CHAPTER 3

MAIN RESULTS

In this chapter, we present some new delay-dependent criterion for asymptotic stability for uncertain neutral system based on Lyapunov stability theory and linear matrix inequality (LMI) techniques. In section 3.1, we derive new sufficient condition for asymptotic stability of neutral system without matric uncertainties. Then, we extend the result to establish asymptotic stability of neutral system with matric uncertainties in section 3.2. Some numerical simulations are given to illustrate the effectiveness of our theoretical results.

Consider the following neutral system with time-varying delay:

$$\dot{x}(t) - C\dot{x}(t-d) = (A + \Delta A(t))x(t) + (B + \Delta B(t))x(t - \tau(t)) \tag{3.1}$$

$$x(t) = \phi(t),\ t \in [-h, 0]$$

where $x(t) \in R^n$ is state at time t defined by $x_t(\theta) = x(t+\theta), \forall\theta \in [-h, 0]$, $C, A, B \in R^{n\times n}$ are constant matrices and initial vector $\phi \in C_h$, $\Delta A(t)$ and $\Delta B(t)$ are unknown real matrices of appropriate dimensions representing the system 's time-varying parameter uncertainties and satisfy

$$[\Delta A(t) \quad \Delta B(t)] = EF(t)[G \quad G_1] \tag{3.2}$$

where E, G, G_1 are known real constant matrices of appropriate dimensions, and $F(t)$ is an unknown matrix function satisfying

$$F^T(t)F(t) \leq I, \forall t \geq 0. \tag{3.3}$$

the delay d is positive constant and the delay $\tau(t)$ is a time-varying continuous function which satisfies

$$0 \leq \tau_m \leq \tau(t) \leq \tau_M, \tag{3.4}$$

where τ_m, τ_M are constants and $h = \max\{d, \tau_M\}$. Now we present a delay-dependent asymptotic stability condition for system (3.1).

Let $\tau_e = \frac{1}{2}(\tau_M + \tau_m)$ and $\rho = \frac{1}{2}(\tau_M - \tau_m)$. Then $\tau(t)$ can be expressed as

$$\tau(t) = \tau_e + \rho\xi(t), \tag{3.5}$$

where

$$\xi(t) = \begin{cases} \frac{2\tau(t)-(\tau_M+\tau_m)}{\tau_M-\tau_m}, & \tau_M > \tau_m, \\ 0, & \tau_M = \tau_m. \end{cases}$$

Obviously, $|\xi(t)| \leq 1$. For this case, $\tau(t)$ is a function belonging to the interval $[\tau_e - \rho, \tau_e + \rho]$, where ρ can be taken as the range of variation of the time-varying delay $\tau(t)$. Let operator $D : C([-d, 0], R^n) \to R^n$ is defined as $D(x_t) = x(t) - Cx(t-d)$ where $\|C\| < 1$. We first investigate delay-dependent criterion for asymptotic stability for neutral system without matric uncertainties of system (3.1)

$$\dot{x}(t) = Ax(t) + Bx(t - \tau(t)) + C\dot{x}(t - d). \tag{3.6}$$

Using the fact that

$$x(t - \tau_e) - x(t - \tau(t)) = \int_{t-\tau(t)}^{t-\tau_e} \dot{x}(s)ds$$

system (3.6) can be rewritten as

$$\dot{x}(t) = Ax(t) + Bx(t - \tau_e) - B\int_{t-\tau(t)}^{t-\tau_e} \dot{x}(s)ds + C\dot{x}(t - d). \tag{3.7}$$

3.1 Delay-dependent Criterion for Asymptotic Stability of Neutral System without Matrices Uncertainties

In this section, we deal with the problem for asymptotic stability of the zero solution of neutral system without matric uncertainties.

Theorem 3.1.1. *For given nonnegative scalars τ_m and τ_M, system (3.6) is asymptotically stable for any $\tau(t)$ satisfying $0 \leq \tau_m \leq \tau(t) \leq \tau_M$, if there exist matrices*

$P > 0, Q > 0, Q_1 > 0, Q_2 > 0, R > 0, S > 0, Y > 0$ and $L_i, K_i, M_i (i = 1, 2, 3, 4, 5)$ of appropriate dimensions such that

$$\Sigma_1 = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} & 2dM_1^T & 2\tau_e L_1^T & \rho K_1^T B \\ * & \phi_{22} & \phi_{23} & \phi_{24} & \phi_{25} & 2dM_2^T & 2\tau_e L_2^T & \rho K_2^T B \\ * & * & \phi_{33} & \phi_{34} & \phi_{35} & 2dM_3^T & 2\tau_e L_3^T & \rho K_3^T B \\ * & * & * & \phi_{44} & \phi_{45} & 2dM_4^T & 2\tau_e L_4^T & \rho K_4^T B \\ * & * & * & * & \phi_{55} & 2dM_5^T & 2\tau_e L_5^T & \rho K_5^T B \\ * & * & * & * & * & -2dY & 0 & 0 \\ * & * & * & * & * & * & -2\tau_e R & 0 \\ * & * & * & * & * & * & * & -\rho S \end{pmatrix} < 0 \qquad (3.8)$$

where

$\phi_{11} = Q + K_1^T A + A^T K_1 + L_1 + L_1^T + M_1^T + M_1 + Q_1$

$\phi_{12} = K_1^T B - L_1^T + A^T K_2 + L_2 + M_2$

$\phi_{13} = -K_1^T + A^T K_3 + L_3 + M_3 + P$

$\phi_{14} = K_1^T C + A^T K_4 + L_4 + M_4 - PC$

$\phi_{15} = A^T K_5 + L_5 + M_5 - M_1$

$\phi_{22} = -Q + K_2^T B + B^T K_2 - L_2^T - L_2$

$\phi_{23} = -K_2^T + B^T K_3 - L_3$

$\phi_{24} = K_2^T C + B^T K_4 - L_4$

$\phi_{25} = B^T K_5 - L_5 - M_2$

$\phi_{33} = dY + \tau_e R + \rho S - K_3^T - K_3 + Q_2$

$\phi_{34} = K_3^T C - K_4$

$\phi_{35} = -K_5 - M_3 - PC$

$\phi_{44} = K_4^T C + C^T K_4 - Q_2$

$\phi_{45} = C^T K_5 - M_4 + CPC^T$

$\phi_{55} = -M_5^T - M_5 - Q_1$

Proof. We prove the Theorem 3.1.1 is true for three cases, namely , $\tau_m \leq \tau(t) < \tau_e$; $\tau(t) = \tau_e$; and $\tau_e < \tau(t) \leq \tau_M$

Case I: $\tau_m \leq \tau(t) < \tau_e$

Choose a Lyapunov-Krasovskii functional candidate as

$$\begin{aligned} V_1(x_t) =& D^T(x_t)PD(x_t) + \int_{t-\tau_e}^{t} x^T(s)Qx(s)ds + \int_{t-d}^{t} x^T(s)Q_1x(s)ds \\ &+ \int_{t-d}^{t} \dot{x}^T(s)Q_2\dot{x}(s)ds + \int_{-\tau_e}^{0} ds \int_{t+s}^{t} \dot{x}^T(\theta)R\dot{x}(\theta)d\theta \\ &+ \int_{-\tau_e}^{-\tau_m} ds \int_{t+s}^{t} \dot{x}^T(\theta)S\dot{x}(\theta)d\theta + \int_{-d}^{0} ds \int_{t+s}^{t} \dot{x}^T(\theta)Y\dot{x}(\theta)d\theta \end{aligned} \tag{3.9}$$

where $P > 0, Q > 0, Q_1 > 0, Q_2 > 0, R > 0, S > 0$ and $Y > 0$. Taking the derivative of $V_1(x_t)$ with respect to t along the trajectory of (3.7) yields

$$\begin{aligned} \dot{V}_1(x_t) =& 2D^T(x_t)P\dot{D}(x_t) + x^T(t)Qx(t) - x^T(t-\tau_e)Qx(t-\tau_e) \\ &+ x^T(t)Q_1x(t) - x^T(t-d)Q_1x(t-d) + \dot{x}^T(t)Q_2\dot{x}^T(t) \\ &- \dot{x}^T(t-d)Q_2\dot{x}^T(t-d) + \dot{x}^T(t)(dY + \tau_eR + \rho S)\dot{x}(t) - \int_{t-\tau_e}^{t} \dot{x}^T(s)R\dot{x}(s)ds \\ &- \int_{t-\tau_e}^{t-\tau_m} \dot{x}^T(s)S\dot{x}(s)ds - \int_{t-d}^{t} \dot{x}^T(s)Y\dot{x}(s)ds \end{aligned} \tag{3.10}$$

