Chapter 3
Main Results

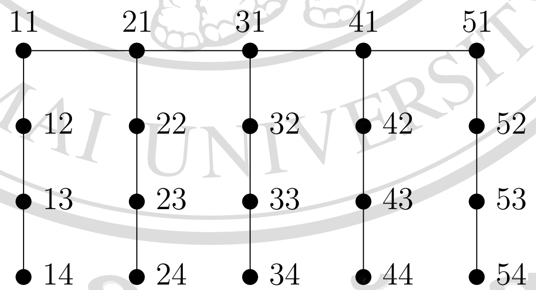
This chapter is divided into 2 sections. In section 3.1, we study the determinant of the adjacency matrices of graphs $P_n(P_m)$. In section 3.2, we study the determinant of the adjacency matrices of graphs $C_n(P_m)$.

3.1 Determinant of the Adjacency Matrices of Graph $P_n(P_m)$

Definition 3.1.1. Let m, n be positive integers such that $n \geq 1$. Let $P_n(P_m)$ be a graph with the vertex set $V(P_n(P_m)) = \{ij | i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$ and the edge set $E(P_n(P_m)) = \{\{ij, ik\} | \{j, k\} \in E(P_m)\} \cup \{\{i1, k1\} | \{i, k\} \in E(P_n)\}$ we call this graph the path P_n of paths P_m.

Example 3.1.2. The path P_5 of paths P_4.

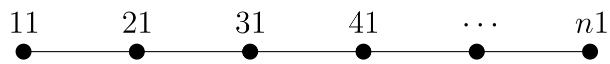
$P_5(P_4)$:



We can see that

Lemma 3.1.3. *$P_n(P_1) \cong P_n$ for all positive integer n.*

$P_n(P_1)$:



For a graph Γ with n vertices, m edges and c components, we denoted the number of vertices, the number of edges and the number of components of a graph Γ by $v(\Gamma)$, $e(\Gamma)$ and $c(\Gamma)$, respectively.

Lemma 3.1.4. *Let Γ be a spanning elementary subgraph of a graph G. If each component of graph Γ is K_2 then $e(\Gamma) = c(\Gamma) = \frac{v(\Gamma)}{2}$.*

Proof : *Let Γ be a spanning elementary subgraph of a graph G and every components of Γ is K_2. Assume that Γ consists of r components, then*

$v(\Gamma) = 2r$, $e(\Gamma) = r$ *and* $c(\Gamma) = r$.

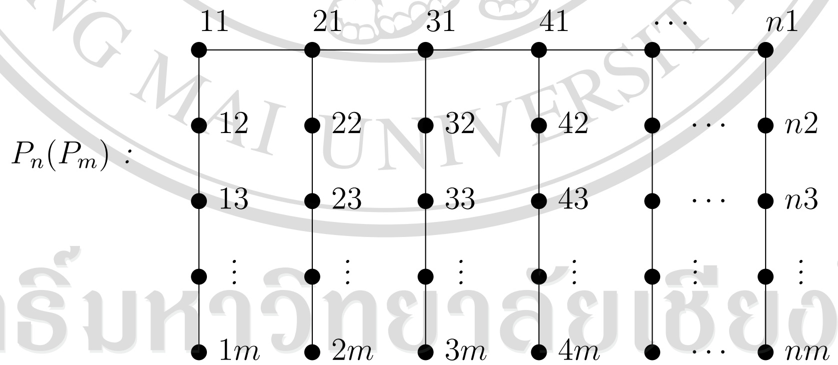
Therefore,

$e(\Gamma) = c(\Gamma) = \frac{v(\Gamma)}{2}$. □

Theorem 3.1.5. *For positive integers $n, m \geq 1$*

$$\det(A(P_n(P_m)) = \begin{cases} 0 & \text{if } nm \text{ is odd,} \\ (-1)^{\frac{nm}{2}} & \text{if } nm \text{ is even.} \end{cases}$$

Proof :



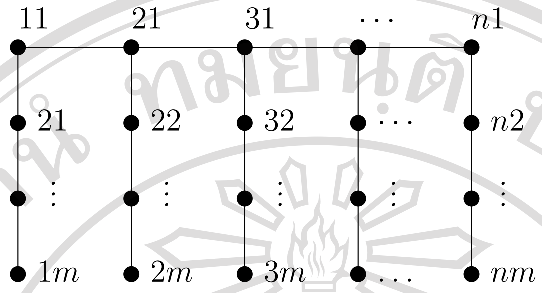
We consider the following two cases :

