

CHAPTER 3 MAIN RESULTS

We concerned with the oscillatory behavior properties of the second order nonlinear delay differential equation with damping term in two forms. So we will discuss each form in 2 parts.

Part 1

We started with the second order nonlinear delay differential equation with damping term of the form

$$[r(t)k_1(x(t), x'(t))] + p(t)k_2(x(t), x'(t))x'(t) + q(t)f(x(t)) = 0 \quad (1)$$

where $r \in C^1([t_0, \infty); (0, \infty))$, $p, q \in C([t_0, \infty); \mathbb{R})$, $k_1 \in C^1(\mathbb{R}^2; \mathbb{R}^2)$, $k_2 \in C(\mathbb{R}^2; \mathbb{R}^2)$, $f \in C(\mathbb{R}; \mathbb{R})$ and $xf(x) > 0$, $x \neq 0$. Throughout this work we suppose that the following conditions hold:

$$(C_1) \quad vk_1(u, v) \geq \alpha_1 k_1^2(u, v), \text{ for some constant } \alpha_1 > 0 \text{ and } (u, v) \in \mathbb{R}^2,$$

$$(C_2) \quad vk_2(u, v) \geq \alpha_2 k_1(u, v), \text{ for some constant } \alpha_2 > 0 \text{ and } (u, v) \in \mathbb{R}^2.$$

Using general weight functions from the class W , introduced by Philos [3].

Let $D_0 = \{(t, s) : t > s \geq t_0\}$ and $D = \{(t, s) : t \geq s \geq t_0\}$. The function $H \in C(D; \mathbb{R})$ is said to belong to the class W if

$$(i) \quad H(t, t) = 0 \text{ for } t \geq t_0 \text{ and } H(t, s) > 0 \text{ for } (t, s) \in D_0.$$

$$(ii) \quad H \text{ has a continuous nonpositive partial derivative on } D_0 \text{ with respect to the}$$

second variable such that

$$\frac{\partial}{\partial s} H(t, s) = -h(t, s) \sqrt{H(t, s)} \text{ for } (t, s) \in D_0 \quad (2)$$

where h is a nonnegative and continuous function on D .

Theorem 1: Let (C_1) and (C_2) hold. Assume that $f'(x)$ exists and

$$f'(x) \geq \mu \quad (3)$$

for some $\mu > 0$ and for all $x \neq 0$. Suppose that there exists a function $g \in C^1([t_0, \infty); \mathbb{R})$ such that for some $\beta \geq 1$ and $H \in W$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left(H(t, s) \psi(s) - \frac{\beta}{4\mu\alpha_1} v(s)r(s)(h(t, s))^2 \right) ds = \infty \quad (4)$$

where

$$v(t) = \exp\left[-2\mu\alpha_1 \int_{t_0}^t \left(g(s) - \frac{\alpha_2 p(s)}{2\mu\alpha_1 r(s)} \right) ds\right] \quad (5)$$

and

$$\psi(t) = v(t)[q(t) + \mu\alpha_1 r(t)g^2(t) - \alpha_2 p(t)g(t) - (r(t)g(t))']. \quad (6)$$

Then Equation (1) is oscillatory.

Proof: Let $x(t)$ be a nonoscillatory solution of Equation (1). Then there exists a $T \geq t_0$ such that $x(t) \neq 0$ for all $t \geq T$. Without loss of generality, we may assume that $x(t) > 0$ for all $t \geq T$.

Define

$$u(t) = v(t)r(t) \left[\frac{k_1(x(t), x'(t))}{f(x(t))} + g(t) \right] \quad (7)$$

where $v(t)$ is given by (5). Differentiating (7), we have

$$\begin{aligned} u'(t) &= v'(t)r(t) \left[\frac{k_1(x(t), x'(t))}{f(x(t))} + g(t) \right] + v(t) \left[\frac{r(t)k_1(x(t), x'(t))}{f(x(t))} + r(t)g(t) \right]' \\ &= \frac{v'(t)}{v(t)} u(t) + v(t) \left[\frac{(r(t)k_1(x(t), x'(t)))'}{f(x(t))} - \frac{r(t)k_1(x(t), x'(t))f'(x(t))x'(t)}{(f(x(t)))^2} + (r(t)g(t))' \right] \\ &= \frac{v'(t)}{v(t)} u(t) + v(t) \left[\frac{-p(t)k_2(x(t), x'(t))}{f(x(t))} - q(t) - \frac{r(t)k_1(x(t), x'(t))f'(x(t))x'(t)}{(f(x(t)))^2} + (r(t)g(t))' \right] \\ &\leq \frac{v'(t)}{v(t)} u(t) - \alpha_2 \frac{p(t)v(t)k_1(x(t), x'(t))}{f(x(t))} - \mu\alpha_1 \frac{v(t)r(t)k_1^2(x(t), x'(t))}{(f(x(t)))^2} + v(t)[(r(t)g(t))' - q(t)] \\ &\leq \frac{v'(t)}{v(t)} u(t) + v(t)[(r(t)g(t))' - q(t)] - \alpha_2 p(t)v(t) \left[\frac{u(t)}{v(t)r(t)} - g(t) \right] - \mu\alpha_1 v(t)r(t) \left[\frac{u(t)}{v(t)r(t)} - g(t) \right]^2 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{v'(t)}{v(t)}u(t) + v(t)\left[(r(t)g(t))' - q(t)\right] - \alpha_2 \frac{p(t)u(t)}{r(t)} + \alpha_2 v(t)p(t)g(t) \\
&\quad - \mu\alpha_1 v(t)r(t) \left[\left(\frac{u(t)}{v(t)r(t)} \right)^2 - 2 \frac{g(t)u(t)}{v(t)r(t)} + (g(t))^2 \right] \\
&\leq \frac{v'(t)}{v(t)}u(t) + v(t)\left[(r(t)g(t))' - q(t) + \alpha_2 p(t)g(t)\right] - \alpha_2 \frac{p(t)u(t)}{r(t)} - \mu\alpha_1 \frac{(u(t))^2}{v(t)r(t)} + 2\mu\alpha_1 g(t)u(t) \\
&\quad - \mu\alpha_1 v(t)r(t)(g(t))^2 \\
&\leq \left[-2\mu\alpha_1 g(t) + \alpha_2 \frac{p(t)}{r(t)} \right] u(t) - \alpha_2 \frac{p(t)}{r(t)} u(t) + 2\mu\alpha_1 g(t)u(t) - \mu\alpha_1 \frac{(u(t))^2}{v(t)r(t)} \\
&\quad - v(t)\left[q(t) + \mu\alpha_1 r(t)(g(t))^2 - \alpha_2 p(t)g(t) - (r(t)g(t))' \right] \\
&= -\psi(t) - \frac{\mu\alpha_1}{v(t)r(t)}(u(t))^2
\end{aligned}$$