Since

$$-\int_{t-\tau_e}^{t-\tau_m} \dot{x}^T(s)S\dot{x}(s)ds \leq -\int_{t-\tau_e}^{t-\tau(t)} \dot{x}^T(s)S\dot{x}(s)ds \tag{3.11}$$

and from the following equalities :

$$\begin{aligned} &2[x^T(t)K_1^T + x^T(t-\tau_e)K_2^T + \dot{x}^T(t)K_3^T + \dot{x}^T(t-d)K_4^T + x^T(t-d)K_5^T]\times \\ &[Ax(t) + Bx(t-\tau_e) - B\int_{t-\tau(t)}^{t-\tau_e} \dot{x}(s)ds + C\dot{x}(t-d) - \dot{x}(t)] = 0 \end{aligned} \tag{3.12}$$

$$\begin{aligned} &2[x^T(t)L_1^T + x^T(t-\tau_e)L_2^T + \dot{x}^T(t)L_3^T + \dot{x}^T(t-d)L_4^T + x^T(t-d)L_5^T]\times \\ &[x(t) - x(t-\tau_e) - \int_{t-\tau_e}^{t} \dot{x}(s)ds] = 0 \end{aligned} \tag{3.13}$$

$$\begin{aligned} &2[x^T(t)M_1^T + x^T(t-\tau_e)M_2^T + \dot{x}^T(t)M_3^T + \dot{x}^T(t-d)M_4^T + x^T(t-d)M_5^T]\times \\ &[x(t) - x(t-d) - \int_{t-d}^{t} \dot{x}(s)ds] = 0 \end{aligned} \tag{3.14}$$

where $K_i, L_i, M_i (i = 1, 2, 3, 4, 5)$ are some matrices of appropriate dimensions, then we have

$$\begin{aligned}
\dot{V}_1(x_t) \leq & 2D^T(x_t)P\dot{D}(x_t) + x^T(t)Qx(t) - x^T(t-\tau_e)Qx(t-\tau_e) \\
& + x^T(t)Q_1x(t) - x^T(t-d)Q_1x(t-d) + \dot{x}^T(t)Q_2\dot{x}^T(t) - \dot{x}(t-d)Q_2\dot{x}(t-d) \\
& + \dot{x}^T(t)(dY + \tau_e R + \rho S)\dot{x}(t) - \int_{t-\tau_e}^{t} \dot{x}^T(s)R\dot{x}(s)ds \\
& - \int_{t-\tau_e}^{t-\tau(t)} \dot{x}^T(s)S\dot{x}(s)ds - \int_{t-d}^{t} \dot{x}^T(s)Y\dot{x}(s)ds \\
& + 2[x^T(t)K_1^T + x^T(t-\tau_e)K_2^T + \dot{x}^T(t)K_3^T + \dot{x}^T(t-d)K_4^T + x^T(t-d)K_5^T] \\
& \times [Ax(t) + Bx(t-\tau_e) - B\int_{t-\tau(t)}^{t-\tau_e} \dot{x}(s)ds + C\dot{x}(t-d) - \dot{x}(t)] \\
& + 2[x^T(t)L_1^T + x^T(t-\tau_e)L_2^T + \dot{x}^T(t)L_3^T + \dot{x}^T(t-d)L_4^T + x^T(t-d)L_5^T] \\
& \times [x(t) - x(t-\tau_e) - \int_{t-\tau_e}^{t} \dot{x}(s)ds] \\
& + 2[x^T(t)M_1^T + x^T(t-\tau_e)M_2^T + \dot{x}^T(t)M_3^T + \dot{x}^T(t-d)M_4^T + x^T(t-d)M_5^T] \\
& \times [x(t) - x(t-d) - \int_{t-d}^{t} \dot{x}(s)ds] \\
= & 2[x(t) - Cx(t-d)]^T P[\dot{x}(t) - C\dot{x}(t-d)] + x^T(t)Qx(t) - x^T(t-\tau_e)Qx(t-\tau_e) \\
& + x^T(t)Q_1x(t) - x^T(t-d)Q_1x(t-d) + \dot{x}^T(t)Q_2\dot{x}^T(t) - \dot{x}(t-d)Q_2\dot{x}(t-d) \\
& + \dot{x}^T(t)(dY + \tau_e R + \rho S)\dot{x}(t) - \int_{t-\tau_e}^{t} \dot{x}^T(s)R\dot{x}(s)ds \\
& - \int_{t-\tau_e}^{t-\tau(t)} \dot{x}^T(s)S\dot{x}(s)ds - \int_{t-d}^{t} \dot{x}^T(s)Y\dot{x}(s)ds \\
& + 2[x^T(t)K_1^T + x^T(t-\tau_e)K_2^T + \dot{x}^T(t)K_3^T + \dot{x}^T(t-d)K_4^T + x^T(t-d)K_5^T] \\
& \times [Ax(t) + Bx(t-\tau_e) - B\int_{t-\tau(t)}^{t-\tau_e} \dot{x}(s)ds + C\dot{x}(t-d) - \dot{x}(t)] \\
& + 2[x^T(t)L_1^T + x^T(t-\tau_e)L_2^T + \dot{x}^T(t)L_3^T + \dot{x}^T(t-d)L_4^T + x^T(t-d)L_5^T] \\
& \times [x(t) - x(t-\tau_e) - \int_{t-\tau_e}^{t} \dot{x}(s)ds] \\
& + 2[x^T(t)M_1^T + x^T(t-\tau_e)M_2^T + \dot{x}^T(t)M_3^T + \dot{x}^T(t-d)M_4^T + x^T(t-d)M_5^T] \\
& \times [x(t) - x(t-d) - \int_{t-d}^{t} \dot{x}(s)ds]
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{d}\int_{t-d}^{t}\omega^T(t,s)\phi_1\omega(t,s)ds+\frac{1}{\tau_e}\int_{t-\tau_e}^{t}\omega^T(t,s)\phi_2\omega(t,s)ds\\
&+\frac{1}{\tau_e-\tau(t)}\int_{t-\tau_e}^{t-\tau(t)}\omega^T(t,s)\phi_3\omega(t,s)ds
\end{aligned}
\tag{3.15}
$$

where

$\omega^T(t,s)=[x^T(t) \quad x^T(t-\tau_e) \quad \dot{x}^T(t) \quad \dot{x}^T(t-d) \quad x^T(t-d) \quad -\dot{x}^T(s)]$

$$
\phi_1=\frac{1}{2}\begin{pmatrix}
\phi_{11}+Z_{11} & \phi_{12}+Z_{12} & \phi_{13}+Z_{13} & \phi_{14}+Z_{14} & \phi_{15}+Z_{15} & 2dM_1^T\\
* & \phi_{22}+Z_{22} & \phi_{23}+Z_{23} & \phi_{24}+Z_{24} & \phi_{25}+Z_{25} & 2dM_2^T\\
* & * & \phi_{33}+Z_{33} & \phi_{34}+Z_{34} & \phi_{35}+Z_{35} & 2dM_3^T\\
* & * & * & \phi_{44}+Z_{44} & \phi_{45}+Z_{45} & 2dM_4^T\\
* & * & * & * & \phi_{55}+Z_{55} & 2dM_5^T\\
* & * & * & * & * & -2dY
\end{pmatrix}
$$

$$
\phi_2=\frac{1}{2}\begin{pmatrix}
\phi_{11}+Z_{11} & \phi_{12}+Z_{12} & \phi_{13}+Z_{13} & \phi_{14}+Z_{14} & \phi_{15}+Z_{15} & 2\tau_eL_1^T\\
* & \phi_{22}+Z_{22} & \phi_{23}+Z_{23} & \phi_{24}+Z_{24} & \phi_{25}+Z_{25} & 2\tau_eL_2^T\\
* & * & \phi_{33}+Z_{33} & \phi_{34}+Z_{34} & \phi_{35}+Z_{35} & 2\tau_eL_3^T\\
* & * & * & \phi_{44}+Z_{44} & \phi_{45}+Z_{45} & 2\tau_eL_4^T\\
* & * & * & * & \phi_{55}+Z_{55} & 2\tau_eL_5^T\\
* & * & * & * & * & -2\tau_eR
\end{pmatrix}
$$

$$
\phi_3=\begin{pmatrix}
-Z_{11} & -Z_{12} & -Z_{13} & -Z_{14} & -Z_{15} & (\tau(t)-\tau_e)K_1^TB\\
* & -Z_{22} & -Z_{23} & -Z_{24} & -Z_{25} & (\tau(t)-\tau_e)K_2^TB\\
* & * & -Z_{33} & -Z_{34} & -Z_{35} & (\tau(t)-\tau_e)K_3^TB\\
* & * & * & -Z_{44} & -Z_{45} & (\tau(t)-\tau_e)K_4^TB\\
* & * & * & * & -Z_{55} & (\tau(t)-\tau_e)K_5^TB\\
* & * & * & * & * & -|\tau(t)-\tau_e|S
\end{pmatrix}
$$

