CHAPTER 3

FIRST DEGREE APPROXIMATION

In this chapter, we will present some results in the approximation of the cone of copositive matrices theory, we divided into 2 sections. In section 3.1, we shall establish some theory of the sum of squares decompositions. In section 3.2, we will present a system of linear matrix inequalities LMI's in case $r = 1$ for approximating the copositive programming. For detail of the prove see [3].

Since any $\mathbf{y} \in \mathbb{R}^n_+$ can be written as $\mathbf{y} = \mathbf{x} \circ \mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$, we can represent the copositivity requirement for an $n \times n$ symmetric matrix M as

$$P(\mathbf{x}) := (\mathbf{x} \circ \mathbf{x})^T M (\mathbf{x} \circ \mathbf{x}) = \sum_{i,j=1}^{n} M_{ij} x_i^2 x_j^2 \geq 0 \quad for\ all \ \ \mathbf{x} \in \mathbb{R}^n. \tag{3.1}$$

We can represent the polynomial P as a homogeneous polynomial of degree four, where the coefficients of $(x_i x_j)(x_k x_l)$ are nonzero for $i \neq j \neq k \neq l$.

$$P(\mathbf{x}) = \tilde{\mathbf{x}}^T \tilde{M} \tilde{\mathbf{x}} \tag{3.2}$$

where $\tilde{\mathbf{x}} = [x_1^2, ..., x_n^2, x_1 x_2, x_1 x_3, ..., x_{n-1} x_n]^T$ and $\tilde{M}$ is a symmetric matrix of order $n + \frac{1}{2} n(n-1)$, then $\tilde{M}$ is not uniquely determined. The non-uniqueness follows from the identities:

$$\begin{aligned}
(x_i x_j)^2 &= (x_i)^2 (x_j)^2 \\
(x_i x_j)(x_i x_k) &= (x_i)^2 (x_j x_k) \\
(x_i x_j)(x_k x_l) &= (x_i x_k)(x_j x_l) = (x_i x_l)(x_j x_k).
\end{aligned}$$

3.1 Sum of Squares Decompositions : S.O.S

Polynomial $P(\mathbf{x})$ is called the sum of squares decompositions (S.O.S) if and only if $P(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x})^2$ for some polynomial functions $f_i(\mathbf{x})$, $i = 1, ..., n$.

Note. For any $\mathbf{x} \in \mathbb{R}^n$ and any multi-index $\mathbf{m} \in \mathbb{N}_0^n$ (where $\mathbb{N}_0 = \{0, 1, 2, ...\}$)we define $|\mathbf{m}| = \sum_i m_i$ and $\mathbf{x}^{\mathbf{m}} = \prod_i x_i^{m_i}$ the corresponding monomial of degree $|\mathbf{m}|$.

And $I^n(s) = \{\mathbf{m} \in \mathbb{N}_0^n : |\mathbf{m}| = s\}$ refers to the set of all possible exponents of monomials of degree s (there are $d = \begin{pmatrix} n+s-1 \\ s \end{pmatrix}$) and $2I^n(s) = \{2\mathbf{m} : \mathbf{m} \in I^n(s)\}$. Finally, given a set of multi-indices I and a vector $\mathbf{x} \in \mathbb{R}^n$, we define $[\mathbf{x}^{\mathbf{m}}]_{\mathbf{m}\in I}$ as the vector with components $\mathbf{x}^{\mathbf{m}} = \prod_i x_i^{m_i}$ for each $\mathbf{m} \in I$.

Lemma 3.1.1. *If $\overline{P}(x)$ is a homogeneous polynomial of degree $2s$ in n variables $\mathbf{x} = [x_1, ..., x_n]^T$, which has a representation*

$$\overline{P}(x) = \sum_{i=1}^{l} f_i(\mathbf{x})^2$$

for some polynomials $f_i(\mathbf{x})(i = 1, ..., n)$,then there are polynomials $h_i(\mathbf{x})$ which are homogeneous of degree s for all i such that $\overline{P}(x) = \sum_{i=1}^{t} h_i(\mathbf{x})^2$ with $1 \leqslant t \leqslant l$. Further, $\overline{P}$ has a s.o.s representation as above if and only if there is a symmetric positive-semidefinite matrix $d \times d$ matrix $\tilde{M} \in S_d^+$ such that

$$\overline{P}(\mathbf{x}) = \tilde{\mathbf{x}}^T \tilde{M} \tilde{\mathbf{x}}$$

where $d = \begin{pmatrix} n+s-1 \\ s \end{pmatrix}$ and $\tilde{\mathbf{x}} = [\mathbf{x}^k]_{k\in I^n(s)} \in \mathbb{R}^d$.