where $\psi(t)$ is defined by (6). Hence, we obtain

$$\psi(t) \leq -u'(t) - \frac{\mu\alpha_1}{v(t)r(t)}(u(t))^2. \quad (8)$$

Multiplying (8) by $H(t,s)$ and integrating from T to t , we have for some $\beta \geq 1$

$$\begin{aligned}
\int_T^t H(t,s)\psi(s)ds &\leq -\int_T^t H(t,s)u'(s)ds - \int_T^t \mu\alpha_1 \frac{H(t,s)}{v(s)r(s)}(u(s))^2 ds \\
&= -H(t,s)u(s)\Big|_T^t - \int_T^t \left[\frac{\partial}{\partial s} H(t,s)u(s) + \mu\alpha_1 \frac{H(t,s)}{v(s)r(s)}(u(s))^2 \right] ds \\
&= H(t,T)u(T) - \int_T^t \left[h(t,s)\sqrt{H(t,s)u(s)} + \mu\alpha_1 \frac{H(t,s)}{v(s)r(s)}(u(s))^2 \right] ds \\
&= H(t,T)u(T) - \int_T^t \left[\sqrt{\frac{\mu\alpha_1 H(t,s)}{\beta v(s)r(s)}}u(s) + \sqrt{\frac{\beta v(s)r(s)}{4\mu\alpha_1}}h(t,s) \right]^2 ds \\
&\quad + \frac{\beta}{4\mu\alpha_1} \int_T^t v(s)r(s)(h(t,s))^2 ds - \int_T^t \frac{\mu\alpha_1(\beta-1)}{\beta v(s)r(s)}H(t,s)(u(s))^2 ds.
\end{aligned}$$

Writing the latter inequality in the form

$$\int_T^t \left[H(t,s)\psi(s) - \frac{\beta}{4\mu\alpha_1} v(s)r(s)(h(t,s))^2 \right] ds \leq H(t,T)u(T) - \int_T^t \frac{\mu\alpha_1(\beta-1)}{\beta v(s)r(s)} H(t,s)(u(s))^2 ds$$

$$-\int_{\tau}^t \left[\sqrt{\frac{\mu\alpha_1 H(t,s)}{\beta v(s)r(s)}} u(s) + \sqrt{\frac{\beta v(s)r(s)}{4\mu\alpha_1}} h(t,s) \right]^2 ds. \quad (9)$$

Using the property of $H(t,s)$, we conclude from (19) that, for all $t \geq T$

$$\int_{\tau}^t \left[H(t,s)\psi(s) - \frac{\beta}{4\mu\alpha_1} v(s)r(s)(h(t,s))^2 \right] ds \leq H(t,T)|u(T)| \\ \leq H(t,t_0)|u(T)|.$$

Hence, for $t \geq T \geq t_0$

$$\int_{\tau}^t \left[H(t,s)\psi(s) - \frac{\beta}{4\mu\alpha_1} v(s)r(s)(h(t,s))^2 \right] ds \\ \leq H(t,T)|u(T)| + \int_{t_0}^T \left[H(t,s)\psi(s) - \frac{\beta}{4\mu\alpha_1} v(s)r(s)(h(t,s))^2 \right] ds \\ \leq H(t,t_0)|u(T)| + \int_{t_0}^T H(t,s)\psi(s) ds \\ \leq H(t,t_0)|u(T)| + H(t,t_0) \int_{t_0}^T |\psi(s)| ds \\ \leq H(t,t_0) \left[|u(T)| + \int_{t_0}^T |\psi(s)| ds \right]. \quad (10)$$

By (10) we get

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)\psi(s) - \frac{\beta}{4\mu\alpha_1} v(s)r(s)(h(t,s))^2 \right] ds \leq |u(T)| + \int_{t_0}^T |\psi(s)| ds$$

which contradicts (4). Therefore, we have proved that all solutions of Equation (1) are oscillatory.

Theorem 2: Let (C_1) , (C_2) and (3) hold. Suppose that

$$0 < \inf_{s \geq t_0} \left[\liminf_{t \rightarrow \infty} \frac{H(t,s)}{H(t,t_0)} \right] \leq \infty \quad (11)$$

Assume that there exist function $g \in C^1([t_0, \infty); \mathbb{R})$ and $\phi \in C([t_0, \infty); \mathbb{R})$ such that for all $t \geq T \geq t_0$ and for some $\beta \geq 1$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{\tau}^t \left[H(t, s) \psi(s) - \frac{\beta}{4\mu\alpha_1} v(s)r(s)(h(t, s))^2 \right] ds \geq \phi(T) \quad (12)$$

where $v(t), \psi(t)$ are as in Theorem 1, and suppose further that

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \frac{(\phi_+(s))^2}{v(s)r(s)} ds = \infty \quad (13)$$

where $\phi_+(t) = \max\{\phi(t), 0\}$. Then Equation (1) is oscillatory.

