Chapter 2

Basic Concepts

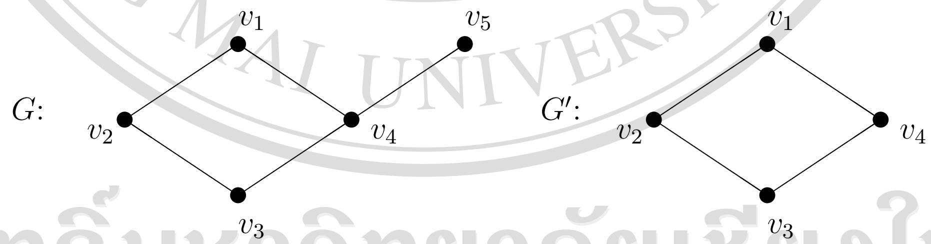
In this chapter we collect some important basic knowledge that will be needed for an understanding of the research work. Although the details are included in some cases, many of the fundamental principles of graph are stated without proof.

2.1 Basic Concepts on Graphs

Let us start with some basic definitions.

Definition 2.1.1. A graph G consists of a nonempty set of vertices, $V(G)$ and a set of edges $E(G)$ which is a subset of the set $\{\{x,y\} \subseteq V(G) \mid x \neq y\}$. A graph G' is called a subgraph of graph G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. If u and v are vertices of G, and $\{u,v\}$ belongs to $E(G)$, we say that u and v are adjacent.

Example 2.1.2. Consider the following graphs G and G',



Then

$$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$$
$$E(G) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}\}$$

and

$$V(G') = \{v_1, v_2, v_3, v_4\}$$
$$E(G') = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}\}.$$

Therefore, G' is a subgraph of the graph G

Definition 2.1.3. For each positive integer $n \geq 1$. A path of length n, we denote by P_n , is a graph with the vertex set $V(P_n) = \{1, 2, \ldots, n\}$ and edge set $E(P_n) = \{\{x, y\} \subseteq V(P_n) \mid |x - y| = 1\}$.

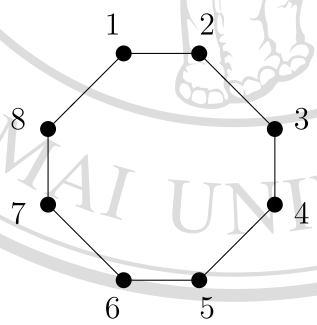
Definition 2.1.4. For each positive integer $n \geq 3$. A cycle of length n, we denote by C_n , is a graph with the vertex set $V(C_n) = \{1, 2, \ldots, n\}$ and edge set $E(C_n) = \{\{x, y\} \subseteq V(P_n) \mid |x - y| = 1\} \cup \{\{1, n\}\}$.

Definition 2.1.5. For each positive integer $n \geq 1$. A complete graph of n points, we denote by K_n, is a graph with the vertex set $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(K_n) = \{\{v_i, v_j\} \mid i, j = 1, 2, \ldots, n, i \neq j\}$.

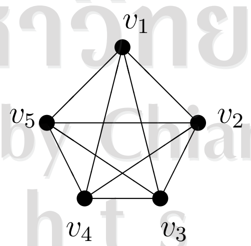
Example 2.1.6. Examples of path, cycle and complete graph.

P_4:

C_8:



K_5:

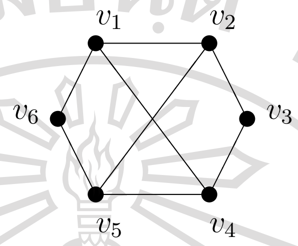


Definition 2.1.7. The number of all vertices which adjacent to the vertex v is called the degree of v, denote by $d(v)$.

Definition 2.1.8. A graph G is said to be regular of degree k (or k-regular) if each of its vertices has degree k.

Example 2.1.9. Consider the degree of each vertices of the following graph.

G :

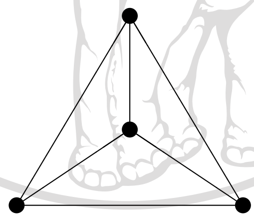


Then,

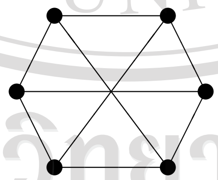
$$d(v_1) = d(v_2) = d(v_4) = d(v_5) = 3,\ d(v_3) = d(v_6) = 2.$$

Example 2.1.10. The followings graphs are 3-regular graphs.

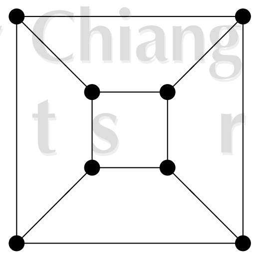
G_1 :



G_2 :



G_3 :

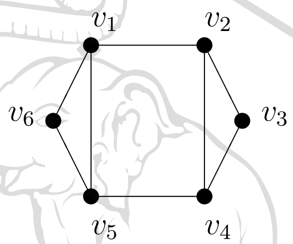


Definition 2.1.11. A graph G is said to be connected if each pair u, v of distinct vertices, there exist a sequence $a_1, a_2, a_3, \ldots, a_n$ in $V(G)$ such that $a_1 = u, a_n = v$ and $\{a_i, a_{i+1}\} \in E(G)$ for all $i = 1, 2, \ldots, n-1$.