$Z_{11}, Z_{22}, Z_{33}, Z_{44}, Z_{55} > 0, Z_{12}, Z_{13}, Z_{14}, Z_{15}, Z_{23}, Z_{24}, Z_{25}, Z_{34}, Z_{35}$ and Z_{45} are some parameter matrices of appropriate dimensions. From (3.15) if $\phi_1 < 0, \phi_2 < 0$ and $\phi_3 < 0$, then $\dot{V}_1(x_t) \le -\lambda_1\|x(t)\|^2$ for some $\lambda_1 > 0$. Pre and post-multiplying both sides of $\phi_3 < 0$ by diag$\{I, I, I, I, I, sgn(\tau(t) - \tau_e)\}$, we get that

$$\begin{pmatrix} -Z_{11} & -Z_{12} & -Z_{13} & -Z_{14} & -Z_{15} & \rho K_1^T B \\ * & -Z_{22} & -Z_{23} & -Z_{24} & -Z_{25} & \rho K_2^T B \\ * & * & -Z_{33} & -Z_{34} & -Z_{35} & \rho K_3^T B \\ * & * & * & -Z_{44} & -Z_{45} & \rho K_4^T B \\ * & * & * & * & -Z_{55} & \rho K_5^T B \\ * & * & * & * & * & -\rho S \end{pmatrix} < 0 \tag{3.16}$$

By Lemma 2.3.1 (Schur Complement [2]) implies $\phi_3 < 0$. Let $\phi_4 = \phi_1 + \phi_2$, in light of Lemma 2.3.4 (3.8) holds if and only if $\phi_4 < 0$ and (3.16) simultaneously hold, and in light of Lemma 2.3.4 again $\phi_4 < 0$ if and only if $\phi_1 < 0$ and $\phi_2 < 0$. Then (3.8) hold if and only if $\phi_1 < 0, \phi_2 < 0$ and (3.16) simultaneously hold. It's obvious that $V_1(x_t) \geq \lambda_{min}(P)\|D(x_t)\|^2$.

Consider $\quad D^T(x_t)PD(x_t)$

$$\begin{aligned}
&\leq \lambda_{max}(P)\|D(x_t)\|^2 \\
&\leq \lambda_{max}(P)\|x(t) - Cx(t-d)\|^2 \\
&\leq \lambda_{max}(P)[2\|x(t)\|^2 + 2\|Cx(t-d)\|^2] \\
&\leq 2\lambda_{max}(P)[\|x(t)\|^2 + \|Cx(t-d)\|^2] \\
&\leq 2\lambda_{max}(P)[\|x_t\|^2 + \|C\|^2\|x_t\|^2] \\
&\leq 2\lambda_{max}(P)[1 + \|C\|^2]\|x_t\|^2 \\
&= a\|x_t\|^2; a = 2\lambda_{max}(P)[1 + \|C\|^2] > 0.
\end{aligned}$$

Consider $\int_{t-\tau_e}^{t} x^T(s)Qx(s)ds$

$$\begin{aligned}
&\leq \int_{t-\tau_e}^{t} \lambda_{max}(Q)\|x(s)\|^2 ds \\
&\leq \lambda_{max}(Q) \int_{t-\tau_e}^{t} \|x(s)\|^2 ds \\
&\leq \lambda_{max}(Q) \int_{t-\tau_e}^{t} \sup_{t-\tau_M \leq s \leq t} \|x(s)\|^2 ds \\
&\leq \tau_e \lambda_{max}(Q) \sup_{t-\tau_M \leq s \leq t} \|x(s)\|^2 \\
&= \tau_e \lambda_{max}(Q)\|x_t\|^2 \\
&= b\|x_t\|^2; b = \tau_e \lambda_{max}(Q) > 0.
\end{aligned}$$

Consider $\int_{t-d}^{t} x^T(s)Q_1x(s)ds$

$$\leq \int_{t-d}^{t} \lambda_{max}(Q_1)\|x(s)\|^2 ds$$
$$\leq d\lambda_{max}(Q_1)\|x(s)\|^2$$
$$\leq d\lambda_{max}(Q_1)\|x_t\|^2$$
$$= c\|x_t\|^2; c = d\lambda_{max}(Q_1) > 0.$$

Consider $\int_{t-d}^{t} \dot{x}^T(s)Q_2\dot{x}(s)ds$

$$= x^T(t)Q_2x(t) - x^T(t-d)Q_2x(t-d)$$
$$\leq \lambda_{max}(Q_2)\|x(t)\|^2$$
$$\leq \lambda_{max}(Q_2)\|x_t\|^2$$
$$= d\|x_t\|^2; d = \lambda_{max}(Q_2) > 0.$$

Consider $\int_{-\tau_e}^{0} \int_{t+s}^{t} \dot{x}^T(\theta)R\dot{x}(\theta)d\theta ds$

$$= \int_{-\tau_e}^{0} [x^T(t)Rx(t) - x^T(t+s)Rx(t+s)]ds$$
$$= \int_{-\tau_e}^{0} x^T(t)Rx(t)ds - \int_{-\tau_e}^{0} x^T(t+s)Rx(t+s)ds$$
$$\leq \int_{-\tau_e}^{0} \lambda_{max}(R)\|x(t)\|^2 ds$$
$$\leq \lambda_{max}(R)\|x(t)\|^2 . \tau_e$$
$$\leq \tau_e\lambda_{max}(R)\|x_t\|^2 = e\|x_t\|^2; e = \tau_e\lambda_{max}(R) > 0.$$

Consider $\int_{-\tau_e}^{-\tau_m} \int_{t+s}^{t} \dot{x}^T(\theta)S\dot{x}(\theta)d\theta ds$

$$= \int_{-\tau_e}^{-\tau_m} [x^T(t)Sx(t) - x^T(t+s)Sx(t+s)]ds$$
$$= \int_{-\tau_e}^{-\tau_m} x^T(t)Sx(t)ds - \int_{-\tau_e}^{-\tau_m} x^T(t+s)Sx(t+s)ds$$
$$\leq \int_{-\tau_e}^{-\tau_m} \lambda_{max}(S)\|x(t)\|^2 ds$$
$$= \lambda_{max}(S)\|x(t)\|^2(-\tau_m + \tau_e)$$
$$\leq (\tau_e - \tau_m)\lambda_{max}(S)\|x_t\|^2 = f\|x_t\|^2; f = (\tau_e - \tau_m)\lambda_{max}(S) > 0.$$

Finally consider $\int_{-d}^{0}\int_{t+s}^{t}\dot{x}^T(\theta)Y\dot{x}(\theta)d\theta ds$

$$
\begin{aligned}
&=\int_{-d}^{0}[x^T(t)Yx(t)-x^T(t+s)Yx(t+s)]ds\\
&=\int_{-d}^{0}x^T(t)Yx(t)ds-\int_{-d}^{0}x^T(t+s)Yx(t+s)ds\\
&\leq\int_{-d}^{0}\lambda_{max}(Y)\|x(t)\|^2ds\\
&=d\lambda_{max}(Y)\|x(t)\|^2\\
&\leq d\lambda_{max}(Y)\|x_t\|^2=g\|x_t\|^2;g=d\lambda_{max}(Y)>0.
\end{aligned}
$$

Hence, we conclude that $V_1(x_t)\leq(a+b+c+d+e+f+g)\|x_t\|^2$. Since operator $D(x_t)$ stable. Therefore, solution $x=0$ of system (3.6) is asymptotically stable according to Theorem (2.3.2)

Case II $\tau(t)=\tau_e$

For this case, we choose a Lyapunov-Krasovskii functional candidate as

$$
\begin{aligned}
V_2(x_t)=&D^T(x_t)PD(x_t)+\int_{t-\tau_e}^{t}x^T(s)Qx(s)ds+\int_{t-d}^{t}x^T(s)Q_1x(s)ds\\
&+\int_{t-d}^{t}\dot{x}^T(s)Q_2\dot{x}(s)ds+\int_{-\tau_e}^{0}ds\int_{t+s}^{t}\dot{x}^T(\theta)R\dot{x}(\theta)d\theta\\
&+\int_{-d}^{0}\int_{t+s}^{t}\dot{x}^T(\theta)Y\dot{x}(\theta)d\theta \qquad (3.17)
\end{aligned}
$$

where $P>0,\ \ Q>0,\ \ Q_1>0,\ \ Q_2>0,\ \ R>0$ and $Y>0$ are the same as those in (3.9). Taking the derivative of $V_2(x_t)$ with respect to t along any trajectory of solution of (3.7) yields

$$
\begin{aligned}
\dot{V}_2(x_t)=&2D^T(x_t)P\dot{D}(x_t)+x^T(t)Qx(t)-x^T(t-\tau_e)Qx(t-\tau_e)+x^T(t)Q_1x(t)\\
&-x^T(t-d)Q_1x(t-d)+\dot{x}^T(t)Q_2\dot{x}^T(t)-\dot{x}(t-d)Q_2\dot{x}(t-d)\\
&+\dot{x}^T(t)(dY+\tau_eR)\dot{x}(t)-\int_{t-\tau_e}^{t}\dot{x}^T(s)R\dot{x}(s)ds\\
&-\int_{t-d}^{t}\dot{x}^T(s)Y\dot{x}(s)ds. \qquad (3.18)
\end{aligned}
$$

