CHAPTER 2

PRELIMINARIES

In this chapter, we give some basic definitions, notations, lemmas and results which will be used in the later chapters. For more details of such lemmas and definition see [1] and [9].

2.1 Notations

The following are notations of a few important convex cones which we use in this thesis.

$\mathbb{N}$ $-$ the set of all non-negative integer numbers,

$\mathbb{R}$ $-$ the set of all real numbers,

$\mathbb{R}^n$ $-$ the n dimensional Euclidean space,

$\mathbb{R}^n_+$ $-$ the set of all $n \times n$ positive real matrices,

$\mathbb{R}^{n\times n}$ $-$ the set of all $n \times n$ real matrices,

$diag(\cdot)-$ the (block) diagonal matrix,

$Diag(\cdot)-$ the vector defined by the diagonal elements,

$trX-$ the trace of square matrix X, defined as the sum of its diagonal elements,

$< X, Y >$ $-$ the inner product: $< X, Y >= tr(X^TY)$,

$G \bullet H = tr(G^TH)$,

$\mathbf{x} \circ \mathbf{x}$ $-$ the Hadamard product: $\mathbf{x} \circ \mathbf{x} = [x_1^2\ ,\ x_2^2\ ,\ x_3^2\ ,\ \dots\ ,\ x_n^2]^T$,

where $\mathbf{x} = [x_1\ ,\ x_2\ ,\ x_3\ ,\ \dots\ ,\ x_n]^T$ and $x_i \in \mathbb{R}$,

$\begin{pmatrix} k \\ n \end{pmatrix}$ $-$ the binomial coefficient, $\begin{pmatrix} k \\ n \end{pmatrix} \doteq \dfrac{n!}{(n-k)!k!}$,

min - minimize,

max - maximize,

P - the primal problem,

D - the dual problem,

LP - the linear programming,

SDP - the semidefinite programming.

2.2 Inner Product

Definition 2.2.1. *Let V be a vector space over the field F, then $\langle \cdot, \cdot \rangle : V \times V \longrightarrow F$ is called inner product on V if for all $u, v, w \in V$ and $k \in R$ satisfies four basic properties:*

1. $\langle v, v \rangle \geqslant 0$ *and* $\langle v, v \rangle = 0$ *if and only if* $v = 0$
2. $\langle u, v \rangle = \langle v, u \rangle$
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
4. $\langle ku, v \rangle = k\langle u, v \rangle$.

2.3 Types of Matrix

A^T − the transpose of matrix A,

S_n − the $n \times n$ symmetric matrices,

S_n^+ − the $n \times n$ symmetric positive semidefinite matrices,

$S_n^+ = \{X \in S_n, \mathbf{y}^T X \mathbf{y} \geq 0 \ , \forall \mathbf{y} \in \mathbb{R}^n\}$,

C_n − the $n \times n$ symmetric copositive matrices,

$C_n = \{X \in S_n, \mathbf{y}^T X \mathbf{y} \geq 0 \ , \forall \mathbf{y} \in \mathbb{R}^n_+\}$,

C_n^* − the $n \times n$ symmetric completely positive matrices,

$C_n^* = \{x = \sum_{i=1}^k \mathbf{y}_i \mathbf{y}_i^T \ , \mathbf{y}_i \in \mathbb{R}^n_+, \ (i = 1, 2, ..., k)\}$,

N_n − the $n \times n$ symmetrical nonnegative matrices,

$N_n = \{X \in S_n, X_{ij} \geq 0, \ (i, j = 1, 2, ..., n)\}$,

D_n − the $n \times n$ symmetric doubly nonnegative matrices: $D_n = S_n^+ \cap N_n$,

$X \succ 0$ − means that X is a symmetric completely positive matrices,

$X \succcurlyeq 0$ − means that X is a symmetric positive definite matrices,

$X \succcurlyeq Y$ − means that $X - Y \succcurlyeq 0$,

Definition 2.3.1. *(Trace) The sum of its diagonal elements $A = [a_{ij}]_{n,n}$ defined as*

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

Definition 2.3.2. *(Symmetric Matrix) A real $n \times n$ matrix A is called symmetric if*

$$A^T = A$$

which is a vector space with dimension $n(n+1)/2$.