Case I : if nm is odd :

Since this case the graph $P_n(P_m)$ is a tree with odd vertices, $\det(A(P_n(P_m)) = 0$ *(by Theorem 2.2.9).*

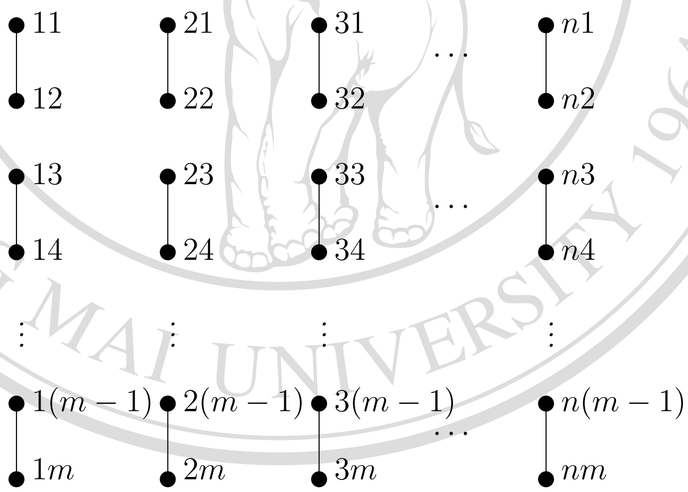
Case II : if nm is even :

In the case m is even,

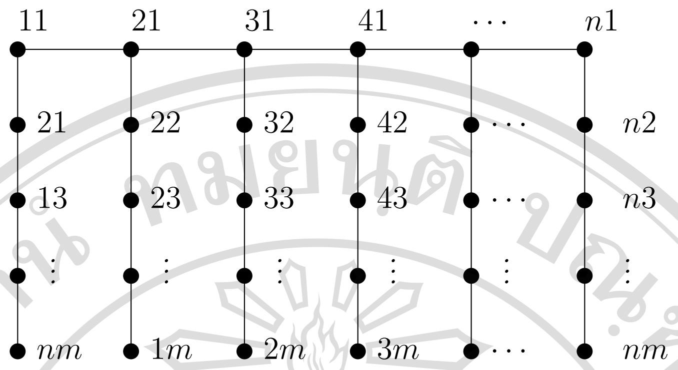


(*this picture* $n = 5, m = 4$)

There exists only one spanning elementary subgraph Γ *where every components is in the form* $\{i(2j-1), i(2j)\}, i = 1, 2, \ldots, n\ , j = 1, 2, \ldots, \frac{m}{2}$.

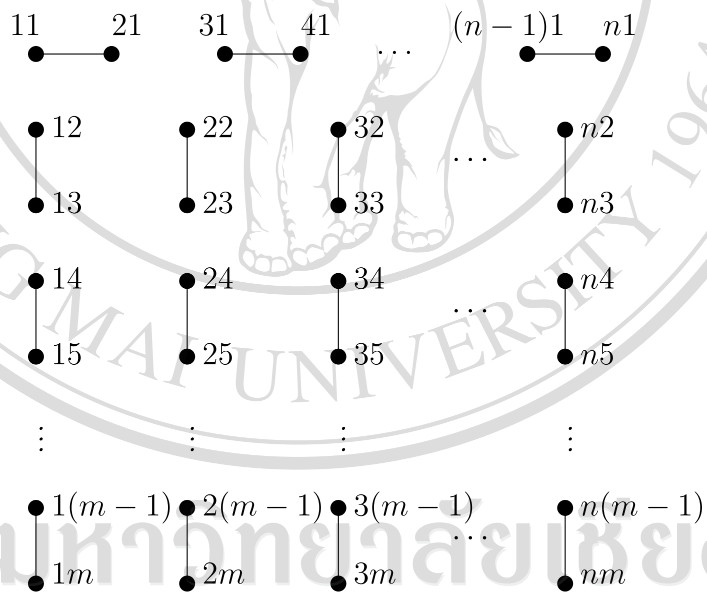


In the case m is odd, n is even,



(*this picture* $n = 6, m = 5$)

There also exists only one spanning elementary subgraph Γ *where every components is in one of the forms* $\{i(2j), i(2j+1)\}$ *or* $\{(2k-1)1, (2k)1\}, i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, \frac{m-1}{2},$ $k = 1, 2, \ldots, \frac{n}{2}$.