The following equalities hold:

$$
\begin{aligned}
&2[x^T(t)K_1^T+x^T(t-\tau_e)K_2^T+\dot{x}^T(t)K_3^T+\dot{x}^T(t-d)K_4^T+x^T(t-d)K_5^T]\times\\
&[Ax(t)+Bx(t-\tau_e)+C\dot{x}(t-d)-\dot{x}(t)]=0 \qquad (3.19)
\end{aligned}
$$

$$2[x^T(t)L_1^T + x^T(t-\tau_e)L_2^T + \dot{x}^T(t)L_3^T + \dot{x}^T(t-d)L_4^T + x^T(t-d)L_5^T] \times [x(t) - x(t-\tau_e) - \int_{t-\tau_e}^{t} \dot{x}(s)ds] = 0 \tag{3.20}$$

$$2[x^T(t)M_1^T + x^T(t-\tau_e)M_2^T + \dot{x}^T(t)M_3^T + \dot{x}^T(t-d)M_4^T + x^T(t-d)M_5^T] \times [x(t) - x(t-d) - \int_{t-d}^{t} \dot{x}(s)ds] = 0 \tag{3.21}$$

where $K_i(i = 1, 2, 3, 4, 5)$ are the same as those in (3.12), $L_i(i = 1, 2, 3, 4, 5)$ are the same as those in (3.13), $M_i(i = 1, 2, 3, 4, 5)$ are the same as those in (3.14). Then from (3.18), (3.19), (3.20), (3.21). we have

$$\begin{aligned}
\dot{V}_2(x_t) =& 2D^T(x_t)P\dot{D}(x_t) + x^T(t)Qx(t) - x^T(t-\tau_e)Qx(t-\tau_e) + x^T(t)Q_1x(t) \\
&- x^T(t-d)Q_1x(t-d) + \dot{x}^T(t)Q_2\dot{x}^T(t) - \dot{x}(t-d)Q_2\dot{x}(t-d) \\
&+ \dot{x}^T(t)(dY + \tau_eR)\dot{x}(t) - \int_{t-\tau_e}^{t} \dot{x}^T(s)R\dot{x}(s)ds - \int_{t-d}^{t} \dot{x}^T(s)Y\dot{x}(s)ds \\
&+ 2[x^T(t)K_1^T + x^T(t-\tau_e)K_2^T + \dot{x}^T(t)K_3^T + \dot{x}^T(t-d)K_4^T + x^T(t-d)K_5^T] \\
&\times [Ax(t) + Bx(t-\tau_e) + C\dot{x}(t-d) - \dot{x}(t)] \\
&+ 2[x^T(t)L_1^T + x^T(t-\tau_e)L_2^T + \dot{x}^T(t)L_3^T + \dot{x}^T(t-d)L_4^T + x^T(t-d)L_5^T] \\
&\times [x(t) - x(t-\tau_e) - \int_{t-\tau_e}^{t} \dot{x}(s)ds] \\
&+ 2[x^T(t)M_1^T + x^T(t-\tau_e)M_2^T + \dot{x}^T(t)M_3^T + \dot{x}^T(t-d)M_4^T + x^T(t-d)M_5^T] \\
&\times [x(t) - x(t-d) - \int_{t-d}^{t} \dot{x}(s)ds] \\
=& \frac{1}{\tau_e}\int_{t-\tau_e}^{t} \omega^T(t,s)\tilde{\phi}_1\omega(t,s)ds \\
&+ \frac{1}{d}\int_{t-d}^{t} \omega^T(t,s)\tilde{\phi}_2\omega(t,s)ds
\end{aligned} \tag{3.22}$$

where

$$\tilde{\phi}_1 = \begin{pmatrix}
\phi_{11}+Z_{11} & \phi_{12}+Z_{12} & \phi_{13}+Z_{13} & \phi_{14}+Z_{14} & \phi_{15}+Z_{15} & \tau_eL_1^T \\
* & \phi_{22}+Z_{22} & \phi_{23}+Z_{23} & \phi_{24}+Z_{24} & \phi_{25}+Z_{25} & \tau_eL_2^T \\
* & * & \phi_{33}+Z_{33} & \phi_{34}+Z_{34} & \phi_{35}+Z_{35} & \tau_eL_3^T \\
* & * & * & \phi_{44}+Z_{44} & \phi_{45}+Z_{45} & \tau_eL_4^T \\
* & * & * & * & \phi_{55}+Z_{55} & \tau_eL_5^T \\
* & * & * & * & * & -\tau_eR
\end{pmatrix}$$

$$\tilde{\phi}_2 = \begin{pmatrix} -Z_{11} & -Z_{12} & -Z_{13} & -Z_{14} & -Z_{15} & dM_1^T \\ * & -Z_{22} & -Z_{23} & -Z_{24} & -Z_{25} & dM_2^T \\ * & * & -Z_{33} & -Z_{34} & -Z_{35} & dM_3^T \\ * & * & * & -Z_{44} & -Z_{45} & dM_4^T \\ * & * & * & * & -Z_{55} & dM_5^T \\ * & * & * & * & * & -dY \end{pmatrix}.$$

From (3.22), if $\tilde{\phi}_1 < 0$ and $\tilde{\phi}_2 < 0$, then $\dot{V}_2(x_t) \leq -\lambda_2 \|x(t)\|^2$, for some $\lambda_2 > 0$. Notice that (3.8) implies $\tilde{\phi}_1 < 0$ and $\tilde{\phi}_2 < 0$. As in *Case I*, $V(x_t) \leq (a+b+c+d+e+g)\|x_t\|^2$. Since operator $D(x_t)$ stable, the solution $x = 0$ of system (3.6) is asymptotically stable according to Theorem 2.3.2 .

Case III $\tau_e < \tau(t) < \tau_M$

For this case, we choose the Lyapunov-Krasovskii functional candidate as

$$\begin{aligned} V_3(x_t) =& D^T(x_t)PD(x_t) + \int_{t-\tau_e}^{t} x^T(s)Qx(s)ds + \int_{t-d}^{t} x^T(s)Q_1x(s)ds \\ &+ \int_{t-d}^{t} \dot{x}^T(s)Q_2\dot{x}(s)ds + \int_{-\tau_e}^{0} ds \int_{t+s}^{t} \dot{x}^T(\theta)R\dot{x}(\theta)d\theta \\ &+ \int_{-\tau_M}^{-\tau_e} ds \int_{t+s}^{t} \dot{x}^T(\theta)S\dot{x}(\theta)d\theta + \int_{-d}^{0} ds \int_{t+s}^{t} \dot{x}^T(\theta)Y\dot{x}(\theta)d\theta \end{aligned} \tag{3.23}$$

where $P > 0, Q > 0, Q_1 > 0, Q_2 > 0, R > 0, S > 0$ and $Y > 0$ are the same as those in (3.9). Similar to *Case I*, we can prove that system (3.6) is asymptotically stable. The proof is complete.

3.2 Delay-dependent Criterion for Asymptotic Stability for Uncertain Neutral System

In this section, we deal with the problem for asymptotic stability of the zero solution for uncertain neutral system,

$$\begin{aligned}\dot{x}(t) =&(A+\Delta A(t))x(t)+(B+\Delta B(t))x(t-\tau_e)\\ &-(B+\Delta B(t))\int_{t-\tau(t)}^{t-\tau_e}\dot{x}(s)ds+C\dot{x}(t-d).\end{aligned} \tag{3.24}$$

Theorem 3.2.1 *For given nonnegative scalars τ_m and τ_M, system (3.1) is asymptotically stable for any $\tau(t)$ satisfying $0 \leq \tau_m \leq \tau(t) \leq \tau_M$ if there exist some matrices $P, Q, Q_1, Q_2, R, S, Y > 0$ and $L_i, K_i, M_i (i = 1, 2, 3, 4, 5)$ such that*

$$\Sigma = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} & \xi_{15} & 2dM_1^T & 2\tau_e L_1^T & \rho K_1^T(B+\Delta B(t)) \\ * & \xi_{22} & \xi_{23} & \xi_{24} & \xi_{25} & 2dM_2^T & 2\tau_e L_2^T & \rho K_2^T(B+\Delta B(t)) \\ * & * & \xi_{33} & \xi_{34} & \xi_{35} & 2dM_3^T & 2\tau_e L_3^T & \rho K_3^T(B+\Delta B(t)) \\ * & * & * & \xi_{44} & \xi_{45} & 2dM_4^T & 2\tau_e L_4^T & \rho K_4^T(B+\Delta B(t)) \\ * & * & * & * & \xi_{55} & 2dM_5^T & 2\tau_e L_5^T & \rho K_5^T(B+\Delta B(t)) \\ * & * & * & * & * & -2dY & 0 & 0 \\ * & * & * & * & * & * & -2\tau_e R & 0 \\ * & * & * & * & * & * & * & -\rho S \end{pmatrix} < 0 \tag{3.25}$$

where

$\xi_{11} = \phi_{11} + K_1^T\Delta A(t) + \Delta A(t)^T K_1 + Z_{11}$

$\xi_{12} = \phi_{12} + \Delta A^T(t)K_2 + K_1^T\Delta B(t) + Z_{12}$

$\xi_{13} = \phi_{13} + \Delta A(t)^T K_3 + Z_{13}$

$\xi_{14} = \phi_{14} + \Delta A(t)^T K_4 + Z_{14}$

$\xi_{15} = \phi_{15} + \Delta A(t)^T K_5 + Z_{15}$

$\xi_{22} = \phi_{22} + \Delta B(t)^T K_2 + K_2^T\Delta B(t) + Z_{22}$

$\xi_{23} = \phi_{23} + \Delta B(t)^T K_3 + Z_{23}$

$\xi_{24} = \phi_{24} + \Delta B(t)^T K_4 + Z_{24}$

$\xi_{25} = \phi_{25} + \Delta B(t)^T K_5 + Z_{25}$

Proof We consider three cases, as in Theorem 3.1.1. Replacing $A = A + \Delta A(t)$

and $B = B + \Delta B(t)$.