Definition 2.3.3. *(Ortonormal matrix) A real* $n \times n$ *matrix* A *is called ortonormal matrix if*

$$A^{-1} = A^T.$$

Remark 2.3.4. $S^n_+ = \{X \in S^n | X \succcurlyeq 0\}$ *is a closed convex cone in* $\mathbb{R}^{n^2}$ *of dimension* $n \times (n+1)/2$.

Recall the following properties of symmetric matrices:

Lemma 2.3.5. *A symmetric matrix is positive semidefinite (definite) matrix if all of its eigenvalues are nonnegative (positive)*

Lemma 2.3.6. *A symmetric matrix is negative semidefinite (definite) matrix if all of its eigenvalues are nonpositive (negative).*

Lemma 2.3.7. *If* $X \in S^n$, *then* $X = QDQ^T$ *for some orthonormal matrix* Q *and some diagonal matrix* D.

Lemma 2.3.8. *If* $X = QDQ^T$ *as above, then the columns of* Q *form a set of* n *orthogonal eigenvectors of* X, *whose eigenvalues are the corresponding diagonal entries of* D.

Lemma 2.3.9. $X \succcurlyeq 0$ *if and only if* $X = QDQ^T$ *where the eigenvalues (i.e., the diagonal entries of* D*) are all nonnegative.*

Lemma 2.3.10. $X \succ 0$ *if and only if* $X = QDQ^T$ *where the eigenvalues (i.e., the diagonal entries of* D*) are all positive.*

Lemma 2.3.11. *If* $X \succcurlyeq 0$ *and if* $X_{ii} = 0$, *then* $X_{ij} = X_{ji} = 0$ *for all* $j = 1, ..., n$.

2.4 Linear Programming Problem: LP

Linear programming problem is an optimization problem(maximizing or minimizing) of a linear function subject to linear constraints. The constraints may be equalities or inequalities.

Consider the linear programming problem in standard form:

$$\min \quad \sum_{j=1}^{n} c_j x_j \qquad \text{(objective function)}$$

$$\text{subject to the constraint} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i$$

$$x_j \geq 0 \quad , \; i = 1, ..., m \;\; \text{and} \;\; j = 1, ..., n$$

LP can be written as the matrix form:

$$\min \quad c^T x \qquad \text{(objective function)}$$

$$\text{subject to the constraint} \quad Ax = b$$

$$x \geq 0$$

where

- $x \in \mathbb{R}^n$ is the design vector,
- $y \in \mathbb{R}^m$ is the design vector,
- $c \in \mathbb{R}^n$ is a given vector of coefficients of the objective function $c^T x$,
- A is a given $m \times n$ constraint matrix,
- $b \in \mathbb{R}^m$ is a given right-hand side of the constraints.

Definition 2.4.1. *If x satisfies $Ax = b, x \geq 0$, then x is called a feasible solution . And the set of all feasible solution is called a feasible set.*

Definition 2.4.2. *(LP) is called feasible if its feasible set $F = \{x | Ax - b \geq 0\}$ is nonempty set such that a point $x \in F$ is called a feasible solution to (LP). (Conversely, (LP) is called infeasible.)*

Definition 2.4.3. *(LP) is called bounded below if its objective $c^T x$ is bounded below on F and unbounded below if for all $\lambda \in \mathbb{R}$ there exists x^* such that $c^T x^* \leq \lambda$.*

Definition 2.4.4. *(LP) is called solvable if it is feasible and bounded below and the optimal value is attained,i.e., there exists $x \in F$ with $c^T x = z^*$. An x of this type is called an optimal solution to (LP).*

Equivalent form

A minimum problem can be changed to a maximum problem by multiplying the objective function by -1

$$\min \quad c^T x \Leftrightarrow \quad \max -c^T x.$$

Similarly, constraints of the form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad \Leftrightarrow \quad \sum_{j=1}^{n} a_{ij}x_j \geqslant b_i$$

can be changed into the form

$$\sum_{j=1}^{n}(-a_{ij})x_j \leqslant -b_i.$$

Duality

For every linear programming problem, there is a dual linear programming problem with which it is intimately connected. We call the first problem that *primal problem* (P) and second problem that *dual problem* (D).

$$\begin{array}{lll} (P) & & \min \quad c^T x \\ & & Ax = b, \\ & & x \geq 0 \\ & & \\ (D) & & \max \quad b^T y \\ & & A^T y + s = c, \\ & & s \geq 0 \end{array}$$

where

- $x \in \mathbb{R}^n$ is the design vector,
- $y \in \mathbb{R}^m$ is the design vector,
- $c \in \mathbb{R}^n$ is a given vector of coefficients of the objective function $c^T x$,
- A is a given $m \times n$ constraint matrix,
- $b \in \mathbb{R}^m$ is a given right-hand side of the constraints.