Lemma 3.1.2. *Let $\overline{P}(x) = \sum_{\mathbf{m}\in I^n(s)} A_{\mathbf{m}} \mathbf{x}^{2m}$ be a homogeneous polynomial of degree $2s$ in n variables $\mathbf{x} = [x_1, ..., x_n]^T$ and define $\tilde{M} \in S_d$ and $\tilde{\mathbf{x}} \in \mathbb{R}^d$ as in Lemma (3.1.1). Then $\overline{P}(x) = \tilde{\mathbf{x}}^T \tilde{M} \tilde{\mathbf{x}}$ if and only if*

$$\sum_{(\mathbf{j},\mathbf{k})\in[I^n(s)]^2 : \mathbf{j}+\mathbf{k}=2\mathbf{m}} \tilde{M}_{\mathbf{j},\mathbf{k}} = A_{\mathbf{m}} \quad forall \quad \mathbf{m} \in I^n(s) \tag{3.3}$$

$$\sum_{(\mathbf{j},\mathbf{k})\in[I^n(s)]^2 : \mathbf{j}+\mathbf{k}=\mathbf{n}} \tilde{M}_{\mathbf{j},\mathbf{k}} = 0 \quad forall \quad \mathbf{n} \in I^n(2s)\backslash 2I^n(s). \tag{3.4}$$

We define the cone $K_n^0 := S_n^+ + N_n = D_n^*$, the cone dual dual to that of all doubly nonnegative matrices.

Theorem 3.1.1. (Parrilo [8]). *$P(\mathbf{x}) = (\mathbf{x} \circ \mathbf{x})^T M (\mathbf{x} \circ \mathbf{x})$ allows for a polynomial s.o.s if and only if $M \in K_n^0$, i.e., if and only if $M = S + T$ for matrices $S \in S_n^+$ and $T \in N_n$.*

Higher order sufficient conditions can be derived by the polynomial:

$$P^{(r)}(\mathbf{x}) = P(\mathbf{x})(\sum_{k=1}^{n} x_k^2)^r = \sum_{i,j=1}^{n} M_{ij} x_i^2 x_j^2 (\sum_{k=1}^{n} x_k^2)^r \tag{3.5}$$

and we can consider $P^{(r)}(\mathbf{x})$ has a sum of squares decomposition (S.O.S) from Lemma (3.1.1).

Definition 3.1.1. (De Klerk and Pasechnik [5]). *The convex cone K_n^r consists of the matrices for which $P^{(r)}(\mathbf{x})$ in (3.5) allows a polynomial sum of squares decomposition.*

Obviously, these cones are contained in each other: $K_n^r \subseteq K_n^{r+1}$ for all r. This follows from

$$P^{(r+1)}(\mathbf{x}) = \sum_k x_k^2 P^{(r)}(\mathbf{x}) = \sum_{i,k} [f_i(\mathbf{x}) x_k]^2.$$

By explicitly calculating the coefficients $A_{\mathbf{m}}(M)$ of the homogeneous polynomial $P^{(r)}(\mathbf{x})$ of degree $2(r+2)$ and summarizing the above auxiliary results, we arrive at a characterization of K_n^r which has not appeared in the literature before.