Therefore, in the case nm is even,

$e(\Gamma) = c(\Gamma) = \frac{v(\Gamma)}{2} = \frac{nm}{2}$.

Then

$r(\Gamma) = nm - \frac{nm}{2} = \frac{nm}{2}$

$s(\Gamma) = \frac{nm}{2} - nm + \frac{nm}{2} = 0,$

and

$$\det(A(P_n(P_m))) = (-1)^{\frac{nm}{2}}(2)^0 = (-1)^{\frac{nm}{2}}$$

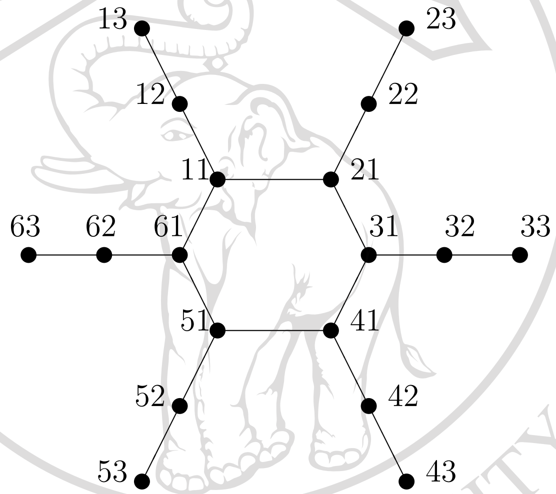
□

3.2 Determinant of the Adjacency Matrices of Graph $C_n(P_m)$

Definition 3.2.1. Let m, n be positive integers such that $n \geq 3$. Let $C_n(P_m)$ be a graph with the vertex set $V(C_n(P_m)) = \{ij | i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$ and the edge set $E(C_n(P_m)) = \{\{ij, ik\} | \{j, k\} \in E(P_m)\} \cup \{\{i1, k1\} | \{i, k\} \in E(C_n)\}$ we call this graph, the cycle C_n of paths P_m.

Example 3.2.2. The cycle C_6 of paths P_3.

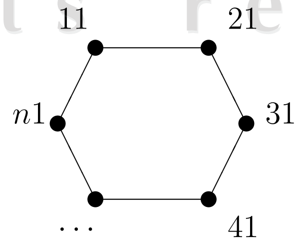
$C_6(P_3)$:



We can see that

Lemma 3.2.3. $C_n(P_1) \cong C_n$ *for all integer* $n \geq 3$.

$C_n(P_1)$:

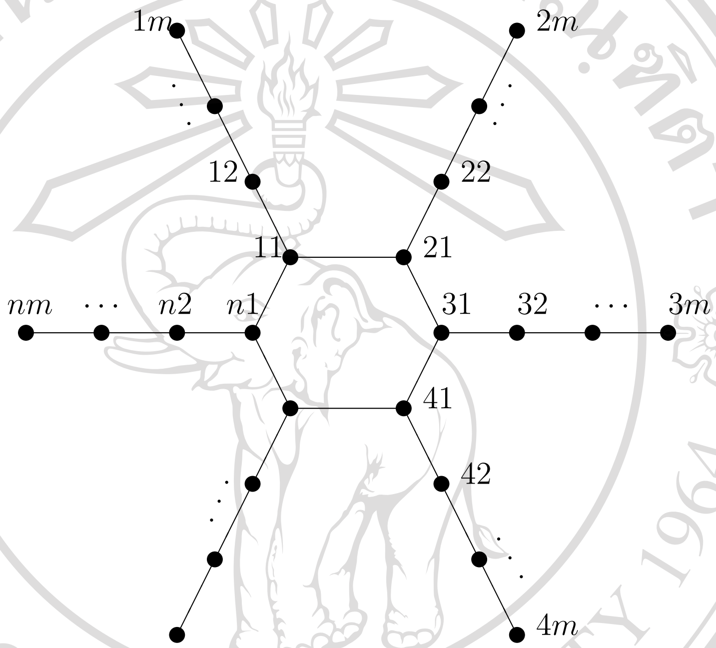


Theorem 3.2.4. *For positive integers $n \geq 3, m \geq 1$,*

$$\det(A(C_n(P_m)) = \begin{cases} (-1)^{\frac{nm}{2}} & \text{if } m \text{ is even,} \\ 2(-1)^{\frac{nm+n-2}{2}} & \text{if } m \text{ is odd and } n \text{ is odd,} \\ 2(-1)^{\frac{nm}{2}}[1-(-1)^{\frac{n}{2}}] & \text{if } m \text{ is odd and } n \text{ is even.} \end{cases}$$

Proof :

$C_n(P_m)$:



(*this picture $n = 6, m = 4$*)

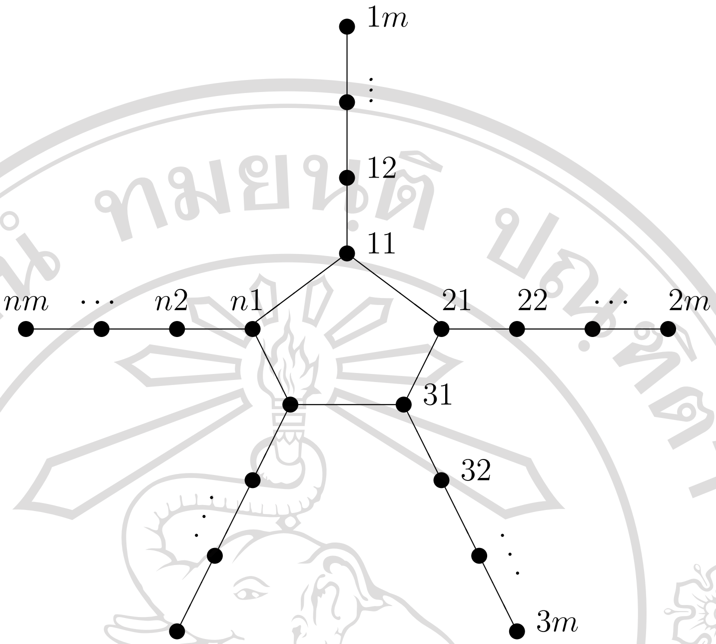
We consider the following three cases :

Case I : if m is even

Case II : if m, n are odd

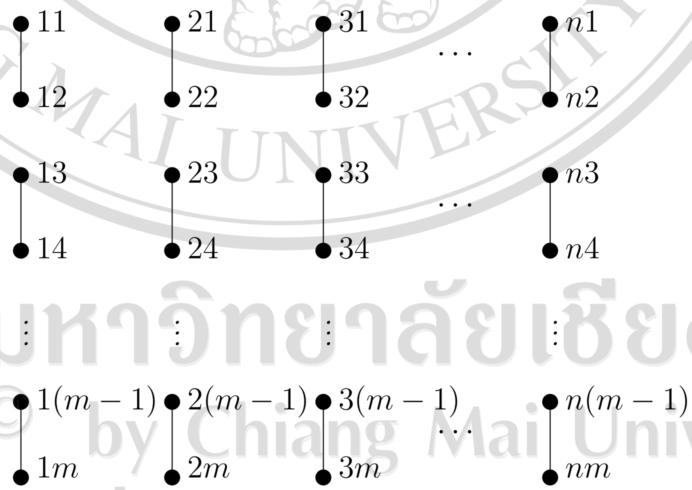
Case III : if m is odd, n is even

In the first case, m is even :



(*this picture* $n = 5, m = 4$)

There exists only one spanning elementary subgraph Γ *where every components is in the form* $\{i(2j-1), i(2j)\}, i = 1, 2, \ldots, n\ , j = 1, 2, \ldots, \frac{m}{2}$.



