definition of oscillation, which is used in this thesis; this is the one most frequently used in the literature.

Definition: A nontrivial solution y (implying a regular solution always) is said to be oscillatory if it has a arbitrarily large zeros for $t \geq t_0$, that is, there exists a sequence of zeros $\{t_n\}$(i.e.,(t_n)=0) of y such that $\lim_{n\to\infty} t_n = 0$. Otherwise, y is said to be nonoscillatory.

For nonoscillatory solutions there exist t_1 such that $y(t) \neq 0$ for all $t \geq t_1$

Definition of Increasing and Decreasing Functions

Definition : Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be continuous function. Then, for $x_1, x_2 \in I$

(1) function *f* is said to be increasing on I if $x_1 \leq x_2$ then $f(x_1) \leq f(x_2)$,

(2) function *f* is said to be strictly increasing on I if $x_1 < x_2$ then, $f(x_1) < f(x_2)$

(3) function *f* is said to be decreasing on I if $x_1 \leq x_2$ then $f(x_1) \geq f(x_2)$,

(4) function *f* is said to be strictly decreasing on I if $x_1 < x_2$ then $f(x_1) > f(x_2)$.

Theorem Let $f : I \to \mathbb{R}$ be differentiable on the interval I. Then

(1) *f* is increasing on I if and only if $f'(x) \geq 0$ for all $x \in I$,

(2) *f* is decreasing on I if and only if $f'(x) \leq 0$ for all $x \in I$.

Corollary Let $f : I \to \mathbb{R}$ be differentiable on the interval I. Then

(1) if f' is positive on I, then *f* is strictly increasing on I,

(2) if f' is negative on I, then *f* is strictly increasing on I.

The completeness property of $\mathbb{R}$

Definition 1: Let *S* be a nonempty subset of $\mathbb{R}$.

(1) An element $u \in \mathbb{R}$ is said to be an upper bound of *S* if $s \leq u$ for all $s \in S$.

(2) An element $w \in \mathbb{R}$ is said to be a lower bound of *S* if $w \leq s$ for all $s \in S$.

Definition 2: Let *S* be a nonempty subset of $\mathbb{R}$.

(1) If *S* is bounded above, then an upper bound of *S* is said to be a supremum (or a least upper bound) of *S* if it is less than any other upper bound of *S*.

(2) If *S* is bounded below, then an lower bound of *S* is said to be a infimum (or a greatest lower bound) of *S* if it is greater than any other lower bound of *S*.

Theorem 1: (Supremum Property)

Every nonempty set of real number which has an upper bound has a supremum.

Theorem 2: (Infimum Property)

Every nonempty set of real number which has an lower bound has an infimum.

Definition 3: Let $X = (x_n)$ be a bounded sequence in $\mathbb{R}$.

(1) The limit superior of X, which we denote by

$$\limsup X, \limsup (x_n) \text{ or } \overline{\lim}(x_n),$$

is the infimum of the set V of $v \in \mathbb{R}$ such that there are at most a finite number of $n \in N$ such that $v < x_n$.

(2) The limit inferior of X, which we denote by

$$\liminf X, \liminf (x_n) \text{ or } \underline{\lim}(x_n),$$

is the supremum of the set W of $w \in \mathbb{R}$ such that there are at most a finite number of $m \in N$ such that $x_m < w$.

Theorem 3: If $X=(x_n)$ is bounded sequence in $\mathbb{R}$, then the following statements are equivalent for a real number x^*.

(1) $x^*=\limsup(x_n)$.

(2) If $\varepsilon>0$, then there are at most a finite number of $n\in N$ such that $x^*+\varepsilon<x_n$, but there are an finite number such that $x^*-\varepsilon<x_n$.

(3) If $v_m=\sup\{x_n : n\geq m\}$, then $x^*=\inf\{v_m : n\in N\}$.

(4) If $v_m=\sup\{x_n : n\geq m\}$, then $x^*=\lim(v_m)$.