Theorem 3.1.2. (Bomze [3]). *Let $n, r \in \mathbb{N}$, $d = \begin{pmatrix} n+r+1 \\ r+2 \end{pmatrix}$, $\mathbf{m}(i,j) = \mathbf{m} - \mathbf{e}^i - \mathbf{e}^j$ for any $\mathbf{m} \in \mathbb{R}^n$ and introduce the multinomial coefficients*

$$\begin{aligned} c(\mathbf{m}) &= |\mathbf{m}|!/\prod_i (m_i)! \quad \text{if} \quad \mathbf{m} \in \mathbb{N}_0^n, \\ c(\mathbf{m}) &= 0 \qquad\qquad\quad \text{if} \quad \mathbf{m} \in \mathbb{R}^n \setminus \mathbb{N}_0^n. \end{aligned} \tag{3.6}$$

For a symmetric matrix $M \in S_n$, define

$$A_{\mathbf{m}}(M) = \sum_{i,j} c(\mathbf{m}(i,j)) M_{ij}. \tag{3.7}$$

Then $M \in K_n^r$ if and only if there is a symmetric positive-semidefinite $d \times d$ $\tilde{M} \in S_d^+$ such that

$$\begin{aligned} \sum_{(\mathbf{j},\mathbf{k}) \in [I^n(r+2)]^2 : \mathbf{j}+\mathbf{k}=2\mathbf{m}} \tilde{M}_{\mathbf{j},\mathbf{k}} &= A_{\mathbf{m}}(M) \quad \text{for all} \quad \mathbf{m} \in I^n(r+2) \\ \sum_{(\mathbf{j},\mathbf{k}) \in [I^n(r+2)]^2 : \mathbf{j}+\mathbf{k}=\mathbf{n}} \tilde{M}_{\mathbf{j},\mathbf{k}} &= 0 \qquad\qquad \text{for all} \quad \mathbf{n} \in I^n(2r+4) \setminus 2I^n(r+2) \end{aligned} \tag{3.8}$$

Lemma 3.1.3. (Bomze [3]). *Let M be an arbitrary $n \times n$ matrix and denote by $diagM^{(i)} = [M_{ii}]_i \in \mathbb{R}^n$ the vector obtained by extracting the diagonal elements of M. If $A_{\mathbf{m}}(M)$ is defined as in (3.7), then*

$$A_{\mathbf{m}}(M) = \frac{c(\mathbf{m})}{s(s-1)}[\mathbf{m}^T M\mathbf{m} - \mathbf{m}^T diagM] \quad \text{for all} \quad \mathbf{m} \in I^n(s), \quad s \in \mathbb{N}. \tag{3.9}$$

For $M = E_n$, we have, from $\mathbf{m}^T E_n \mathbf{m} = (\mathbf{e}^T\mathbf{m})^2 = |\mathbf{m}|^2$, thus

$$A_{\mathbf{m}}(E_n) = \frac{c(\mathbf{m})}{s(s-1)}[s^2 - s] = c(\mathbf{m}) \quad \text{for all} \quad \mathbf{m} \in I^n(s), \quad s \in \mathbb{N}. \tag{3.10}$$

Parrilo [8] showed that $M \in K_n^1$ if the following system of linear matrix inequalities has a solution

$$\begin{aligned} M - M^{(i)} &\in S_n^+ &&, i = 1, ..., n, \\ M_{ii}^{(i)} &= 0 &&, i = 1, ..., n, \\ M_{ii}^{(j)} + 2M_{ij}^{(i)} &= 0 &&, i \neq j, \\ M_{jk}^{(i)} + M_{ik}^{(j)} + M_{ij}^{(k)} &\geq 0 &&, i < j < k \end{aligned}$$

where $M^{(i)} \in S_n$ for $i = 1, ..., n$.

3.2 First Degree Approximation of the Cone of Copositive Matrices

In this section, we present system of linear matrix inequalities in case $r = 1$ for approximating the copositive programming.

Theorem 3.2.1. *$M \in K_n^1$ if and only if there are n symmetric $n \times n$ matrices $M^{(i)} \in S_n$ for $i = 1, ..., n$ such that the system of LMI's is satisfied.*

$$\begin{aligned} M - M^{(i)} &\in S_n^+ &&, i = 1, ..., n, \\ M_{ii}^{(i)} &= 0 &&, i = 1, ..., n, \\ M_{ii}^{(j)} + 2M_{ij}^{(i)} &= 0 &&, i \neq j, \\ M_{jk}^{(i)} + M_{ik}^{(j)} + M_{ij}^{(k)} &\geq 0 &&, i < j < k \end{aligned} \tag{3.11}$$

where $M^{(i)} \in S_n$ for $i = 1, ..., n$.