Proof : We proceed as in the proof of Theorem 1, assuming, without loss of generality, that there exists a solution $x(t)$ of Equation (1) such that $x(t) > 0$ on $[T, \infty)$, for some $T \geq t_0$.

Introducing the function $u(t)$ through (7), where $v(t)$ is defined by (5), we arrive at the inequality (9), which yields

$$\begin{aligned} & \frac{1}{H(t, T)} \int_{\tau}^t \left(H(t, s) \psi(s) - \frac{\beta}{4\mu\alpha_1} v(s)r(s)h^2(t, s) \right) ds \\ & \leq u(T) - \frac{1}{H(t, T)} \int_{\tau}^t \frac{\mu\alpha_1(\beta-1)}{\beta v(s)r(s)} H(t, s) u^2(s) ds - \frac{1}{H(t, T)} \int_{\tau}^t \left[\sqrt{\frac{\mu\alpha_1 H(t, s)}{\beta v(s)r(s)}} u(s) + \sqrt{\frac{\beta v(s)r(s)}{4\mu\alpha_1}} \right]^2 ds \\ & \leq u(T) - \frac{1}{H(t, T)} \int_{\tau}^t \frac{\mu\alpha_1(\beta-1)}{\beta v(s)r(s)} H(t, s) u^2(s) ds. \end{aligned}$$

Therefore, for $t > T \geq t_0$, we have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{\tau}^t \left[H(t, s) \psi(s) - \frac{\beta}{4\mu\alpha_1} v(s)r(s)(h(t, s))^2 \right] ds \\ & \leq u(T) - \liminf_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{\tau}^t \frac{\mu\alpha_1(\beta-1)}{\beta v(s)r(s)} H(t, s) (u(t))^2 ds. \end{aligned}$$

It follows from (12) that

$$\phi(T) \leq u(T) - \liminf_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_{\tau}^t \frac{\mu\alpha_1(\beta-1)}{\beta v(s)r(s)} H(t, s) (u(t))^2 ds.$$

That is,

$$u(T) \geq \phi(T) + \liminf_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_{\tau}^t \frac{\mu\alpha_1(\beta-1)}{\beta v(s)r(s)} H(t, s) (u(t))^2 ds$$

Consequently, for all $t > T \geq T_0$

$$u(T) \geq \phi(T) \quad (14)$$

and

$$\liminf_{t \rightarrow \infty} \frac{1}{H(t, T_0)} \int_{T_0}^t \frac{H(t, s)}{v(s)r(s)} (u(t))^2 ds \leq \frac{\beta}{\mu \alpha_1 (\beta - 1)} [u(T_0) - \phi(T_0)] < \infty \quad (15)$$

In order to prove that

$$\int_{T_0}^{\infty} \frac{(u(s))^2}{v(s)r(s)} ds < \infty. \quad (16)$$

Suppose the contrary, that is,

$$\int_{T_0}^{\infty} \frac{(u(s))^2}{v(s)r(s)} ds = \infty. \quad (17)$$

Assumption (11) implies existence of a $\rho > 0$ such that

$$\inf_{s \geq T_0} \left[\liminf_{t \rightarrow \infty} \frac{H(t, s)}{H(t, t_0)} \right] > \rho. \quad (18)$$

By (18), we have

$$\liminf_{t \rightarrow \infty} \frac{H(t, s)}{H(t, t_0)} > \rho > 0,$$

and there exists a $T_2 \geq T_1$ such that $\frac{H(t, T_1)}{H(t, t_0)} \geq \rho$ for all $t \geq T_2$. On the other hand, by

virtue of (17), for any positive number k , there exists a $T_1 > T_0$ such that for all $t \geq T_1$,

$$\int_{T_0}^t \frac{u^2(s)}{v(s)r(s)} ds \geq \frac{k}{\rho}.$$

Using integration by part, we conclude that for all $t \geq T_0$,

$$\begin{aligned} & \frac{1}{H(t, T_0)} \int_{T_0}^t \frac{H(t, s)}{v(s)r(s)} u^2(s) ds \\ &= \frac{1}{H(t, T_0)} \int_{T_0}^t H(t, s) \cdot d \left(\int_{T_0}^s \frac{(u(\tau))^2}{v(\tau)r(\tau)} d\tau \right) \\ &= \frac{1}{H(t, T_0)} \left[H(t, s) \int_{T_0}^s \frac{(u(\tau))^2}{v(\tau)r(\tau)} d\tau \Big|_{s=T_0}^t - \int_{T_0}^t \left(\frac{\partial H(t, s)}{\partial S} \right) \left(\int_{T_0}^s \frac{(u(\tau))^2}{v(\tau)r(\tau)} d\tau \right) ds \right] \\ &= \frac{1}{H(t, T_0)} \int_{T_0}^t \left(-\frac{\partial}{\partial S} H(t, s) \right) \left(\int_{T_0}^s \frac{(u(\tau))^2}{v(\tau)r(\tau)} d\tau \right) ds \\ &\geq \frac{k}{p} \frac{1}{H(t, T_0)} \int_{T_0}^t \left(-\frac{\partial}{\partial S} H(t, s) \right) ds \end{aligned}$$

$$\begin{aligned}
&\geq \frac{k}{\rho} \frac{1}{H(t, T_0)} \left[-H(t, s) \right]_{s=T_0}^t \\
&\geq \frac{k}{\rho} \frac{H(t, T_1)}{H(t, T_0)}.
\end{aligned} \tag{19}$$

It follows from (19) that, for all $t \geq T_0$,

$$\frac{1}{H(t, T_0)} \int_{T_0}^t H(t, s) \frac{u^2(s)}{v(s)r(s)} ds \geq \frac{k}{\rho} \frac{H(t, T_1)}{H(t, T_0)} \geq \frac{k}{\rho} \cdot \rho = k$$

Since k is an arbitrary positive constant,

$$\liminf_{t \rightarrow \infty} \frac{1}{H(t, T_0)} \int_{T_0}^t H(t, s) \frac{u^2(s)}{v(s)r(s)} ds = +\infty$$

which contradicts (15). Consequently, (16) holds and by virtue of (14),

$$\int_{T_0}^{\infty} \frac{\Phi_+^2(s)}{v(s)r(s)} ds \leq \int_{T_0}^{\infty} \frac{u^2(s)}{v(s)r(s)} ds < +\infty$$

This contradicts (1). Therefore, Equation (1) is oscillatory.