Note. The inequalities constraints can be written as the equalities constraints by adding *slack variable* (s) such that

$$a_i^T x \leq b \quad \Leftrightarrow \quad a_i^T x + s_i = b_i \quad , s_i \geq 0.$$

Theorem 2.4.1. *(Duality gap) The value of the dual objective at every dual feasible solution y is less or equal to the value of the primal objective at every primal feasible solution x, so that the duality gap*

$$c^T x - b^T y$$

is non-negative at every primal-dual feasible pair (x, y).

Theorem 2.4.2. *(Weak duality) Given a feasible solution* x *and* (y, s) *of (LP), the duality gap is* $c^T x - b^T y = sx \geq 0$. *If* $c^T x - b^T y = 0$, *then* x *and* (y, s) *are each optimal solutions to (LP), and furthermore* $sx = 0$.

Theorem 2.4.3. *(Strong duality) Let* z_P^* *and* z_D^* *denote the optimal objective function values of (LP). Suppose that there exists a feasible solution* $\widehat{x}$ *such that* $\widehat{x} > 0$, *and* $(\widehat{y}, \widehat{s})$ *such that* $\widehat{s} > 0$. *Then (LP) attains optimal solutions, and*

$$z_P^* = z_D^*.$$

2.5 Semidefinite Programming Problem: SDP

Consider a class of well know optimization problems known as semidefinite programming problem and its dual

$$
\begin{array}{lll}
(P) & \min & C \bullet X \\
& s.t. & A_i \bullet X = b_i, \quad i = 1, 2, ..., m \\
& & X \succcurlyeq 0 \\
(D) & \min & b^T y \\
& s.t. & \displaystyle\sum_{i=1}^{m} y_i A_i + Z = C \\
& & Z \succcurlyeq 0
\end{array}
$$

where $C \in S_n, A_i \in S_n, i = 1, 2, \ldots, m$, $b = (b_1, b_2, \ldots, b_m)^T \in R^m$ and $X \in S_n^+, (y, Z) \in R^m \times S_n^+$ is primal and dual feasible solution respectively.

Note. S_n refer to symmetric matrices $n \times n$.

S_n^+ refer to symmetric positive definite matrices $n \times n$.

$X \succcurlyeq 0$ refer to X is a symmetric positive definite matrices.

$X \succ 0$ refer to X is a symmetric completely positive matrices.

$X \succcurlyeq Y$ refer to $X - Y \succcurlyeq 0$

$G \bullet H = tr(G^T H)$

If X satisfied constraint condition $A_i \bullet X = b_i$ and (y, Z) satisfied constraint condition $\sum_{i=1}^{m} y_i A_i + Z = C$ where $i = 1, 2, ..., m$ then we say X and (y, Z) are feasible solutions.

The following theorem states that weak duality must hold for the primal and dual of (SDP):

Theorem 2.5.1. *(Duality gap) The value of the dual objective at every dual feasible solution y is less or equal to the value of the primal objective at every primal feasible solution X, so that the duality gap*

$$C \bullet X - b^T y$$

is non-negative at every primal-dual feasible pair (X, y).

Theorem 2.5.2. *(Weak duality) Given a feasible solution X and (y, Z) of (SDP), the duality gap is $C \bullet X - b^T y = Z \bullet X \geq 0$. If $C \bullet X - b^T y = 0$, then X and (y, Z) are each optimal solutions to (SDP), and furthermore $ZX = 0$.*

Theorem 2.5.3. *(Strong duality) Let z_P^* and z_D^* denote the optimal objective function values of (SDP). Suppose that there exists a feasible solution $\widehat{X}$ such that $\widehat{X} \succ 0$, and $(\widehat{y}, \widehat{Z})$ such that $\widehat{Z} \succ 0$. Then (SDP) attains optimal solutions, and*

$$z_P^* = z_D^*.$$

2.6 Conic Programming: CP

Let $\mathbf{K}$ be a cone in $\mathbb{R}^m$ (convex, pointed, closed, and with nonempty interior). Given an objective $c \in \mathbb{R}^n$, an $m \times n$ constraint matrix A, and a right-hand side $b \in \mathbb{R}^m$, consider the optimization problem

$$\begin{aligned} \min \quad & c^T x \\ & Ax - b \geq_{\mathbf{K}} 0. \end{aligned}$$

Note. $Ax - b \geq_{\mathbf{K}} 0$ means that $Ax - b \in \mathbf{K}$.