Proof. First assume that $M \in K_n^1$. By Theorem 3.1.2 there exists a $\tilde{M} \in S_d^+$ satisfying 3.8 such that

$$P^{(1)}(x) = \sum_{i,j=1}^{n} M_{ij} x_i^2 x_j^2 \Big(\sum_{k=1}^{n} x_k^2 \Big) = \tilde{x}^T \tilde{M} \tilde{x}$$

where $\tilde{x} = [x^k]_{k \in I^n(3)} \in \mathbb{R}^d$ and $d = \begin{pmatrix} n+2 \\ 3 \end{pmatrix}$.

By(3.9) , we have $A_{iii}(M) = M_{ii}$ while $A_{iij}(M) = 2M_{ij} + M_{ii}$ and $A_{ijk}(M) = 2(M_{ij} + M_{ik} + M_{jk})$ if $1 \leqslant i < k \leqslant n$. Similarly, the left-hand side of (3.8) read in case $n = 2m$,

$$\begin{array}{ll} \tilde{M}_{iii,iii} & ,\text{if } \ n = 6e_i \\ \tilde{M}_{iij,iij} + 2\tilde{M}_{iii,ijj} & ,\text{if } \ n = 4e_i + 2e_j \\ \tilde{M}_{ijk,ijk} + 2(\tilde{M}_{iij,jkk} + \tilde{M}_{iik,jjk} + \tilde{M}_{ijj,ikk}) & ,\text{if } \ n = 2(e_i + e_j + e_k), \quad i < j < k \end{array}$$

Now put $S_{jk}^{(i)} = \tilde{M}_{ijj,ikk}$ for all triples (ijk). Then $S^{(i)} \in S_n^+$ since it is a principal submatrix of the positive-semidefinite matrix $\tilde{M}$. Hence setting $M^{(i)} = M - S^{(i)}$ we see that the first condition of 3.11 is satisfied. It remains to show that 3.11 hold. Now

$$\begin{aligned} M_{ii}^{(i)} &= M_{ii} - S_{ii}^{(i)} \\ &= A_{iii}(M) - \tilde{M}_{iii,iii} \\ &= \tilde{M}_{iii,iii} - \tilde{M}_{iii,iii} \\ &\geqslant 0 \end{aligned}$$

and similarly

$$\begin{aligned} M_{ii}^{(j)} + 2M_{ij}^{(i)} &= M_{ii} + 2M_{ij} - S_{ii}^{(j)} - 2S_{ij}^{(i)} \\ &= A_{iij}(M) - \tilde{M}_{iij,iij} - 2\tilde{M}_{iii,ijj} \\ &= \tilde{M}_{iij,iij} + 2\tilde{M}_{iii,ijj} - \tilde{M}_{iij,iij} - 2\tilde{M}_{iii,ijj} \\ &= 0, \end{aligned}$$

whereas

$$
\begin{aligned}
M_{jk}^{(i)} + M_{ik}^{(j)} + M_{ij}^{(k)} &= M_{ij} + M_{ik} + M_{jk} - S_{jk}^{(i)} - S_{ik}^{(j)} - S_{ij}^{(k)} \\
&= \tfrac{1}{2} A_{ijk}(M) - \tilde{M}_{ijj,ikk} - \tilde{M}_{iij,jkk} - \tilde{M}_{iik,jjk} \\
&= \tilde{M}_{iij,iij} + \tilde{M}_{iii,ijj} - \tilde{M}_{iij,iij} - \tilde{M}_{iii,ijj} \\
&= \tfrac{1}{2} \tilde{M}_{ijk,ijk} \\
&\geqslant 0,
\end{aligned}
$$

because the diagonal entries of $\tilde{M}$ cannot be negative. Thus we have constructed a solution to the system of LMI's 3.11.

