

# CHAPTER 2

## PRELIMINARIES

In this chapter, we give some basic definitions, notations, lemmas and results which will be used in the later chapters.

### 2.1 Notations

The following notations that will be used in this thesis

$\mathbb{R}^n$  – the  $n$  dimensional Euclidean space,

$\mathbb{R}^{n \times n}$  – the set of all  $n \times n$  real matrices,

$\|x\|$  – the Euclidean norm of vector  $x$ ,

$\text{diag}\{\cdot\}$  – the block diagonal matrix,

$I$  – the identity matrix,

$A^T$  – the transpose of matrix  $A$ ,

$A > 0, A \geq 0, A < 0, A \leq 0$  – means that  $A$  is symmetric positive definite,

positive semi-definite, negative definite and negative semi-definite,

$\lambda_{\max}(A)$  – maximum eigenvalue of matrix  $A$ ,

$\lambda_{\min}(A)$  – minimum eigenvalue of matrix  $A$ ,

$C_h = C([-h, 0], \mathbb{R}^n), h > 0$  – denotes the Banach space of continuous functions, mapping the interval  $[-h, 0]$  into  $\mathbb{R}^n$ , with the topology of uniform convergence,

$\|x_t\| \in C_h$  defined  $x_t = x(t + \theta), -h \leq \theta \leq 0$  and  $\|x_t\|_{C_h} = \sup_{-h \leq \theta \leq 0} \|x(t + \theta)\|$ ,

$\begin{bmatrix} A & B \\ * & C \end{bmatrix} - *$  represents the symmetric form of matrix, namely  $* = B^T$ .

### 2.2 Lyapunov Function

Consider the system described by

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \quad (2.1)$$

where  $x \in R^n$ ,  $x_i = x_i(t)$ ,  $f = (f_1, f_2, \dots, f_n)$  and  $f_i = f_i(t, x_1, x_2, \dots, x_n)$  for  $i = 1, 2, \dots, n$

**Definition 2.2.1 (Lyapunov Function)** Let  $D$  be a domain  $\mathbb{R}^n$  such that  $\bar{0} \in D$   $V : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ , We say that  $V(x)$  is a Lyapunov function of system (2.1) if the following conditions hold :

- (1)  $V(x)$  is continuous on  $D \subseteq \mathbb{R}^n$ .
- (2)  $V(x)$  is positive definite such that  $V(\bar{0}) = 0$  and  $V(x) > 0$  for  $x \neq \bar{0}$ .
- (3) the derivative of  $V$  with respect to (2.1) is negative semidefinite (i.e.  $\dot{V}(0) = 0$ , and for all  $x$  in  $\|x\| \leq k$ ,  $\dot{V}(x) \leq 0$ ).

## 2.3 Stability

**Definition 2.3.1** A point  $\bar{x}$  is called an *equilibrium point* of equation (2.1) if  $f(t, \bar{x}) = 0$  for all  $t \geq t_0$ . For all purposes of the stability theory we can assume that  $\bar{0}$  is an equilibrium of (2.1).

**Definition 2.3.2** The equilibrium point  $\bar{x}$  of equation (2.1) is called *stable* if, for each  $\epsilon > 0$ , there is  $\delta = \delta(\epsilon, t_0) > 0$  such that  $\|x(t_0)\| < \delta$  implies  $\|x(t)\| < \epsilon$  for all  $t \geq t_0 \geq 0$ .

**Definition 2.3.3** The equilibrium point  $\bar{x}$  of equation (2.1) is called *unstable* if it is not stable.

**Definition 2.3.4** The equilibrium point  $\bar{x}$  of equation (2.1) is called *Asymptotically stable* (denoted A.S.) if it is stable and  $\|x(t)\| \rightarrow 0$  as  $t \rightarrow \infty$ .

**Definition 2.3.5** The equilibrium point  $\bar{x}$  of equation (2.1) is called *Uniformly asymptotically stable* if, for each  $\epsilon > 0$ , there is  $\delta = \delta(\epsilon) > 0$  such that  $\|x(t_0)\| < \delta$  implies  $\|x(t)\| < \epsilon$  and  $\|x(t)\| \rightarrow 0$  as  $t \rightarrow \infty$  for all  $t \geq t_0 \geq 0$ .

**Theorem 2.3.1** [1] The equilibrium point  $\bar{x}$  of equation (2.1) is stable if there exists a Lyapunov function for system (2.1). Moreover, if there exists a Lyapunov

function whose derivative is negative definite, then the equilibrium point  $\bar{x}$  is A.S.