Case I: $\tau_m \le \tau(t) < \tau_e$

Choose a Lyapunov-Krasovskii functional candidate as

$$\begin{aligned} V_1(x_t) =& D^T(x_t)PD(x_t) + \int_{t-\tau_e}^{t} x^T(s)Qx(s)ds + \int_{t-d}^{t} x^T(s)Q_1x(s)ds \\ &+ \int_{t-d}^{t} \dot{x}^T(s)Q_2\dot{x}(s)ds + \int_{-\tau_e}^{0} ds \int_{t+s}^{t} \dot{x}^T(\theta)R\dot{x}(\theta)d\theta \\ &+ \int_{-\tau_e}^{-\tau_m} ds \int_{t+s}^{t} \dot{x}^T(\theta)S\dot{x}(\theta)d\theta + \int_{-d}^{0} ds \int_{t+s}^{t} \dot{x}^T(\theta)Y\dot{x}(\theta)d\theta \end{aligned}$$

where $P > 0, Q > 0, Q_1 > 0, Q_2 > 0, R > 0, S > 0$ and $Y > 0$. Taking the derivative of $V_1(x_t)$ with respect to t along the trajectory of (3.24) yields

$$\begin{aligned} \dot{V}_1(x_t) =& 2D^T(x_t)P\dot{D}(x_t) + x^T(t)Qx(t) - x^T(t-\tau_e)Qx(t-\tau_e) \\ &+ x^T(t)Q_1x(t) - x^T(t-d)Q_1x(t-d) + \dot{x}^T(t)Q_2\dot{x}^T(t) \\ &- \dot{x}^T(t-d)Q_2\dot{x}^T(t-d) + \dot{x}^T(t)(dY + \tau_eR + \rho S)\dot{x}(t) - \int_{t-\tau_e}^{t} \dot{x}^T(s)R\dot{x}(s)ds \\ &- \int_{t-\tau_e}^{t-\tau_m} \dot{x}^T(s)S\dot{x}(s)ds - \int_{t-d}^{t} \dot{x}^T(s)Y\dot{x}(s)ds \end{aligned}$$

Since

$$- \int_{t-\tau_e}^{t-\tau_m} \dot{x}^T(s)S\dot{x}(s)ds \le - \int_{t-\tau_e}^{t-\tau(t)} \dot{x}^T(s)S\dot{x}(s)ds \tag{3.26}$$

and from the following equalities :

$$\begin{aligned} &2[x^T(t)K_1^T + x^T(t-\tau_e)K_2^T + \dot{x}^T(t)K_3^T + \dot{x}^T(t-d)K_4^T + x^T(t-d)K_5^T]\times \\ &[(A + \Delta A(t))x(t) + (B + \Delta B(t))x(t-\tau_e) - (B + \Delta B(t)) \int_{t-\tau(t)}^{t-\tau_e} \dot{x}(s)ds \\ &+ C\dot{x}(t-d) - \dot{x}(t)] = 0 \end{aligned}$$

$$\begin{aligned} &2[x^T(t)L_1^T + x^T(t-\tau_e)L_2^T + \dot{x}^T(t)L_3^T + \dot{x}^T(t-d)L_4^T + x^T(t-d)L_5^T]\times \\ &[x(t) - x(t-\tau_e) - \int_{t-\tau_e}^{t} \dot{x}(s)ds] = 0 \end{aligned}$$

$$\begin{aligned} &2[x^T(t)M_1^T + x^T(t-\tau_e)M_2^T + \dot{x}^T(t)M_3^T + \dot{x}^T(t-d)M_4^T + x^T(t-d)M_5^T]\times \\ &[x(t) - x(t-d) - \int_{t-d}^{t} \dot{x}(s)ds] = 0 \end{aligned}$$

where $K_i, L_i, M_i (i = 1, 2, 3, 4, 5)$ are some matrices of appropriate dimensions, then we have

$$
\begin{aligned}
\dot{V}_1(x_t) \leq & 2D^T(x_t)P\dot{D}(x_t) + x^T(t)Qx(t) - x^T(t-\tau_e)Qx(t-\tau_e) \\
& + x^T(t)Q_1x(t) - x^T(t-d)Q_1x(t-d) + \dot{x}^T(t)Q_2\dot{x}^T(t) - \dot{x}(t-d)Q_2\dot{x}(t-d) \\
& + \dot{x}^T(t)(dY + \tau_eR + \rho S)\dot{x}(t) - \int_{t-\tau_e}^{t} \dot{x}^T(s)R\dot{x}(s)ds \\
& - \int_{t-\tau_e}^{t-\tau(t)} \dot{x}^T(s)S\dot{x}(s)ds - \int_{t-d}^{t} \dot{x}^T(s)Y\dot{x}(s)ds \\
& + 2[x^T(t)K_1^T + x^T(t-\tau_e)K_2^T + \dot{x}^T(t)K_3^T + \dot{x}^T(t-d)K_4^T + x^T(t-d)K_5^T] \\
& \times [(A + \Delta A(t))x(t) + (B + \Delta B(t))x(t-\tau_e) - (B + \Delta B(t)) \int_{t-\tau(t)}^{t-\tau_e} \dot{x}(s)ds \\
& + C\dot{x}(t-d) - \dot{x}(t)] \\
& + 2[x^T(t)L_1^T + x^T(t-\tau_e)L_2^T + \dot{x}^T(t)L_3^T + \dot{x}^T(t-d)L_4^T + x^T(t-d)L_5^T] \\
& \times [x(t) - x(t-\tau_e) - \int_{t-\tau_e}^{t} \dot{x}(s)ds] \\
& + 2[x^T(t)M_1^T + x^T(t-\tau_e)M_2^T + \dot{x}^T(t)M_3^T + \dot{x}^T(t-d)M_4^T + x^T(t-d)M_5^T] \\
& \times [x(t) - x(t-d) - \int_{t-d}^{t} \dot{x}(s)ds] \\
= & 2[x(t) - Cx(t-d)]^TP[\dot{x}(t) - C\dot{x}(t-d)] + x^T(t)Qx(t) - x^T(t-\tau_e)Qx(t-\tau_e) \\
& + x^T(t)Q_1x(t) - x^T(t-d)Q_1x(t-d) + \dot{x}^T(t)Q_2\dot{x}^T(t) - \dot{x}(t-d)Q_2\dot{x}(t-d) \\
& + \dot{x}^T(t)(dY + \tau_eR + \rho S)\dot{x}(t) - \int_{t-\tau_e}^{t} \dot{x}^T(s)R\dot{x}(s)ds \\
& - \int_{t-\tau_e}^{t-\tau(t)} \dot{x}^T(s)S\dot{x}(s)ds - \int_{t-d}^{t} \dot{x}^T(s)Y\dot{x}(s)ds \\
& + 2[x^T(t)K_1^T + x^T(t-\tau_e)K_2^T + \dot{x}^T(t)K_3^T + \dot{x}^T(t-d)K_4^T + x^T(t-d)K_5^T] \\
& \times [(A + \Delta A(t))x(t) + (B + \Delta B(t))x(t-\tau_e) - (B + \Delta B(t)) \int_{t-\tau(t)}^{t-\tau_e} \dot{x}(s)ds \\
& + 2[x^T(t)M_1^T + x^T(t-\tau_e)M_2^T + \dot{x}^T(t)M_3^T + \dot{x}^T(t-d)M_4^T + x^T(t-d)M_5^T] \\
& \times [x(t) - x(t-d) - \int_{t-d}^{t} \dot{x}(s)ds]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{d}\int_{t-d}^{t} \omega^T(t,s)\hat{\phi}_1\omega(t,s)ds + \frac{1}{\tau_e}\int_{t-\tau_e}^{t} \omega^T(t,s)\hat{\phi}_2\omega(t,s)ds \\
&\quad + \frac{1}{\tau_e-\tau(t)}\int_{t-\tau_e}^{t-\tau(t)} \omega^T(t,s)\hat{\phi}_3\omega(t,s)ds \qquad (3.27)
\end{aligned}
$$

where

$\omega^T(t,s) = [x^T(t) \quad x^T(t-\tau_e) \quad \dot{x}^T(t) \quad \dot{x}^T(t-d) \quad x^T(t-d) \quad -\dot{x}^T(s)]$ where

$$
\hat{\phi}_1 = \frac{1}{2}\begin{pmatrix}
\xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} & \xi_{15} & 2dM_1^T \\
* & \xi_{22} & \xi_{23} & \xi_{24} & \xi_{25} & 2dM_2^T \\
* & * & \phi_{33}+Z_{33} & \phi_{34}+Z_{34} & \phi_{35}+Z_{35} & 2dM_3^T \\
* & * & * & \phi_{44}+Z_{44} & \phi_{45}+Z_{45} & 2dM_4^T \\
* & * & * & * & \phi_{55}+Z_{55} & 2dM_5^T \\
* & * & * & * & * & -2dY
\end{pmatrix}
$$