Theorem 3: Let $(C_1) - (C_2)$, (3) and (11) hold. Suppose that there exist functions $g \in C^1([t, \infty), \mathbb{R})$ and $\phi \in C([t_0, \infty), \mathbb{R})$ such that for all $t \geq T \geq t_0$ and for some $\beta > 1$,

$$\liminf_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t \left[H(t, s) \psi(s) - \frac{\beta}{4\mu\alpha_1} v(s)r(s)(h(t, s))^2 \right] ds \geq \phi(T) \tag{20}$$

where $v(t)$, $\psi(t)$ and $\phi(t)$ are the same as in Theorem 2. Assume also that (13) holds.

Then Equation (1) is oscillatory.

Proof: By virtue of

$$\begin{aligned}
\phi(T) &\leq \liminf_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t \left[H(t, s) \psi(s) - \frac{\beta}{4\mu\alpha_1} v(s)r(s)(h(t, s))^2 \right] ds \\
&\leq \limsup_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t \left[H(t, s) \psi(s) - \frac{\beta}{4\mu\alpha_1} v(s)r(s)(h(t, s))^2 \right] ds,
\end{aligned}$$

the conclusion follows immediately from Theorem 2.

Theorem 4: Let (C_1) and (C_2) hold. Suppose that $f(x)$ satisfies

$$\frac{f(x)}{x} \geq \mu \quad (21)$$

for some $\mu > 0$ and for all $x \neq 0$. Assume that there exists a function $g \in C^1([t_0, \infty), \mathbb{R})$ such that for some $\beta \geq 1$ and $H \in W$,

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[H(t, s) \psi(s) - \frac{\beta}{4\alpha_1} v(s) r(s) (h(t, s))^2 \right] ds = \infty \quad (22)$$

where

$$v(t) = \exp \left[-2\alpha_1 \int_{t_0}^t \left(g(s) - \frac{\alpha_2 p(s)}{2\alpha_1 r(s)} \right) ds \right] \quad (23)$$

and

$$\psi(t) = v(t) \left[\mu q(t) + \alpha_1 r(t) (g(t))^2 - \alpha_2 p(t) q(t) - (r(t) g(t))' \right] \quad (24)$$

Then Equation (1) is oscillatory.

Proof: Suppose on the contrary that there is a nonoscillatory solution $x(t)$ of Equation (1).

Without loss of generality, we may assume that $x(t) > 0$ on $[T_0, \infty)$ for some sufficiently large $T \geq t_0$.

Define

$$u(t) = v(t) r(t) \left[\frac{k_1(x(t), x'(t))}{x(t)} + g(t) \right] \quad (25)$$

where $v(t)$ is given by (2). Differentiating (25), we obtain

$$\begin{aligned} u'(t) &= v'(t) r(t) \left[\frac{k_1(x(t), x'(t))}{x(t)} + g(t) \right] + v(t) \left[\frac{r(t) k_1(x(t), x'(t))}{x(t)} + r(t) g(t) \right]' \\ &= \frac{v'(t)}{v(t)} u(t) + v(t) \left[\frac{(r(t) k_1(x(t), x'(t)))'}{x(t)} - \frac{r(t) k_1(x(t), x'(t)) x'(t)}{(x(t))^2} + (r(t) g(t))' \right] \\ &= \frac{v'(t)}{v(t)} u(t) + v(t) \left[-\frac{p(t) k_2(x(t), x'(t)) x'(t)}{x(t)} - q(t) \frac{f(x(t))}{x(t)} - \frac{r(t) k_1(x(t), x'(t)) x'(t)}{(x(t))^2} + (r(t) g(t))' \right] \\ &\leq \frac{v'(t)}{v(t)} u(t) + v(t) \left[-\alpha_2 \frac{p(t) k_1(x(t), x'(t))}{x(t)} - \mu q(t) - \alpha_1 \frac{r(t) (k_1(x(t), x'(t)))^2}{(x(t))^2} + (r(t) g(t))' \right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{v'(t)}{v(t)}u(t) - \alpha_2 p(t)v(t) \left[\frac{u(t)}{v(t)r(t)} - g(t) \right] - \alpha_1 v(t)r(t) \left[\frac{u(t)}{v(t)r(t)} - g(t) \right]^2 + v(t) \left[-\mu q(t) + (r(t)g(t))' \right] \\
&\leq \frac{v'(t)}{v(t)}u(t) - \alpha_2 \frac{p(t)}{r(t)}u(t) + \alpha_2 v(t)p(t)g(t) + v(t) \left[-\mu q(t) + (r(t)g(t))' \right] \\
&\quad - \alpha_1 v(t)r(t) \left[\frac{(u(t))^2}{(v(t)r(t))^2} - \frac{2g(t)u(t)}{v(t)r(t)} + (g(t))^2 \right] \\
&\leq \left[-2\alpha_1 g(t) + \alpha_2 \frac{p(t)}{r(t)} \right] u(t) - \alpha_2 \frac{p(t)}{r(t)}u(t) - \alpha_1 \frac{(u(t))^2}{v(t)r(t)} + 2\alpha_1 g(t)u(t) \\
&\quad - v(t) \left[\mu q(t) + \alpha_1 r(t)(g(t))^2 - \alpha_2 p(t)g(t) - (r(t)q(t))' \right] \\
&\leq -\psi(t) - \alpha_1 \frac{(u(t))^2}{v(t)r(t)}
\end{aligned}$$