Therefore, in the case m is even,

$$e(\Gamma) = c(\Gamma) = \frac{v(\Gamma)}{2} = \frac{nm}{2}.$$

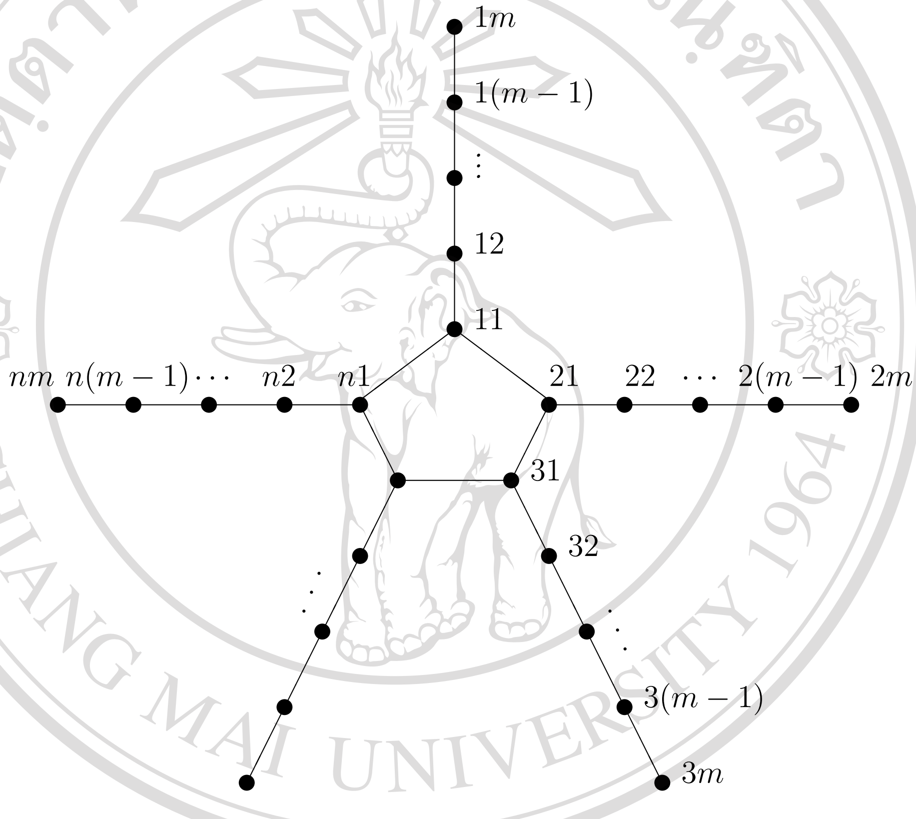
Then

$$r(\Gamma) = nm - \frac{nm}{2} = \frac{nm}{2}$$
$$s(\Gamma) = \frac{nm}{2} - nm + \frac{nm}{2} = 0$$

and

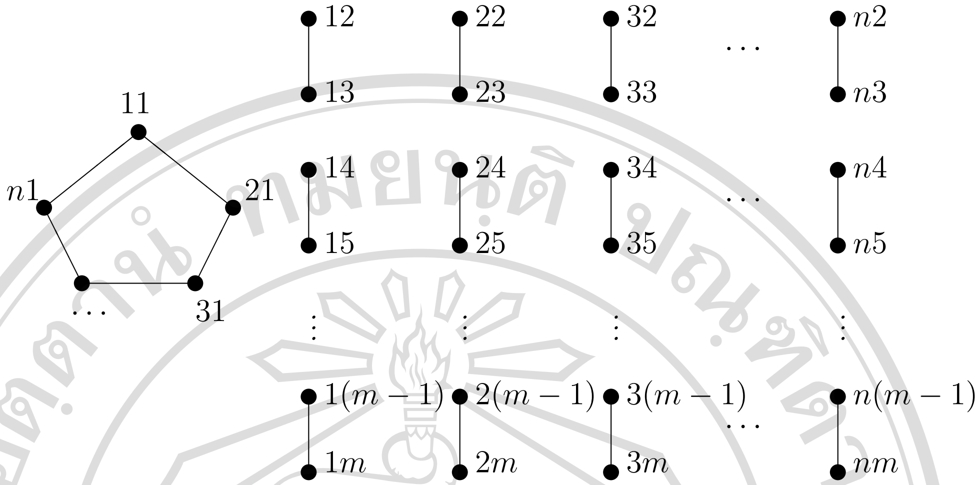
$$\det(A(P_n(P_m))) = (-1)^{\frac{nm}{2}}(2)^0 \ = (-1)^{\frac{nm}{2}}.$$

In the second case, m, n *are odd :*



(*this picture* $n = 5, m = 5$)

There exists only one spanning elementary subgraph Γ *where every components is in one of forms* C_n *or* $\{i(2j), i(2j+1)\}, i = 1, 2, \ldots, n\ , j = 1, 2, \ldots, \frac{m-1}{2}$.