$$
\hat{\phi}_2 = \frac{1}{2}\begin{pmatrix}
\xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} & \xi_{15} & 2\tau_e L_1^T \\
* & \xi_{22} & \xi_{23} & \xi_{24} & \xi_{25} & 2\tau_e L_2^T \\
* & * & \phi_{33}+Z_{33} & \phi_{34}+Z_{34} & \phi_{35}+Z_{35} & 2\tau_e L_3^T \\
* & * & * & \phi_{44}+Z_{44} & \phi_{45}+Z_{45} & 2\tau_e L_4^T \\
* & * & * & * & \phi_{55}+Z_{55} & 2\tau_e L_5^T \\
* & * & * & * & * & -2\tau_e R
\end{pmatrix}
$$

$$
\hat{\phi}_3 = \begin{pmatrix}
-Z_{11} & -Z_{12} & -Z_{13} & -Z_{14} & -Z_{15} & (\tau(t)-\tau_e)K_1^T(B+\Delta B(t)) \\
* & -Z_{22} & -Z_{23} & -Z_{24} & -Z_{25} & (\tau(t)-\tau_e)K_2^T(B+\Delta B(t)) \\
* & * & -Z_{33} & -Z_{34} & -Z_{35} & (\tau(t)-\tau_e)K_3^T(B+\Delta B(t)) \\
* & * & * & -Z_{44} & -Z_{45} & (\tau(t)-\tau_e)K_4^T(B+\Delta B(t)) \\
* & * & * & * & -Z_{55} & (\tau(t)-\tau_e)K_5^T(B+\Delta B(t)) \\
* & * & * & * & * & -|\tau(t)-\tau_e|S
\end{pmatrix}.
$$

$Z_{11}, Z_{22}, Z_{33}, Z_{44}, Z_{55} > 0, Z_{12}, Z_{13}, Z_{14}, Z_{15}, Z_{23}, Z_{24}, Z_{25}, Z_{34}, Z_{35}$ and Z_{45} are some parameter matrices of appropriate dimensions. From (3.26) if $\hat{\phi}_1 < 0, \hat{\phi}_2 < 0$ and $\hat{\phi}_3 < 0$, then $\dot{V}_1(x_t) \leq -\lambda_1\|x(t)\|^2$ for some $\lambda_1 > 0$. Pre and post-multiplying both sides of $\hat{\phi}_3 < 0$ by diag$\{I, I, I, I, I, sgn(\tau(t) - \tau_e)\}$, we get that

$$\begin{pmatrix} -Z_{11} & -Z_{12} & -Z_{13} & -Z_{14} & -Z_{15} & \rho K_1^T(B + \Delta B(t)) \\ * & -Z_{22} & -Z_{23} & -Z_{24} & -Z_{25} & \rho K_2^T(B + \Delta B(t)) \\ * & * & -Z_{33} & -Z_{34} & -Z_{35} & \rho K_3^T(B + \Delta B(t)) \\ * & * & * & -Z_{44} & -Z_{45} & \rho K_4^T(B + \Delta B(t)) \\ * & * & * & * & -Z_{55} & \rho K_5^T(B + \Delta B(t)) \\ * & * & * & * & * & -\rho S \end{pmatrix} < 0 \tag{3.28}$$

By Lemma 2.3.1 (Schur Complement [2]) implies $\hat{\phi}_3 < 0$. Let $\hat{\phi}_4 = \hat{\phi}_1 + \hat{\phi}_2$, in light of Lemma 2.3.4 (3.8) holds if and only if $\hat{\phi}_4 < 0$ and (3.16) simultaneously hold, and in light of Lemma 2.3.4 again $\hat{\phi}_4 < 0$ if and only if $\hat{\phi}_1 < 0$ and $\hat{\phi}_2 < 0$. Then (3.8) hold if and only if $\hat{\phi}_1 < 0, \hat{\phi}_2 < 0$ and (3.16) simultaneously hold. As in *Case I* of Theorem 3.1.1 It's obvious that $V_1(x_t) \geq \lambda_{min}(P)\|D(x_t)\|^2$ and $V_1(x_t) \leq (a+b+c+d+e+f+g)\|x_t\|^2$. Since operator $D(x_t)$ stable. Therefore, solution $x = 0$ of system (3.1) is asymptotically stable according to Theorem (2.3.2)

Case II $\tau(t) = \tau_e$

For this case, we choose a Lyapunov-Krasovskii functional candidate as

$$\begin{aligned} V_2(x_t) =& D^T(x_t)PD(x_t) + \int_{t-\tau_e}^{t} x^T(s)Qx(s)ds + \int_{t-d}^{t} x^T(s)Q_1x(s)ds \\ &+ \int_{t-d}^{t} \dot{x}^T(s)Q_2\dot{x}(s)ds + \int_{-\tau_e}^{0} ds \int_{t+s}^{t} \dot{x}^T(\theta)R\dot{x}(\theta)d\theta \\ &+ \int_{-d}^{0}\int_{t+s}^{t} \dot{x}^T(\theta)Y\dot{x}(\theta)d\theta \end{aligned} \tag{3.29}$$

where $P > 0,\ \ Q > 0,\ \ Q_1 > 0,\ \ Q_2 > 0,\ \ R > 0$ and $Y > 0$ are the same as those in (3.9). Taking the derivative of $V_2(x_t)$ with respect to t along any trajectory of solution of (3.24) yields

$$\begin{aligned} \dot{V}_2(x_t) =& 2D^T(x_t)P\dot{D}(x_t) + x^T(t)Qx(t) - x^T(t-\tau_e)Qx(t-\tau_e) + x^T(t)Q_1x(t) \\ &- x^T(t-d)Q_1x(t-d) + \dot{x}^T(t)Q_2\dot{x}^T(t) - \dot{x}(t-d)Q_2\dot{x}(t-d) \\ &+ \dot{x}^T(t)(dY + \tau_eR)\dot{x}(t) - \int_{t-\tau_e}^{t} \dot{x}^T(s)R\dot{x}(s)ds \\ &- \int_{t-d}^{t} \dot{x}^T(s)Y\dot{x}(s)ds. \end{aligned} \tag{3.30}$$

The following equalities hold:

$$2[x^T(t)K_1^T + x^T(t-\tau_e)K_2^T + \dot{x}^T(t)K_3^T + \dot{x}^T(t-d)K_4^T + x^T(t-d)K_5^T]\times$$
$$[(A+\Delta A(t))x(t) + (B+\Delta B(t))x(t-\tau_e) + C\dot{x}(t-d) - \dot{x}(t)] = 0 \qquad (3.31)$$

$$\begin{aligned}&2[x^T(t)L_1^T + x^T(t-\tau_e)L_2^T + \dot{x}^T(t)L_3^T + \dot{x}^T(t-d)L_4^T + x^T(t-d)L_5^T]\\ &\times [x(t) - x(t-\tau_e) - \int_{t-\tau_e}^{t} \dot{x}(s)ds] = 0\end{aligned} \qquad (3.32)$$

$$\begin{aligned}&2[x^T(t)M_1^T + x^T(t-\tau_e)M_2^T + \dot{x}^T(t)M_3^T + \dot{x}^T(t-d)M_4^T + x^T(t-d)M_5^T]\\ &\times [x(t) - x(t-d) - \int_{t-d}^{t} \dot{x}(s)ds] = 0\end{aligned} \qquad (3.33)$$

where $K_i(i = 1,2,3,4,5)$ are the same as those in (3.12), $L_i(i = 1,2,3,4,5)$ are the same as those in (3.13), $M_i(i = 1,2,3,4,5)$ are the same as those in (3.14). Then from (3.30), (3.31), (3.32), (3.33). we have

$$\begin{aligned}\dot{V}_2(x_t) =& 2D^T(x_t)P\dot{D}(x_t) + x^T(t)Qx(t) - x^T(t-\tau_e)Qx(t-\tau_e) + x^T(t)Q_1x(t)\\
&- x^T(t-d)Q_1x(t-d) + \dot{x}^T(t)Q_2\dot{x}^T(t) - \dot{x}(t-d)Q_2\dot{x}(t-d)\\
&+ \dot{x}^T(t)(dY + \tau_e R)\dot{x}(t) - \int_{t-\tau_e}^{t} \dot{x}^T(s)R\dot{x}(s)ds - \int_{t-d}^{t} \dot{x}^T(s)Y\dot{x}(s)ds\\
&+ 2[x^T(t)K_1^T + x^T(t-\tau_e)K_2^T + \dot{x}^T(t)K_3^T + \dot{x}^T(t-d)K_4^T + x^T(t-d)K_5^T]\\
&\times [(A+\Delta A(t))x(t) + (B+\Delta B(t))x(t-\tau_e) + C\dot{x}(t-d) - \dot{x}(t)]\\
&+ 2[x^T(t)L_1^T + x^T(t-\tau_e)L_2^T + \dot{x}^T(t)L_3^T + \dot{x}^T(t-d)L_4^T + x^T(t-d)L_5^T]\\
&\times [x(t) - x(t-\tau_e) - \int_{t-\tau_e}^{t} \dot{x}(s)ds]\\
&+ 2[x^T(t)M_1^T + x^T(t-\tau_e)M_2^T + \dot{x}^T(t)M_3^T + \dot{x}^T(t-d)M_4^T + x^T(t-d)M_5^T]\\
&\times [x(t) - x(t-d) - \int_{t-d}^{t} \dot{x}(s)ds]\\
=& \frac{1}{\tau_e}\int_{t-\tau_e}^{t} \omega^T(t,s)\tilde{\phi}_1\omega(t,s)ds\\
&+ \frac{1}{d}\int_{t-d}^{t} \omega^T(t,s)\tilde{\phi}_2\omega(t,s)ds\end{aligned} \qquad (3.34)$$