where $\psi(t)$ is defined by (24). Hence, we have

$$\psi(t) \leq -u'(t) - \alpha_1 \frac{(u(t))^2}{v(t)r(t)} \quad (26)$$

Multiplying (26) by $H(t,s)$ and integrating from T to t , we get for some $\beta \geq 1$ and $t \geq T \geq t_0$,

$$\begin{aligned}
&\int_T^t H(t,s)\psi(s)ds \leq -\int_T^t H(t,s)u'(s)ds - \int_T^t \alpha_1 \frac{H(t,s)}{v(s)r(s)}(u(s))^2 ds \\
&= -H(t,s)u(s) \Big|_T^t - \int_T^t \left[-\frac{\partial}{\partial s} H(t,s)u(s) + \alpha_1 \frac{H(t,s)}{v(s)r(s)}(u(s))^2 \right] ds \\
&= H(t,T)u(T) - \int_T^t \left[h(t,s)\sqrt{H(t,s)u(s)} + \alpha_1 \frac{H(t,s)}{v(s)r(s)}(u(s))^2 \right] ds \\
&= H(t,T)u(T) - \int_T^t \left[\sqrt{\frac{\alpha_1 H(t,s)}{\beta v(s)r(s)}}u(s) + \sqrt{\frac{\beta v(s)r(s)}{4\alpha_1}}h(t,s) \right] ds + \frac{\beta}{4\alpha_1} \int_T^t v(s)r(s)(h(t,s))^2 ds \\
&\quad - \int_T^t \frac{\alpha_1(\beta-1)H(t,s)}{\beta v(s)r(s)}(u(s))^2 ds.
\end{aligned}$$

Writing the latter inequality in the form

$$\begin{aligned}
&\int_T^t \left[H(t,s)\psi(s) - \frac{\beta}{4\alpha_1} v(s)r(s)(h(t,s))^2 \right] ds \leq H(t,T)u(T) - \int_T^t \frac{\alpha_1(\beta-1)(u(s))^2}{\beta v(s)r(s)} ds \\
&\quad - \int_T^t \left[\sqrt{\frac{\alpha_1 H(t,s)}{\beta v(s)r(s)}}u(s) + \sqrt{\frac{\beta v(s)r(s)}{4\alpha_1}}h(t,s) \right] ds \quad (27)
\end{aligned}$$

Using the properties of $H(t,s)$, we conclude from (27), for all $t \geq T_0$

$$\int_{T_0}^t \left[H(t,s)\psi(s) - \frac{\beta}{4\alpha_1} v(s)r(s)(h(t,s))^2 \right] ds \leq H(t,T_0)|u(T_0)|$$

$$\leq H(t,t_0)|u(T_0)|.$$

Hence, for all $t \geq T_0$, we get

$$\int_{T_0}^t \left[H(t,s)\psi(s) - \frac{\beta}{4\alpha_1} v(s)r(s)(h(t,s))^2 \right] ds$$

$$\leq H(t,t_0)|u(T_0)| + \int_{T_0}^{T_0} \left[H(t,s)\psi(s) - \frac{\beta}{4\alpha_1} v(s)r(s)(h(t,s))^2 \right] ds$$

$$\leq H(t,t_0)|u(T_0)| + \int_{T_0}^{T_0} H(t,s)\psi(s) ds$$

$$\leq H(t,t_0)|u(T_0)| + H(t,t_0) \int_{T_0}^{T_0} |\psi(s)| ds$$

$$\leq H(t,t_0) \left[|u(T_0)| + \int_{T_0}^{T_0} |\psi(s)| ds \right]. \quad (28)$$

By (28), we get

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t,t_0)} \int_{T_0}^t \left[H(t,s)\psi(s) - \frac{\beta}{4\alpha_1} v(s)r(s)(h(t,s))^2 \right] ds \leq |u(T_0)| + \int_{T_0}^{T_0} |\psi(s)| ds$$

$$< \infty$$

which contradicts (22). Therefore, we have proved that all solutions of Equation (1) is oscillatory.

Part 2

In this part, we add $\tau(t)$ in Equation (1) to change it to the form

$$[r(t)k_1(x(\tau(t)), x'(\tau(t)))]' + p(t)k_2(x(\tau(t)), x'(\tau(t)))x'(\tau(t)) + q(t)f(x(\tau(t))) = 0. \quad (29)$$

Assume conditions (C_1) and (C_2) hold. Moreover, assume that

$$(C_3) \tau'(t) > 0, \tau(t) \leq t \text{ and } \lim_{t \rightarrow \infty} \tau(t) = \infty.$$

Theorem 5: Assume that $f'(x)$ exists and

$$f'(x) \geq \mu$$

for some $\mu > 0$ and for all $x \neq 0$. Suppose that there exists a function $g \in C^1([t_0, \infty); \mathbb{R})$ such that for some $\beta \geq 1$ and $H \in W$,

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t (H(t, s) \psi(s) - \frac{\beta}{4\mu\alpha_1} \frac{v(s)r(s)}{\tau'(s)} (h(t, s))^2) ds = \infty \quad (30)$$

where

$$v(t) = \exp\left[-2\mu\alpha_1 \int_{t_0}^t (\tau'(s)g(s) - \frac{\alpha_2}{2\mu\alpha_1} \frac{p(s)}{r(s)}) ds\right] \quad (31)$$

and

$$\psi(t) = v(t)[q(t) + \mu\alpha_1 r(t)\tau'(t)g^2(t) - \alpha_2 p(t)g(t) - (r(t)g(t))']. \quad (32)$$

Then equation (I) is oscillatory.

Proof : Suppose that $x(t)$ is a nonoscillatory solution of Equation (I). Without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ on $[T, \infty)$ for some sufficient large $T \geq t_0$.