(5) If L is the set of $v\in\mathbb{R}$ such that there exists a subsequence of X which converges to v, then $x^*=\sup L$.

Theorem 4: Let $X=(x_n)$ and $Y=(y_n)$ be bounded sequences of real numbers. Then the following relations hold.

(1) $\liminf(x_n)\leq\limsup(x_n)$.

(2) If $c\geq 0$, then $\liminf(cx_n)=c\liminf(x_n)$ and $\limsup(cx_n)=c\limsup(x_n)$.

(3) If $c\leq 0$, then $\liminf(cx_n)=c\limsup(x_n)$ and $\limsup(cx_n)=c\liminf(x_n)$.

(4) $\liminf(x_n)+\liminf(y_n)\leq\liminf(x_n+y_n)$.

(5) $\limsup(x_n+y_n)\leq\limsup(x_n)+\limsup(y_n)$.

(6) If $x_n\leq y_n$, then $\liminf(x_n)\leq\liminf(y_n)$ and $\limsup(x_n)\leq\limsup(y_n)$.

Theorem 5: (Monotone Convergent Theorem)

(1) Let $X=(x_n)$ be a sequence of real numbers which is monotone increasing in the sense that

$$x_1\leq x_2\leq\ldots\leq x_n\leq\ldots.$$

Then the sequence X converges if and only if it is bounded in which case

$$\lim(x_n)=\sup\{x_n\}.$$

(2) Let $X=(x_n)$ be a sequence of real numbers which is monotone decreasing in the sense that

$$x_1\geq x_2\geq\ldots\geq x_n\geq\ldots.$$

Then the sequence X converges if and only if it is bounded in which case

$$\lim(x_n) = \inf\{x_n\}.$$

The Fundamental Theorem of Calculus

Theorem 1: (The First Fundamental Theorem of Calculus)

Let $f:[a,b] \to \mathbb{R}$ be integrable on [a,b] and let $F:[a,b] \to \mathbb{R}$ satisfied the conditions:

(1) F is continuous on [a,b],

(2) the derivative F' exists and $F'(x)= f(x)$ for all $x \in (a,b)$.

Then $$\int_a^b f\,dx = F(x) - F(b).$$

Corollary 1: Let $F:[a,b] \to \mathbb{R}$ satisfy the conditions:

(1) the derivative F' exists on [a,b],

(2) the function F' is integrable on [a,b].

Then $\int_a^b f\,dx = F(x) - F(b)$ holds with $f = F'$.

Theorem 2: (The Second Fundamental Theorem of Calculus)

Let $F:[a,b] \to \mathbb{R}$ be integrable on [a,b] and let

$$F(x)=\int_a^x f\,dx \text{ for } x \in [a,b];$$

Then F is continuous on [a,b]. More over, if *f is* continuous at a point $c \in [a,b]$, then F is differentiable at c and

$$F'(c)= f(c).$$

Corollary 2: Let $F:[a,b] \to \mathbb{R}$ be integrable on [a,b] and let

$$F(x)=\int_a^x f\,dx \text{ for } x \in [a,b].$$

Then F is differentiable on [a,b] and $F'(x)= f(x)$ for all $x \in [a,b]$.

Theorem 3: (The Combined Fundamental Theorem of Calculus)

Let F and f be continuous functions on [a,b] and let F(a)=0. Then the following statements are equivalent:

(1) $F'(x)=f(x)$ for all $x \in [a,b]$,

(2) $F(x)=\int_a^x f\,dx$ for all $x \in [a,b]$.

Definition 1: Let I = [a,b] be an interval in $\mathbb{R}$.

(1) If $f : I \to \mathbb{R}$, then an antiderivative of f on I is a function $F : I \to \mathbb{R}$, such that $F'(x)=f(x)$ for all $x \in [a,b]$.