where

$$\tilde{\phi}_1 = \begin{pmatrix} \xi_{11}+Z_{11} & \xi_{12}+Z_{12} & \xi_{13}+Z_{13} & \xi_{14}+Z_{14} & \xi_{15}+Z_{15} & \tau_e L_1^T \\ * & \xi_{22}+Z_{22} & \xi_{23}+Z_{23} & \xi_{24}+Z_{24} & \xi_{25}+Z_{25} & \tau_e L_2^T \\ * & * & \xi_{33}+Z_{33} & \xi_{34}+Z_{34} & \xi_{35}+Z_{35} & \tau_e L_3^T \\ * & * & * & \xi_{44}+Z_{44} & \xi_{45}+Z_{45} & \tau_e L_4^T \\ * & * & * & * & \xi_{55}+Z_{55} & \tau_e L_5^T \\ * & * & * & * & * & -\tau_e R \end{pmatrix}$$

$$\tilde{\phi}_2 = \begin{pmatrix} -Z_{11} & -Z_{12} & -Z_{13} & -Z_{14} & -Z_{15} & dM_1^T \\ * & -Z_{22} & -Z_{23} & -Z_{24} & -Z_{25} & dM_2^T \\ * & * & -Z_{33} & -Z_{34} & -Z_{35} & dM_3^T \\ * & * & * & -Z_{44} & -Z_{45} & dM_4^T \\ * & * & * & * & -Z_{55} & dM_5^T \\ * & * & * & * & * & -dY \end{pmatrix}.$$

From (3.34), if $\tilde{\phi}_1 < 0$ and $\tilde{\phi}_2 < 0$, then $\dot{V}_2(x_t) \leq -\lambda_2 \|x(t)\|^2$, for some $\lambda_2 > 0$. Notice that (3.24) implies $\tilde{\phi}_1 < 0$ and $\tilde{\phi}_2 < 0$. As in *Case I* in Theorem 3.1.1, $V(x_t) \leq (a+b+c+d+e+g)\|x_t\|^2$. Since operator $D(x_t)$ stable, the solution $x = 0$ of system (3.1) is asymptotically stable according to Theorem 2.3.2 .

Case III $\tau_e < \tau(t) \leq \tau_M$

For this case, we choose the Lyapunov-Krasovskii functional candidate as

$$\begin{aligned} V_3(x_t) =& D^T(x_t)PD(x_t) + \int_{t-\tau_e}^{t} x^T(s)Qx(s)ds + \int_{t-d}^{t} x^T(s)Q_1x(s)ds \\ &+ \int_{t-d}^{t} \dot{x}^T(s)Q_2\dot{x}(s)ds + \int_{-\tau_e}^{0} ds \int_{t+s}^{t} \dot{x}^T(\theta)R\dot{x}(\theta)d\theta \\ &+ \int_{-\tau_M}^{-\tau_e} ds \int_{t+s}^{t} \dot{x}^T(\theta)S\dot{x}(\theta)d\theta + \int_{-d}^{0} ds \int_{t+s}^{t} \dot{x}^T(\theta)Y\dot{x}(\theta)d\theta \end{aligned} \tag{3.35}$$

where $P > 0, Q > 0, Q_1 > 0, Q_2 > 0, R > 0, S > 0$ and $Y > 0$ are the same as those in (3.9). Similar to *Case I*, we can prove that system (3.1) is asymptotically stable. The proof is complete.

Since LMI of theorem 3.2.1 depend on time, which can not be solved by LMI control toolbox. So we give the new theorem 3.2.2 to avoid this problem.

Theorem 3.2.2 *For given nonnegative scalars τ_m and τ_M, system (3.1) is asymptotically stable for any $\tau(t)$ satisfying $0 \leq \tau_m \leq \tau(t) \leq \tau_M$ if there exist constants $\epsilon > 0$ and some matrices $P, Q, Q_1, Q_2, R, S, Y > 0$ and $L_i, K_i, M_i (i = 1, 2, 3, 4, 5)$ such that*

$$\Sigma_2 = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} & 2dM_1^T & 2\tau_e L_1^T & \rho K_1^T B & K_1^T E & \epsilon G^T \\ * & \phi_{22} & \phi_{23} & \phi_{24} & \phi_{25} & 2dM_2^T & 2\tau_e L_2^T & \rho K_2^T B & K_2^T E & \epsilon G_1^T \\ * & * & \phi_{33} & \phi_{34} & \phi_{35} & 2dM_3^T & 2\tau_e L_3^T & \rho K_3^T B & K_3^T E & 0 \\ * & * & * & \phi_{44} & \phi_{45} & 2dM_4^T & 2\tau_e L_4^T & \rho K_4^T B & K_4^T E & 0 \\ * & * & * & * & \phi_{55} & 2dM_5^T & 2\tau_e L_5^T & \rho K_5^T B & K_5^T E & 0 \\ * & * & * & * & * & -2dY & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -2\tau_e R & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\rho S & 0 & 0 \\ * & * & * & * & * & * & * & * & -\epsilon I & 0 \\ * & * & * & * & * & * & * & * & * & -\epsilon I \end{pmatrix} < 0 \tag{3.36}$$

Proof We will sufficient to show that $\Sigma < 0$ if and only if $\Sigma_2 < 0$.

Consider

$$\Sigma = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} & \xi_{15} & 2dM_1^T & 2\tau_e L_1^T & \rho K_1^T (B + \Delta B(t)) \\ * & \xi_{22} & \xi_{23} & \xi_{24} & \xi_{25} & 2dM_2^T & 2\tau_e L_2^T & \rho K_2^T (B + \Delta B(t)) \\ * & * & \xi_{33} & \xi_{34} & \xi_{35} & 2dM_3^T & 2\tau_e L_3^T & \rho K_3^T (B + \Delta B(t)) \\ * & * & * & \xi_{44} & \xi_{45} & 2dM_4^T & 2\tau_e L_4^T & \rho K_4^T (B + \Delta B(t)) \\ * & * & * & * & \xi_{55} & 2dM_5^T & 2\tau_e L_5^T & \rho K_5^T (B + \Delta B(t)) \\ * & * & * & * & * & -2dY & 0 & 0 \\ * & * & * & * & * & * & -2\tau_e R & 0 \\ * & * & * & * & * & * & * & -\rho S \end{pmatrix}$$

Since $\Delta A(t) = EF(t)G$ and $\Delta B(t) = EF(t)G_1$

, then Σ=

$$\begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} & 2dM_1^T & 2\tau_e L_1^T & \rho K_1^T B \\ * & \phi_{22} & \phi_{23} & \phi_{24} & \phi_{25} & 2dM_2^T & 2\tau_e L_2^T & \rho K_2^T B \\ * & * & \phi_{33} & \phi_{34} & \phi_{35} & 2dM_3^T & 2\tau_e L_3^T & \rho K_3^T B \\ * & * & * & \phi_{44} & \phi_{45} & 2dM_4^T & 2\tau_e L_4^T & \rho K_4^T B \\ * & * & * & * & \phi_{55} & 2dM_5^T & 2\tau_e L_5^T & \rho K_5^T B \\ * & * & * & * & * & -2dY & 0 & 0 \\ * & * & * & * & * & * & -2\tau_e R & 0 \\ * & * & * & * & * & * & * & -\rho S \end{pmatrix}$$

$$+ \begin{pmatrix} K_1^T E \\ K_2^T E \\ K_3^T E \\ K_4^T E \\ K_5^T E \\ 0 \\ 0 \\ 0 \end{pmatrix} F(t) \begin{pmatrix} G & G_1 & 0 & 0 & 0 & 0 & 0 & \rho G_1 \end{pmatrix}$$

$$+ \begin{pmatrix} G^T \\ G_1^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \rho G_1^T \end{pmatrix} F^T(t) \begin{pmatrix} E^T K_1 & E^T K_2 & E^T K_3 & E^T K_4 & E^T K_5 & 0 & 0 & 0 \end{pmatrix}. \quad (3.37)$$