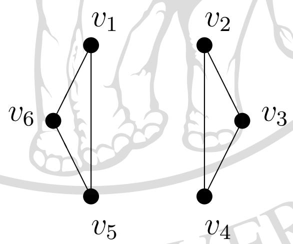
Definition 2.1.12. A subgraph H of the graph G is said to be a component of a graph G if H is a maximal connected subgraph, that is H is a connected subgraph and for any $v \in V(G)\backslash V(H)$, v is not adjacent to any vertices in $V(H)$.

Example 2.1.13.

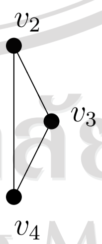
G :



H :



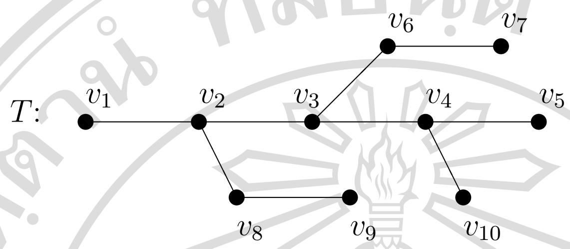
K :



Consider graphs G, H, K above, G and K are connected, but H is not connected graph, since there is no any sequence $a_1, a_2, \ldots, a_n$ in $V(H)$ such that $a_1 = v_1, a_n = v_2$ and $\{a_i, a_{i+1}\} \in E(H)$ for all $i = 1, 2, \ldots, n-1$. Moreover, K is a component of the graph H but not a component of the graph G.

Definition 2.1.14. A graph G is said to be a tree if G is connected and no cycle subgraphs.

Example 2.1.15. Tree.



2.2 The Adjacency Matrices

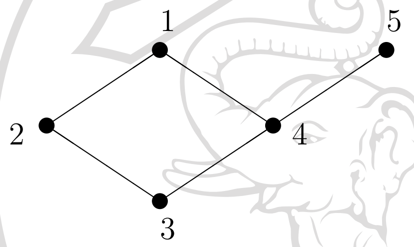
Definition 2.2.1. The adjacency matrices of a graph G is the $n \times n$ matrix $A(G)$ where the entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

We denote $\det(A(G))$, the determinant of the adjacency matrix $A(G)$.

Example 2.2.2. Consider the following graph,

G:



Then $\quad A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$

Note that, The determinant of this matrix is 0 because the first row are the same as the third row.

Definition 2.2.3. An elementary graph is a graph where each component is regular and has degree 1 or 2. In other words, each component is a single edge (K_2) or a cycle (C_n). A spanning elementary subgraph of Γ is an elementary subgraph which contains all vertices of Γ.

Definition 2.2.4. A general graph Γ with n vertices, m edges and c components, The rank of Γ and the co-rank of Γ are, respectively,

$$r(\Gamma) = n - c, s(\Gamma) = m - n + c.$$

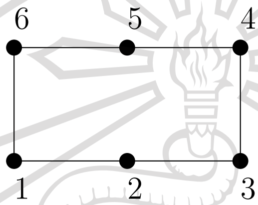
Proposition 2.2.5. **[3]** *Let A be the adjacency matrix of a graph G. Then*

$$\det(A(G)) = \sum (-1)^{r(\Gamma)} 2^{s(\Gamma)}$$

where the summation is over all spanning elementary subgraphs of Γ.

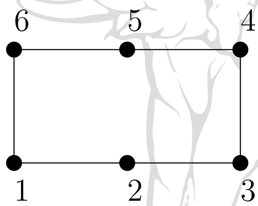
Example 2.2.6. Consider the following graph,

G:



There are altogether three spanning elementary subgraphs of G.

Γ_1:



$r(\Gamma_1) = 6 - 1 = 5$

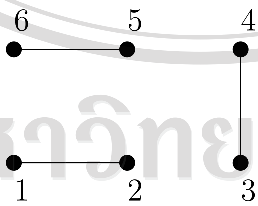
$s(\Gamma_1) = 6 - 6 + 1 = 1$

Γ_2:

$r(\Gamma_2) = 6 - 3 = 3$

$s(\Gamma_2) = 3 - 6 + 3 = 0$

Γ_3:



$r(\Gamma_3) = 6 - 3 = 3$

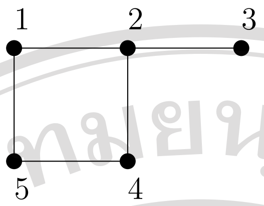
$s(\Gamma_3) = 3 - 6 + 3 = 0$

Then

$$\begin{aligned}\det(A(G)) &= (-1)^5(2)^1 + (-1)^3(2)^0 + (-1)^3(2)^0 \\ &= -2 - 1 - 1 \\ &= -4.\end{aligned}$$

Example 2.2.7. Consider the following graph,

H:



There is no spanning elementary subgraph of H.
Then $\det(A(H)) = 0$.

Theorem 2.2.8. **[5]** *Any trees contains at most one elementary spanning subgraph.*

Theorem 2.2.9. **[5]** *For any tree T, the number of vertices of which is odd, we have*

$$\det(A(T)) = 0.$$

Theorem 2.2.10. **[5]** *For any positive integer $n \geq 1$, the determinant of the adjacency matrices of the path P_n,*

$$\det(A(P_n)) = \begin{cases} (-1)^k & n{=}2k \\ 0 & otherwise. \end{cases}$$

Theorem 2.2.11. **[5]** *For any positive integer $n \geq 3$, the determinant of the adjacency matrices of the cycle C_n,*

$$\det(A(C_n)) = \begin{cases} 0 & n \equiv 0 (mod 4) \\ -4 & n \equiv 2 (mod 4) \\ 2 & otherwise. \end{cases}$$