Define

$$u(t) = v(t)r(t) \left[\frac{k_1(x(\tau(t)), x'(\tau(t)))}{f(x(\tau(t)))} + g(t) \right] \quad (33)$$

where $v(t)$ is given by (31). Differentiating (33), we have

$$\begin{aligned} u'(t) &= v'(t)r(t) \left[\frac{k_1(x(\tau(t)), x'(\tau(t)))}{f(x(\tau(t)))} + g(t) \right] + v(t) \left[\frac{r(t)k_1(x(\tau(t)), x'(\tau(t)))}{f(x(\tau(t)))} + r(t)g(t) \right]' \\ &= \frac{v'(t)}{v(t)} u(t) + v(t) \left[\frac{(r(t)k_1(x(\tau(t)), x'(\tau(t))))'}{f(x(\tau(t)))} - \frac{r(t)k_1(x(\tau(t)), x'(\tau(t)))f'(x(\tau(t)))x'(\tau(t))\tau'(t)}{(f(x(\tau(t))))^2} \right] \\ &\quad + v(t)(r(t)g(t))' \\ &= \frac{v'(t)}{v(t)} u(t) + v(t) \left[\frac{-p(t)k_2(x(\tau(t)), x'(\tau(t)))}{f(x(\tau(t)))} - q(t) \right] \\ &\quad - v(t) \frac{r(t)k_1(x(\tau(t)), x'(\tau(t)))f'(x(\tau(t)))x'(\tau(t))\tau'(t)}{(f(x(\tau(t))))^2} + v(t)(r(t)g(t))' \end{aligned}$$

$$\begin{aligned}
&\leq \frac{v'(t)}{v(t)}u(t) - \alpha_2 \frac{p(t)v(t)k_1(x(\tau(t)), x'(\tau(t)))}{f(x(\tau(t)))} - \mu\alpha_1 \frac{v(t)r(t)\tau'(t)k_1^2(x(\tau(t)), x'(\tau(t)))}{(f(x(\tau(t))))^2} \\
&\quad + v(t)\left[(r(t)g(t))' - q(t)\right] \\
&\leq \frac{v'(t)}{v(t)}u(t) + v(t)\left[(r(t)g(t))' - q(t)\right] - \alpha_2 p(t)v(t)\left[\frac{u(t)}{v(t)r(t)} - g(t)\right] \\
&\quad - \mu\alpha_1 \tau'(t)v(t)r(t)\left[\frac{u(t)}{v(t)r(t)} - g(t)\right]^2 \\
&\leq \frac{v'(t)}{v(t)}u(t) + v(t)\left[(r(t)g(t))' - q(t)\right] - \alpha_2 \frac{p(t)u(t)}{r(t)} + \alpha_2 v(t)p(t)g(t) \\
&\quad - \mu\alpha_1 \tau'(t)v(t)r(t)\left[\left(\frac{u(t)}{v(t)r(t)}\right)^2 - 2\frac{g(t)u(t)}{v(t)r(t)} + (g(t))^2\right] \\
&\leq \frac{v'(t)}{v(t)}u(t) + v(t)\left[(r(t)g(t))' - q(t) + \alpha_2 p(t)g(t)\right] - \alpha_2 \frac{p(t)u(t)}{r(t)} \\
&\quad - \mu\alpha_1 \tau'(t)\frac{(u(t))^2}{v(t)r(t)} + 2\mu\alpha_1 \tau'(t)g(t)u(t) - \mu\alpha_1 \tau'(t)v(t)r(t)(g(t))^2 \\
&\leq \left[-2\mu\alpha_1 \tau'(t)g(t) + \alpha_2 \frac{p(t)}{r(t)}\right]u(t) - \alpha_2 \frac{p(t)}{r(t)}u(t) + 2\mu\alpha_1 \tau'(t)g(t)u(t) \\
&\quad - v(t)\left[q(t) + \mu\alpha_1 \tau'(t)r(t)(g(t))^2 - \alpha_2 p(t)g(t) - (r(t)g(t))'\right] - \mu\alpha_1 \tau'(t)\frac{(u(t))^2}{v(t)r(t)} \\
&= -\psi(t) - \frac{\mu\alpha_1 \tau'(t)}{v(t)r(t)}(u(t))^2
\end{aligned}$$

where $\psi(t)$ is defined by (32). Hence, we obtain

$$\psi(t) \leq -u'(t) - \frac{\mu\alpha_1 \tau'(t)}{v(t)r(t)}(u(t))^2.$$

Multiplying above inequality by $H(t,s)$ and integrating on $[T, t]$, we have for some $\beta \geq 1$

$$\begin{aligned}
&\int_T^t H(t,s)\psi(s)ds \leq -\int_T^t H(t,s)u'(s)ds - \int_T^t \mu\alpha_1 \tau'(t)\frac{H(t,s)}{v(s)r(s)}(u(s))^2 ds \\
&= -H(t,s)u(s)\Big|_T^t - \int_T^t \left[-\frac{\partial}{\partial s} H(t,s)u(s) + \mu\alpha_1 \tau'(t)\frac{H(t,s)}{v(s)r(s)}(u(s))^2 \right] ds \\
&= H(t,T)u(T) - \int_T^t \left[h(t,s)\sqrt{H(t,s)}u(s) + \mu\alpha_1 \tau'(t)\frac{H(t,s)}{v(s)r(s)}(u(s))^2 \right] ds
\end{aligned}$$

$$\begin{aligned}
&= H(t, T)u(T) - \int_{\tau}^t \left[\sqrt{\frac{\mu\alpha_1 \tau'(t)H(t, s)}{\beta v(s)r(s)}} u(s) + \sqrt{\frac{\beta v(s)r(s)}{4\mu\alpha_1 \tau'(t)}} h(t, s) \right]^2 ds \\
&\quad + \frac{\beta}{4\mu\alpha_1} \int_{\tau}^t \frac{v(s)r(s)}{\tau'(t)} (h(t, s))^2 ds - \int_{\tau}^t \frac{\mu\alpha_1 (\beta - 1)}{\beta v(s)r(s)} \tau'(t) H(t, s) (u(s))^2 ds
\end{aligned}$$