The set $\mathbf{K}$ must be a pointed convex cone, i.e., it must satisfy the following conditions:

1. $\mathbf{K}$ is nonempty and closed under addition:

$$a, a' \in \mathbf{K} \Rightarrow a + a' \in \mathbf{K}$$

2. $\mathbf{K}$ is a conic set:

$$a \in \mathbf{K}, \lambda \geq 0 \Rightarrow \lambda a \in K$$

3. $\mathbf{K}$ is pointed:

$$a \in \mathbf{K}, -a \in \mathbf{K} \Rightarrow a = 0$$

Note. 1. Nonstrict inequality

$$a \geq_{\mathbf{K}} b \Longleftrightarrow a - b \geq_{\mathbf{K}} 0 \Longleftrightarrow a - b \in \mathbf{K}$$

2. Strict inequality

$$a >_{\mathbf{K}} b \Longleftrightarrow a - b >_{\mathbf{K}} 0 \Longleftrightarrow a - b \in \mathrm{int}\mathbf{K}$$

where int$\mathbf{K}$ is the interior of cone $\mathbf{K}$.

Conic duality theorem

Consider a conic problem

$$c^* = \min\{c^T x | Ax \geq_{\mathbf{K}} b\} \qquad (CP)$$

along with its conic dual

$$b^* = \max\{b^T y | A^T y = c, y \geq_{\mathbf{K}_*} 0\}. \qquad (D)$$

where $c \in \mathbb{R}^n$, $y \in \mathbf{K}^*$, A is $m \times n$ matrix, $\mathbf{K}$ is a cone in $\mathbb{R}^m$, and $\mathbf{K}^*$ is a dual cone which satisfy the following conditions:

Let $K \subset \mathbb{R}^m$ and $\mathbf{K}$ be a nonempty set,

i. A set $\mathbf{K}^* = \{y \in \mathbb{R}^m : y^T a \geq 0, \forall a \in \mathbf{K}\}$ is a closed convex cone.

ii. If int $\mathbf{K}$ is a nonempty set, then $\mathbf{K}^*$ is pointed.

iii. If a set $\mathbf{K}$ is a closed convex pointed cone, then int $\mathbf{K}$ is a nonempty set.

iv. If a set $\mathbf{K}$ is closed convex cone, then $\mathbf{K}^*$ is a closed convex cone and the dual cone of $\mathbf{K}^*$ is $\mathbf{K}$ $((\mathbf{K}^*)^* = \mathbf{K})$.

Note. 1. The duality is symmetric: the dual problem is conic, and the problem dual to dual is (equivalent to) the primal.

2. The value of the dual objective at every dual feasible solution y is $\leq$ the value of the primal objective at every primal feasible solution x, so that the duality gap

$$c^T x - b^T y$$

is nonnegative at every primal-dual feasible pair (x, y).

3.1. If the primal(CP) is bounded below and strictly feasible (i.e., $Ax >_{\mathbf{K}} b$ for some x), then the dual(D) is solvable and the optimal values in the problem are equal to each other: $c^* = b^*$.

3.2. If the dual(D) is bounded above and strictly feasible (i.e., $y >_{\mathbf{K}_*} 0$ such that $A^T y = c$), then the primal(CP) is solvable and $c^* = b^*$.

4. Assume that at least one of the problems (CP), (D) is bounded and strictly feasible. Then a primal-dual feasible pair (x, y) is a pair of optimal solutions to the respective problems

4.1. *if and only if*

$$b^T y = c^T x$$

4.2. *if and only if*

$$y^T[Ax - b] = 0.$$

We shall refer to conic programming (CP) as a conic problem associated with the cone $\mathbf{K}$. In the case $\mathbf{K} = \mathbb{R}^m$, the conic problem remains to the linear programming (LP) and if $\mathbf{K} = S^n$, it remains to the semidefinite programming (SDP).