Therefore, in the case m, n is odd,

$$e(\Gamma) = n + (\tfrac{n(m-1)}{2})$$

$$c(\Gamma) = 1 + (\tfrac{n(m-1)}{2}).$$

Then

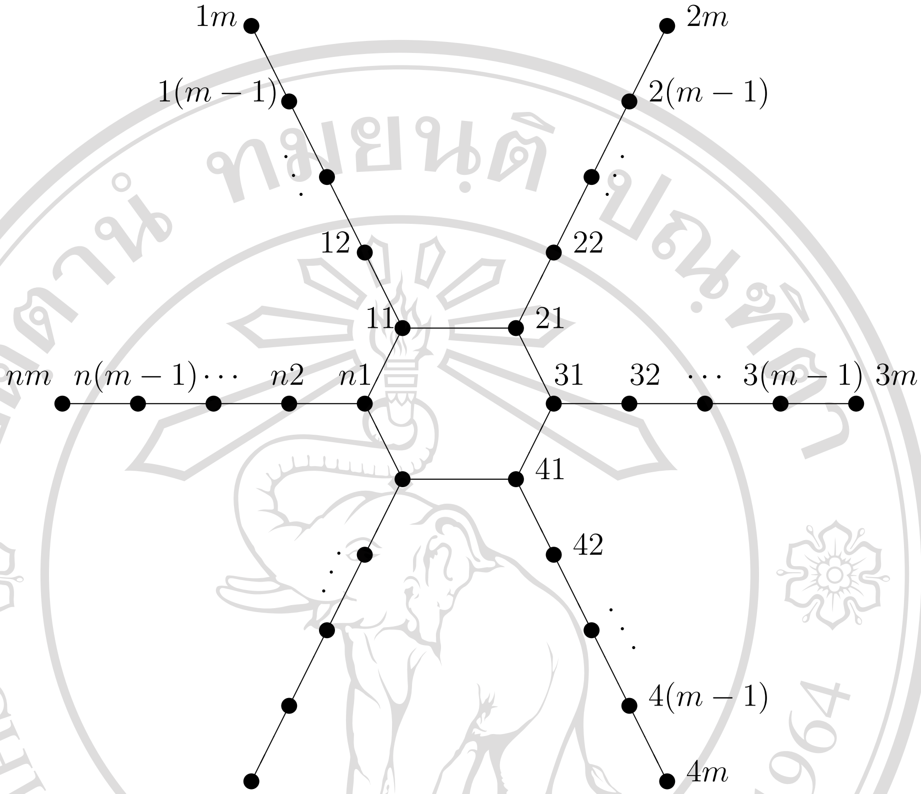
$$\begin{aligned} r(\Gamma) &= nm - [1 + (\tfrac{n(m-1)}{2})] \\ &= nm - \tfrac{2}{2} - \tfrac{nm}{2} + \tfrac{n}{2} \\ &= \tfrac{nm+n-2}{2} \end{aligned}$$

$$\begin{aligned} s(\Gamma) &= [n + (\tfrac{n(m-1)}{2})] - nm + [1 + (\tfrac{n(m-1)}{2})] \\ &= n + n(m-1) - nm + 1 \\ &= n + nm - n - nm + 1 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \det(A(P_n(P_m))) &= (-1)^{\frac{nm+n-2}{2}}(2)^1 \\ &= 2(-1)^{\frac{nm+n-2}{2}} \end{aligned}$$

