CHAPTER 1

INTRODUCTION

Let Q be n by n symmetric matrix. We say that Q is copositive if $x^TQx \geq 0$ for every $x \in \mathbb{R}^n_+$ (i.e. x is a vector in $\mathbb{R}^n$ such that all its components are nonnegative). It is know that the set of all copositive matrices forms the closed convex cone $\mathbb{K}_n$ in the vector subspace S_n of symmetric n by n matrices. S_n is a Euclidean vector space with the scalar product

$$< X, Y >= tr(X^TY). \tag{1.1}$$

The dual cone $\mathbb{K}^*_n$ (with respect to the scalar product (1.1)) consists of the so-called completely positive matrices, i.e. matrices of the form

$$\sum_{i=1}^{n} x^{(i)}[x^{(i)}]^T, m \geq 0 \tag{1.2}$$

where $x^{(i)} \in \mathbb{R}^n_+$, $i = 1, 2, \cdots, n+1$.

The optimization problem of the form:

$$\begin{aligned} \min \quad & tr(C^TX) \\ s.t. \quad & A \bullet X = b \\ & X \in \mathbb{K}_n \end{aligned}$$

is called a copositive programming problem. Many difficult problems in combinatorial optimization are either reduced or approximated with a high accuracy by copositive programming problems. However, the methods of solving for the class of problems are still being developed.

For example, let us consider the partitions problem on graphs. Let $G = (V, E)$ be an undirected graph on n vertices associated with adjacency matrix $A \geq 0$ so $a_{ij} > 0$ implies that the edge $(ij) \in E(G)$ with weight a_{ij}. Given m_1, m_2, and m_3 where $m_1 + m_2 + m_3 = n$, find subset S_1, S_2, and S_3 of $V(G)$ with cardinalities m_1, m_2, and m_3 respectively, such that the total weight edges between S_1 and S_2

is minimal. This problem is NP-complete. In [7], J. Povh and F. Rendl show that this problem can be equivalently reformulated as a linear programming over the cone of completely positive matrices (the dual cone of copositive matrices). The graph partitions appear in many applications for example in circuit board, microchip design, floor planning and analysis of bottlenecks in communication networks. Many approaches have been developed to solve this class of problems. Semidefinite programming turns out to be a very useful approach to get tractable relaxation for the graph partitioning problem. See, for more examples of copositive programming [2], [4], [5], [7]. Despite the fact that (2) - (4) belongs to the class of convex optimization problems, they are very difficult to solve. Indeed, even the problem of checking that matrix $Q \in S_n$ is on $\mathbb{K}_n$ is NP-hard (see e.g. [6]).