Writing the latter inequality in the form

$$\begin{aligned}
\int_{\tau}^t \left[H(t, s)\psi(s) - \frac{\beta}{4\mu\alpha_1} \frac{v(s)r(s)}{\tau'(t)} (h(t, s))^2 \right] ds &\leq H(t, s)u(t) - \int_{\tau}^t \frac{\mu\alpha_1 (\beta - 1)}{\beta v(s)r(s)} \tau'(t) H(t, s) (u(s))^2 ds \\
&\quad - \int_{\tau}^t \left[\sqrt{\frac{\mu\alpha_1 \tau'(t)H(t, s)}{\beta v(s)r(s)}} u(s) + \sqrt{\frac{\beta v(s)r(s)}{4\mu\alpha_1 \tau'(t)}} h(t, s) \right]^2 ds. \quad (34)
\end{aligned}$$

Using the property of $H(t, s)$, we conclude that for all $t \geq T$

$$\begin{aligned}
\int_{\tau}^t \left[H(t, s)\psi(s) - \frac{\beta}{4\mu\alpha_1} \frac{v(s)r(s)}{\tau'(t)} (h(t, s))^2 \right] ds &\leq H(t, T)|u(T)| \\
&\leq H(t, t_0)|u(T)|.
\end{aligned}$$

Hence, for all $t \geq t_0$

$$\begin{aligned}
\int_{t_0}^t \left[H(t, s)\psi(s) - \frac{\beta}{4\mu\alpha_1} \frac{v(s)r(s)}{\tau'(t)} (h(t, s))^2 \right] ds &\leq H(t, t_0)|u(T)| + \int_{t_0}^T \left[H(t, s)\psi(s) - \frac{\beta}{4\mu\alpha_1} \frac{v(s)r(s)}{\tau'(t)} (h(t, s))^2 \right] ds \\
&\leq H(t, t_0)|u(T)| + \int_{t_0}^T H(t, s)\psi(s) ds
\end{aligned}$$

$$\leq H(t, t_0)|u(T)| + H(t, t_0) \int_{t_0}^T |\psi(s)| ds$$

$$\leq H(t, t_0) \left[|u(T)| + \int_{t_0}^T |\psi(s)| ds \right]. \quad (35)$$

Then, we obtain,

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[H(t, s)\psi(s) - \frac{\beta}{4\mu\alpha_1} \frac{v(s)r(s)}{\tau'(t)} (h(t, s))^2 \right] ds \leq |u(T)| + \int_{t_0}^T |\psi(s)| ds$$

$< \infty$.

Which contradict (30). Therefore, we have proved that all solutions of Equation (29) is oscillatory.

Theorem 6: Let $(C_1) - (C_3)$ hold. Suppose that $f(x)$ satisfies

$$\frac{f(x)}{x} \geq \mu \quad (36)$$

for some $\mu > 0$ and for all $x \neq 0$. Suppose that there exists a function $g \in C^1([t, \infty), \mathbb{R})$ such that for some $\beta \geq 1$ and $H \in W$,

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[H(t, s) \psi(s) - \frac{\beta}{4\alpha_1} v(s) r(s) \tau'(t) (h(t, s))^2 \right] ds = \infty \quad (37)$$

where

$$v(t) = \exp\left(-2\alpha_1 \int_{t_0}^t (\tau'(s)g(s) - \frac{\alpha_2 p(s)}{2\alpha_1 r(s)}) ds\right) \quad (38)$$

and

$$\psi(t) = v(t) \left[\mu q(t) + \alpha_1 \tau'(t) r(t) (g(t))^2 - \alpha_2 p(t) q(t) - (r(t)g(t))' \right]. \quad (39)$$

Then Equation (29) is oscillatory.

Proof: Suppose on the contrary that there is a nonoscillatory solution $x(t)$ of Equation (29). Without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ on $[T_0, \infty)$ for some sufficiently large $T \geq t_0$.

Define

$$u(t) = v(t)r(t) \left[\frac{k_1(x(\tau(t)), x'(\tau(t)))}{x(\tau(t))} + g(t) \right] \quad (40)$$

where $v(t)$ is given by (37). Differentiating (39), we obtain

$$\begin{aligned} u'(t) &= v'(t)r(t) \left[\frac{k_1(x(\tau(t)), x'(\tau(t)))}{x(\tau(t))} + g(t) \right] + v(t) \left[\frac{r(t)k_1(x(\tau(t)), x'(\tau(t)))}{x(\tau(t))} + r(t)g(t) \right]' \\ &= \frac{v'(t)}{v(t)} u(t) + v(t) \left[\frac{(r(t)k_1(x(\tau(t)), x'(\tau(t))))'}{x(\tau(t))} - \frac{r(t)k_1(x(\tau(t)), x'(\tau(t)))x'(\tau(t))\tau'(t)}{(x(\tau(t)))^2} + (r(t)g(t))' \right] \\ &= \frac{v'(t)}{v(t)} u(t) + v(t) \left[-\frac{p(t)k_2(x(\tau(t)), x'(\tau(t)))x'(\tau(t))}{x(\tau(t))} - q(t) - \frac{f(x(\tau(t)))}{x(\tau(t))} \right] \end{aligned}$$