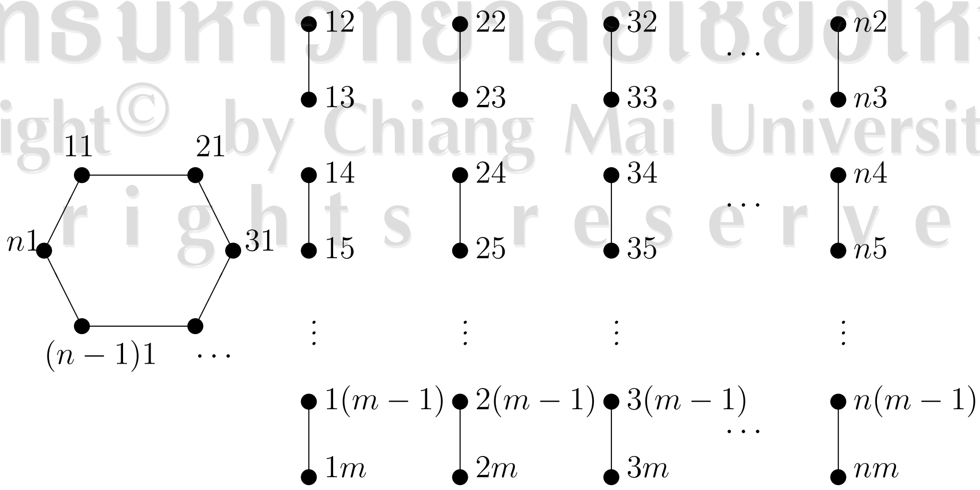
In the third case, m is odd and n is even :



(*this picture* $n = 6, m = 5$)

There exists three spanning elementary subgraphs Γ_1, Γ_2 *and* Γ_3 *where every components of* Γ_1 *is in the form* C_n *or* $\{i(2j), i(2j+1)\}, i = 1, 2, \ldots, n\ , j = 1, 2, \ldots, \frac{m-1}{2}$.

Γ_1 :



Therefore, in the case Γ_1,

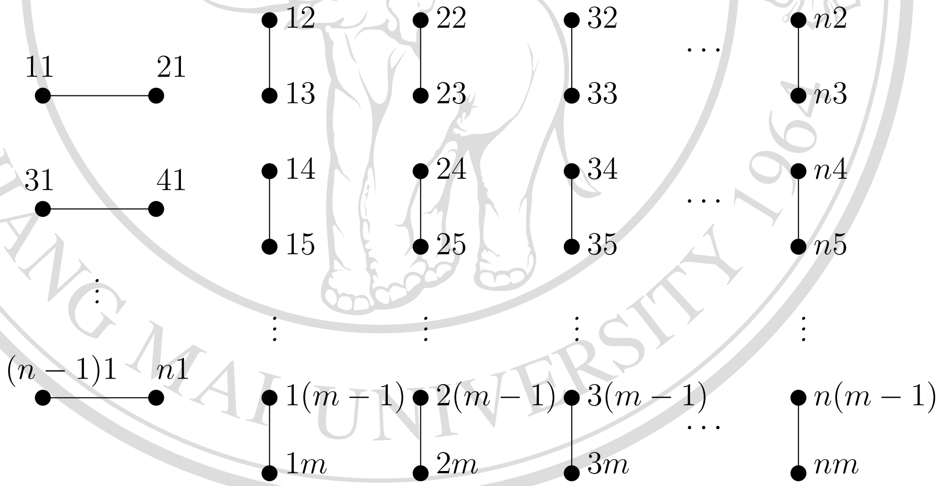
$$e(\Gamma_1) = n + (\tfrac{n(m-1)}{2})$$
$$c(\Gamma_1) = 1 + (\tfrac{n(m-1)}{2}).$$

Then

$$r(\Gamma_1) = nm - [1 + (\tfrac{n(m-1)}{2})] = nm - \tfrac{2}{2} - \tfrac{nm}{2} + \tfrac{n}{2} = \tfrac{nm+n-2}{2} \text{ and}$$
$$\begin{aligned} s(\Gamma_1) &= [n + (\tfrac{n(m-1)}{2})] - nm + [1 + (\tfrac{n(m-1)}{2})] \\ &= n + n(m-1) - nm + 1 \\ &= n + nm - n - nm + 1 \\ &= 1. \end{aligned}$$