Using Lemma (2.3.3), $\Sigma < 0$ if and only if

$$\Theta_1 = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} & \xi_{15} & 2dM_1^T & 2\tau_e L_1^T & \rho K_1^T B \\ * & \xi_{22} & \xi_{23} & \xi_{24} & \xi_{25} & 2dM_2^T & 2\tau_e L_2^T & \rho K_2^T B \\ * & * & \phi_{33} & \phi_{34} & \phi_{35} & 2dM_3^T & 2\tau_e L_3^T & \rho K_3^T B \\ * & * & * & \phi_{44} & \phi_{45} & 2dM_4^T & 2\tau_e L_4^T & \rho K_4^T B \\ * & * & * & * & \phi_{55} & 2dM_5^T & 2\tau_e L_5^T & \rho K_5^T B \\ * & * & * & * & * & -2dY & 0 & 0 \\ * & * & * & * & * & * & -2\tau_e R & 0 \\ * & * & * & * & * & * & * & -\rho S \end{pmatrix}$$

$$+\frac{1}{\epsilon}\begin{pmatrix} K_1^T E \\ K_2^T E \\ K_3^T E \\ K_4^T E \\ K_5^T E \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} E^T K_1 & E^T K_2 & E^T K_3 & E^T K_4 & E^T K_5 & 0 & 0 & 0 \end{pmatrix}$$

$$+\epsilon \begin{pmatrix} G^T \\ G_1^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \rho G_1^T \end{pmatrix} \begin{pmatrix} G & G_1 & 0 & 0 & 0 & 0 & 0 & \rho G_1 \end{pmatrix} < 0. \tag{3.38}$$

Using Lemma 2.3.1, we get that $\Theta_1 < 0$ if and only if (3.36) hold.

Example 3.2.1 Consider neutral system without matric uncertainties with time-varying delay $\tau(t) = \sin^2(t)$, and $d = 0.4$ where

where

$$A = \begin{pmatrix} -1.0407 & -0.0049 \\ 0.0049 & -2.0508 \end{pmatrix}, B = \begin{pmatrix} -0.0585 & -0.0240 \\ -0.006 & -0.0706 \end{pmatrix}, C = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.4 \end{pmatrix}$$

By using the Matlab LMI toolbox, we can solve for matrices P, Q, Q_1, Q_2, R, S, Y and $L_i, K_i, M_i (i = 1, 2, 3, 4, 5)$ which satisfy the criterion of Theorem 3.1.1, thus the zero solution of system (3.6) is asymptotically stable.

Example 3.2.2 Consider uncertain neutral system with time-varying delay $\tau(t) = sin^2(t), d = 0.4$

where

$$A = \begin{pmatrix} -1.0407 & -0.0049 \\ 0.0049 & -2.0508 \end{pmatrix}, B = \begin{pmatrix} -0.0585 & -0.0240 \\ -0.006 & -0.0706 \end{pmatrix}, C = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.4 \end{pmatrix}$$

$$E = \begin{pmatrix} 0.2 & 0.2 \\ 0.1 & -0.2 \end{pmatrix}, G_1 = \begin{pmatrix} -0.1 & -0.1 \\ 0.1 & 0.2 \end{pmatrix}, F = \begin{pmatrix} -0.9 & 0 \\ 0 & 0.7 \end{pmatrix}, G = E.$$

By using the Matlab LMI toolbox, we can solve for matrices P, Q, Q_1, Q_2, R, S, Y, $L_i, K_i, M_i (i = 1, 2, 3, 4, 5)$ and constants $\epsilon_1, \epsilon_2, \epsilon_3$ which satisfies the criterion of Theorem 3.2.1, hence the zero solution of system (3.1) is asymptotically stable.

In numerical simulation, the two cases are given with the initial state $\phi(t) = [0.1; 0.1]^T$ for $t \in [-0.7, 0]$. Fig.3.1. shows the response of state variable x_1 with step size 0.01. Fig.3.2. shows the time response of state variable x_2 with step size 0.01. Hence, we might be obtain that the effect of uncertainties in neutral system will make differences to the dynamics of neutral system.

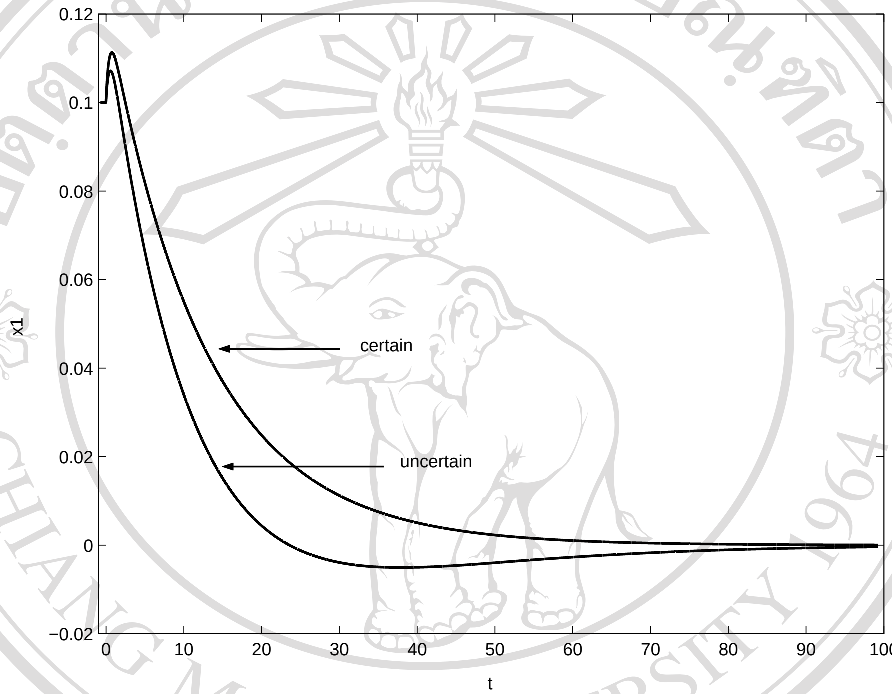

Figure 3.1: The trajectories of state variable x_1 of systems (3.1) and (3.6) in Examples 3.2.1. and 3.2.2.

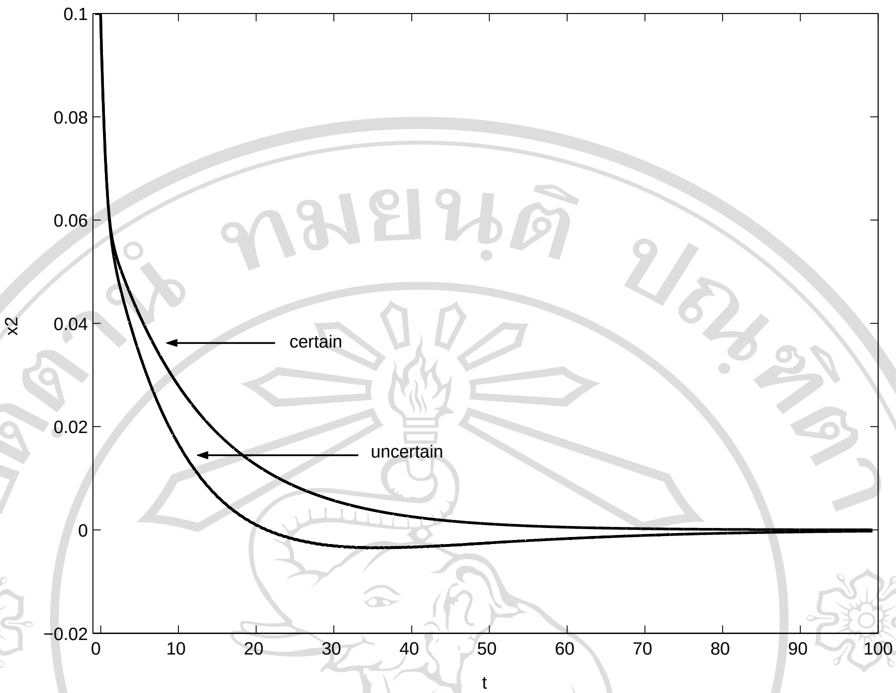

Figure 3.2: The trajectories of state variable x_2 of systems (3.1) and (3.6) in Examples 3.2.1. and 3.2.2.

Table 3.1: Maximum upper bounds of the time-varying delay τ_M for fixed constant delay d.

d	0.1	0.2	0.3	0.4	0.5	0.6	0.7
τ_M	0.8250	0.8210	0.8200	0.8160	0.8040	0.7990	0.7950

Table 3.2: Comparing some previous results with this paper

Results	Maximum allowable value of h
[3]	0.3
[5]	0.5658
[9]	0.74
Result of this paper(Constant delays $\tau(t) = \tau \geq 0$)	0.7910

Example 3.2.3 Consider the stability of the nominal system (3.6) with $h = \max\{d, \tau_M\}$ and $\tau(t)$ satisfy (3.4)

$$A = \begin{pmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{pmatrix}, B = \begin{pmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{pmatrix}, C = \begin{pmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{pmatrix}$$

In the case of $\mu = 0$ (μ is the upper bound of the derivative of time-varying delay), some upper bounds for delays that guarantee asymptotic stability of system in [3, 5, 9] are provided in Table 3.2.