$$\begin{aligned}
& \left. -\frac{r(t)k_1(x(\tau(t)),x'(\tau(t)))x'(\tau(t))\tau'(t)}{(x(\tau(t)))^2} + (r(t)g(t))' \right] \\
\leq & \frac{v'(t)}{v(t)}u(t) + v(t) \left[-\alpha_2 \frac{p(t)k_1(x(\tau(t)),x'(\tau(t)))}{x(\tau(t))} - \mu q(t) \right. \\
& \left. -\alpha_1 \tau'(t) \frac{r(t)(k_1(x(\tau(t)),x'(\tau(t))))^2}{(x(\tau(t)))^2} + (r(t)g(t))' \right] \\
= & \frac{v'(t)}{v(t)}u(t) - \alpha_2 p(t)v(t) \left[\frac{u(t)}{v(t)r(t)} - g(t) \right] - \alpha_1 \tau'(t)v(t)r(t) \left[\frac{u(t)}{v(t)r(t)} - g(t) \right]^2 \\
& + v(t) \left[-\mu q(t) + (r(t)g(t))' \right] \\
= & \frac{v'(t)}{v(t)}u(t) - \alpha_2 \frac{p(t)}{r(t)}u(t) + \alpha_2 v(t)p(t)g(t) + v(t) \left[-\mu q(t) + (r(t)g(t))' \right] \\
& - \alpha_1 \tau'(t)v(t)r(t) \left[\frac{(u(t))^2}{(v(t)r(t))^2} - \frac{2g(t)u(t)}{v(t)r(t)} + (g(t))^2 \right] \\
\leq & \left[-2\alpha_1 g(t) + \alpha_2 \frac{p(t)}{r(t)} \right] u(t) - \alpha_2 \frac{p(t)}{r(t)}u(t) - \alpha_1 \tau'(t) \frac{(u(t))^2}{v(t)r(t)} + 2\alpha_1 \tau'(t)g(t)u(t) \\
& - v(t) \left[\mu q(t) + \alpha_1 \tau'(t)r(t)(g(t))^2 - \alpha_2 p(t)g(t) - (r(t)q(t))' \right] \\
= & -\psi(t) - \alpha_1 \tau'(t) \frac{(u(t))^2}{v(t)r(t)}
\end{aligned}$$

where $\psi(t)$ is defined by (38). Hence, we have

$$\psi(t) \leq -u'(t) - \alpha_1 \tau'(t) \frac{(u(t))^2}{v(t)r(t)}. \quad (41)$$

Multiplying above by $H(t,s)$ and integrating from T to t , we get for some $\beta \geq 1$ and

$$\begin{aligned}
& t \geq T \geq T_0, \\
& \int_T^t H(t,s)\psi(s)ds \leq -\int_T^t H(t,s)u'(s)ds - \int_T^t \alpha_1 \tau'(t) \frac{H(t,s)}{v(s)r(s)} (u(s))^2 ds \\
& = -H(t,s)u(s) \Big|_T^t - \int_T^t \left[-\frac{\partial}{\partial s} H(t,s)u(s) + \alpha_1 \tau'(t) \frac{H(t,s)}{v(s)r(s)} (u(s))^2 \right] ds \\
& = H(t,T)u(T) - \int_T^t \left[h(t,s)\sqrt{H(t,s)u(s)} + \alpha_1 \tau'(t) \frac{H(t,s)}{v(s)r(s)} (u(s))^2 \right] ds \\
& = H(t,T)u(T) - \int_T^t \left[\sqrt{\frac{\alpha_1 \tau'(t)H(t,s)}{\beta v(s)r(s)}} u(s) + \sqrt{\frac{\beta v(s)r(s)}{4\alpha_1 \tau'(t)}} h(t,s) \right] ds + \frac{\beta}{4\alpha_1} \int_T^t \frac{v(s)r(s)}{\tau'(t)} (h(t,s))^2 ds
\end{aligned}$$

$$-\int_{\tau}^t \frac{\alpha_1 (\beta - 1) \tau'(t) H(t, s)}{\beta v(s) r(s)} (u(s))^2 ds.$$

Writing the latter inequality in the form

$$\begin{aligned} & \int_{\tau}^t \left[H(t, s) \psi(s) - \frac{\beta v(s) r(s)}{4\alpha_1 \tau'(t)} (h(t, s))^2 \right] ds \\ & \leq H(t, T) u(T) - \int_{\tau}^t \frac{\alpha_1 \tau'(t) (\beta - 1) (u(s))^2}{\beta v(s) r(s)} ds - \int_{\tau}^t \left[\sqrt{\frac{\alpha_1 \tau'(t) H(t, s)}{\beta v(s) r(s)}} u(s) + \sqrt{\frac{\beta v(s) r(s)}{4\alpha_1 \tau'(t)}} h(t, s) \right]^2 ds. \end{aligned}$$

Using the properties of $H(t, s)$, we conclude that above inequality for all $t \geq T_0$

$$\begin{aligned} & \int_{\tau_0}^t \left[H(t, s) \psi(s) - \frac{\beta v(s) r(s)}{4\alpha_1 \tau'(t)} (h(t, s))^2 \right] ds \leq H(t, T_0) |u(T_0)| \\ & \leq H(t, t_0) |u(T_0)|. \end{aligned}$$

Hence, for all $t \geq T_0$, we get

$$\begin{aligned} & \int_{t_0}^t \left[H(t, s) \psi(s) - \frac{\beta v(s) r(s)}{4\alpha_1 \tau'(t)} (h(t, s))^2 \right] ds \\ & \leq H(t, t_0) |u(T_0)| + \int_{t_0}^t \left[H(t, s) \psi(s) - \frac{\beta v(s) r(s)}{4\alpha_1 \tau'(t)} (h(t, s))^2 \right] ds \\ & \leq H(t, t_0) |u(T_0)| + \int_{t_0}^t H(t, s) \psi(s) ds \\ & \leq H(t, t_0) |u(T_0)| + H(t, t_0) \int_{t_0}^t |\psi(s)| ds \\ & \leq H(t, t_0) \left[|u(T_0)| + \int_{t_0}^t |\psi(s)| ds \right]. \end{aligned} \tag{42}$$

By (42), we get

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[H(t, s) \psi(s) - \frac{\beta v(s) r(s)}{4\alpha_1 \tau'(t)} (h(t, s))^2 \right] ds \leq |u(T_0)| + \int_{t_0}^{T_0} |\psi(s)| ds$$

$< \infty$

which contradicts (37). Therefore, we have proved that all solutions of Equation (29) is oscillatory.