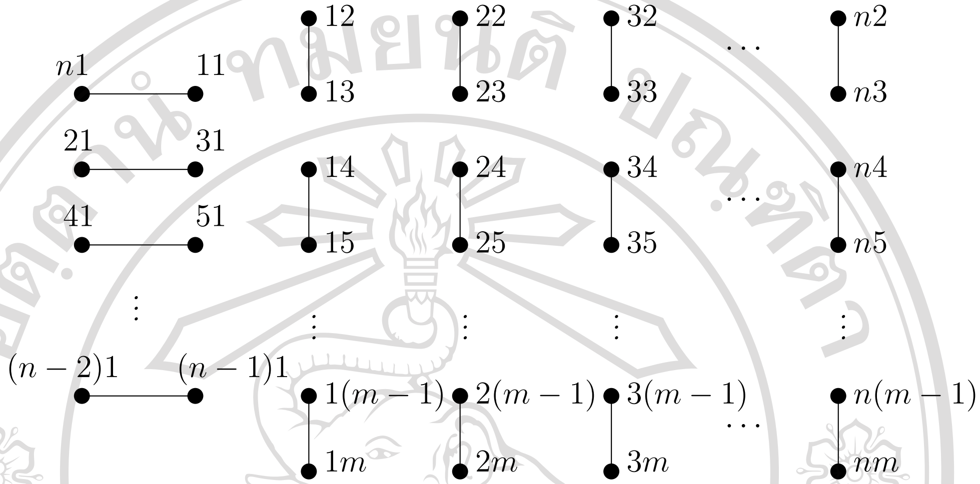
Every components of Γ_2 *is in the form* $\{i(2j), i(2j+1)\}$ *or* $\{(2k-1)1, (2k)1\}, i = 1, 2, \ldots, n\ , j = 1, 2, \ldots, \frac{m-1}{2},\ k = 1, 2, \ldots, \frac{n}{2}$.

Γ_2 :



And every components of Γ_3 *is in the form* $\{i(2j), i(2j{+}1)\}$ *or* $\{(2k)1, (2k{+}1)1\}, i = 1, 2, \ldots, n\ , j = 1, 2, \ldots, \frac{m-1}{2},\ k = 1, 2, \ldots, \frac{n}{2} - 1.$

Γ_3 :



Therefore, in the cases Γ_2, Γ_3,

$e(\Gamma_2) = e(\Gamma_3) = \frac{v(\Gamma_2)}{2} = \frac{nm}{2}$

$c(\Gamma_2) = c(\Gamma_3) = \frac{v(\Gamma_3)}{2} = \frac{nm}{2}.$

Then

$r(\Gamma_2) = r(\Gamma_3) = nm - \frac{nm}{2} = \frac{nm}{2}.$

$s(\Gamma_2) = r(\Gamma_3) = \frac{nm}{2} - nm + \frac{nm}{2} = 0\ .$

Thus

$$\begin{aligned}
\det(A(P_n(P_m))) &= (-1)^{\frac{nm+n-2}{2}}(2)^1 + (-1)^{\frac{nm}{2}}2^0 + (-1)^{\frac{nm}{2}}2^0 \\
&= 2(-1)^{\frac{nm+n-2}{2}} + 2(-1)^{\frac{nm}{2}} \\
&= 2(-1)^{\frac{nm}{2}}[(-1)^{\frac{n-2}{2}} + 1] \\
&= 2(-1)^{\frac{nm}{2}}[(-1)^{\frac{n}{2}}(-1)^{\frac{-2}{2}} + 1] \\
&= 2(-1)^{\frac{nm}{2}}[1 - (-1)^{\frac{n}{2}}]\ .
\end{aligned}$$

□

Example 3.2.5. $\det(A(P_{100}(P_{20})))$ and $\det(A(P_{91}(P_{91})))$.

$$\det(A(P_{100}(P_{20}))) = (-1)^{\frac{100\cdot 20}{2}} = 1.$$
$$\det(A(P_{91}(P_{91}))) = 0.$$

Example 3.2.6. $\det(A(C_{91}(P_{100})))$, $\det(A(C_{91}(P_{91})))$ and $\det(A(C_{100}(P_{91})))$.

$$\det(A(C_{91}(P_{100}))) = (-1)^{\frac{91\cdot 100}{2}} = (-1)^{4550} = 1.$$
$$\det(A(C_{91}(P_{91}))) = 2(-1)^{\frac{(91)(91)+91-2}{2}} = 2(-1)^{4185} = -2.$$
$$\det(A(C_{100}(P_{91}))) = 2(-1)^{\frac{(100)(91)}{2}}[1-(-1)^{\frac{100}{2}}] = 0.$$