(2) If $f : I \to \mathbb{R}$ is integrable on I, then the function $F : I \to \mathbb{R}$ define by

$$F(x)=\int_a^x f\,dx \text{ for all } x \in I$$

is called the indefinite integral of f on I.

Theorem 4: (Integration by Parts)

If $f,g : [a,b] \to \mathbb{R}$ are integrable on [a,b] and have antiderivative F,G on [a,b], then

$$\int_a^b F(x)g(x)dx = (F(b)G(b)\text{-}F(a)G(a)) - \int_a^b f(x)G(x)dx.$$

Theorem 5: (First Substitution Theorem)

Let $J=[\alpha,\beta]$ and let $\varphi : J \to \mathbb{R}$ have a continuous derivative on J. If f is continuous on an interval I containing $\varphi(J)$, then

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx.$$

Theorem 6: (Second Substitution Theorem)

Let $J=[\alpha,\beta]$ and let $\varphi : J \to \mathbb{R}$ have a continuous derivative such that $\varphi'(t) \neq 0$ for $t \in J$. Let I be an interval containing $\varphi(J)$, and let $\psi : I \to \mathbb{R}$ be the function inverse to φ. If $f : I \to \mathbb{R}$ is continuous on I, then

$$\int_\alpha^\beta f(\varphi(t))dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\psi'(x)dx.$$

The Riccati Equation

If the substitution $z = \frac{r(t)u'(t)}{u(t)}$ is made in the self-adjoint differential equation

$$(r(t)u'(t))' + q(t)u(t) = 0$$

where r(t) and q(t) are continuous on an interval [a,b], we obtaib

$$(z(t)u(t))' + q(t)u(t) = 0,$$

or

$$z'(t) + \frac{1}{r(t)} z^2(t) + q(t) = 0. \tag{1}$$

Equation (1) is a Riccati Equation. The general Riccati equation is usually written as

$$z'(t) + a(t)z(t) + z^2(t) + c(t) = 0, \tag{2}$$

where we shall suppose a(t), b(t) and c(t) are continuous on the interval [a,b]. Equation (2) is only apparently more general than equation (1), since the substitution in (2) of

$$w(t) = e^{\int_a^t a(s)ds} z(t) \tag{3}$$

reduces this equation to

$$w'(t) + q(t)w^2(t) + p(t) = 0, \tag{4}$$

where $q(t) = b(t)e^{-\int_a^t a(s)ds}$ and $p(t) = c(t)e^{\int_a^t a(s)ds}$.

If $b(t) = 0$, equation (2) is, of course, linear and it is immediately integrable. If $b(t) \neq 0$ on any subinterval of [a,b], to study the solution of (2) we may employ the substitution (3) to reduce (2) to the form (4). The substitution $q(t)w(t) = \frac{u'(t)}{u(t)}$ then reduces (4) to the form (1), where $r(t) = \frac{1}{q(t)}$. The zero of $q(t)$ are the singular points of the differential equation (1). It will be observed that these successive substitutions may be replaced by the substitution $b(t)z(t) = \frac{u'(t)}{u(t)}$.

<u>Example</u> 3: Study the solutions of the Riccati equation

$$w'(t) - w^2(t) - 1 = 0.$$

This equation is already in the form (4), where $q(t) = -1$ and $p(t) = -1$. The substitution $-w(t) = \frac{u'(t)}{u(t)}$ leads then to the linear self-adjoint differential equation.

$$u''(t) + u(t) = 0,$$

the general solution of which is $c_1 \sin t + c_2 \cos t$. The null solution ($c_1 = c_2 = 0$) leads to no solution w. All other solutions u provide solutions

$$w(t) = \frac{c_1 \cos t - c_2 \sin t}{c_1 \sin t + c_2 \cos t} \quad (c_1 \text{ and } c_2 \text{ not both zero}).$$

of above equation. The choice $c_1 = 0$ leads to the particular solution $w(t) = \tan t$.