**Definition 2.3.6** The operator  $D$  is said to be stable if solution  $\bar{x} = 0$  of the homogeneous difference equation  $D(x_t) = 0$ ,  $t \geq 0$  is stable where  $D : C_h \rightarrow R^n$ .

**Definition 2.3.7** suppose  $f : C_h \rightarrow R^n$ ,  $D : C_h \rightarrow R^n$  are given continuous functions. The relation

$$\frac{d}{dt}D(t, x_t) = f(t, x_t),$$

is called the neutral differential equation. The function  $D$  will be called the operator for the neutral differential equation.

**Theorem 2.3.2** [5] Suppose  $D$  is stable,  $f : C_h \rightarrow R^n$  and suppose  $u(s), v(s)$  and  $w(s)$  are continuous, nonnegative and nondecreasing with  $u(s), v(s) > 0$  for  $s \neq 0$  and  $u(0) = v(0) = 0$ . If there is a continuous function  $V : C_h \rightarrow R^n$  shuh that

$$\begin{aligned} u(\|D(x_t)\|) &\leq V(x_t) \leq v(\|x_t\|_{C_h}), \\ \dot{V}(x_t) &\leq -w(\|D(x_t)\|), \end{aligned}$$

If  $w(s) > 0$  for  $s > 0$ , then the solution  $x = 0$  of the neutral differential equation is uniformly asymptotically stable. The same conclusion holds if the upper bound on  $\dot{V}(x_t)$  is given by  $-w\|x(t)\|$ .

**Lemma 2.3.1 (Schur Complement [2])** Given constant symmetric matrices  $Q$ ,  $S$  and  $R \in \mathbb{R}^{n \times n}$  where  $R(x) < 0$ ,  $Q(x) = Q^T(x)$  and  $R(x) = R^T(x)$  we have

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} < 0 \Leftrightarrow Q(x) - S(x)R^{-1}(x)S^T(x) < 0.$$

**Lemma 2.3.2** [5] Suppose  $\lambda_{min}(Q)$  is minimum eigenvalue of matrix  $Q$  and  $\lambda_{max}(Q)$  is maximum eigenvalue of matrix  $Q$ . The following inequalities hold:

$$\lambda_{min}(Q)x^T x \leq x^T Q x \leq \lambda_{max}(Q)x^T x,$$

for symmetric matrix  $Q \in R^{n \times n}$  for all  $x \in R^n$ .

**Lemma 2.3.3 [9]** Let  $U, V, W$  and  $M$  be real matrices of appropriate dimensions with  $M$  satisfying  $M = M^T$ , then  $M + UVW + W^T V^T U^T < 0$  for all  $V^T V \leq I$  if and only if there exists a scalar  $\epsilon > 0$  such that  $M + \epsilon^{-1} U U^T + \epsilon W^T W < 0$ .

**Lemma 2.3.4 [5]** There exists a symmetric matrix  $X$  such that

$$\begin{pmatrix} P_1 - LXLT^T & Q_1 \\ Q_1^T & R_1 \end{pmatrix} < 0 \quad \text{and}$$

$$\begin{pmatrix} P_2 + X & Q_2 \\ Q_2^T & R_2 \end{pmatrix} < 0$$

if and only if

$$\begin{pmatrix} P_1 + LP_2L^T & Q_1 & LQ_2 \\ Q_1^T & R_1 & 0 \\ Q_2^T L^T & 0 & R_2 \end{pmatrix} < 0$$

## 2.4 Types of Matrix

**Definition 2.4.1 (Symmetric Matrix)** A real  $n \times n$  matrix  $A$  is called *symmetric* if

$$A^T = A.$$

**Definition 2.4.2 (Positive Definite Matrix)** A real  $n \times n$  matrix  $A$  is called *positive definite* if

for all nonzero vectors  $x \in \mathbb{R}^n$ . It is called *positive semidefinite* if

$$x^T A x \geq 0.$$

**Definition 2.4.3 (Negative Definite Matrix)** A real  $n \times n$  matrix  $A$  is called *negative definite* if

$$x^T A x < 0$$

for all nonzero vectors  $x \in \mathbb{R}^n$ . It is called *negative semidefinite* if

$$x^T A x \leq 0.$$

The follows result are well known

**Lemma 2.4.1** [7] A symmetric matrix is positive semidefinite (definite) matrix if all of its eigenvalues are nonnegative (positive).

**Lemma 2.4.2** [7] A symmetric matrix is negative semidefinite (definite) matrix if all of its eigenvalues are nonpositive (negative).



ลิขสิทธิ์มหาวิทยาลัยเชียงใหม่  
Copyright<sup>©</sup> by Chiang Mai University  
All rights reserved