Conversely, assume that a solution to 3.11 is given. Observe that

$$
\begin{aligned}
P^{(1)}(x) &= \sum_{i=1}^{n} x_i^2 (x \circ x)^T M (x \circ x) \\
&= \sum_{i=1}^{n} x_i^2 (x \circ x)^T (M - M^{(i)})(x \circ x) + \sum_{i=1}^{n} x_i^2 (x \circ x)^T (M^{(i)})(x \circ x).
\end{aligned}
$$

The first sum is obviously a s.o.s., since $M - M^{(i)} \in S_n^+$ for every i. The second sum can likewise be written as a s.o.s. because of

$$
\begin{aligned}
&\sum_{i=1}^{n} x_i^2 (x \circ x)^T M^{(i)} (x \circ x) \\
&\quad = \sum_{i,j,k} M_{jk}^{(i)} x_i^2 x_j^2 x_k^2 \\
&\quad = \sum_{i} M_{ii}^{(i)} x_i^6 + \sum_{i \neq j} (M_{ii}^{(j)} + 2M_{ij}^{(i)}) x_i^4 x_j^2 \\
&\qquad + \sum_{i<j<k} (2M_{jk}^{(i)} + 2M_{ik}^{(j)} + 2M_{ij}^{(k)}) x_i^2 x_j^2 x_k^2 \\
&\quad = \sum_{i} (\sqrt{M_{ii}^{(i)}} x_i^3)^2 + \sum_{i \neq j} (\sqrt{M_{ii}^{(j)} + 2M_{ij}^{(i)}} x_i^2 x_j)^2 \\
&\qquad + \sum_{i<j<k} (\sqrt{2(M_{jk}^{(i)} + M_{ik}^{(j)} + M_{ij}^{(k)})} x_i x_j x_k)^2,
\end{aligned}
$$

where we have used the non-negativity of the last condition of 3.11 to obtain the last equality. Note that the first two sums of the last expression vanish due to the second and the third condition of 3.11. Thus $P^{(1)}(\mathbf{x})$ is represented as a s.o.s. □

We can change the SDP approximations of the copositive cone to arrive at a series of LP approximations of copositive cone. These approximation are weaker than the SDP, but can be solved more easily.

Note. If the polynomial $P^{(r)}(\mathbf{x})$ has only nonnegative coefficient, then it is already allows a sum of squares decomposition.

Definition 3.2.1. (De Klerk and Pasechnik [5]) *The convex cone C_n^r consists of the matrices for which $P^{(r)}(\mathbf{x})$ in (3.5) has no nonnegative coefficient. Hence for any r, we have $C_n^r \subseteq K_n^r$.*

Obviously, these cones are contained in each other: $C_n^r \subseteq C_n^{r+1}$ for all r.

Theorem 3.2.2. *For any $\mathbf{m} \in \mathbb{R}^n$, define $Diag\mathbf{m}$ as the $n \times n$ diagonal matrix containing $\mathbf{m}$ as its diagonal, i.e., satisfying $diag(Diag\mathbf{m}) = \mathbf{m}$. Then for all $R \in \mathbb{N}_0$ and $n \in \mathbb{N}$,*

$$\begin{aligned} C_n^r &= \{M \in S_n : \mathbf{m}^T M \mathbf{m} - \mathbf{m}^T diagnal M \geq 0 \text{ for all } \mathbf{m} \in I^n(r+2)\} \\ &= \{M \in S_n : \langle \mathbf{m}\mathbf{m}^T - Diag\mathbf{m}, M \rangle \geq 0 \text{ for all } \mathbf{m} \in I^n(r+2)\}. \end{aligned}$$

We can also establish an alternative characterization of C_n^r similar to Theorem 3.2.1

Theorem 3.2.3. *$M \in C_n^1$ if and only if there are n symmetric $n \times n$ matrices $M^{(i)} \in S_n$ for $i = 1, ..., n$ such that the following system of linear inequalities has a solution:*

$$\begin{aligned} M - M^{(i)} &\in N_n &&, i = 1, ..., n, \\ M_{ii}^{(i)} &= 0 &&, i = 1, ..., n, \\ M_{ii}^{(j)} + 2M_{ij}^{(i)} &= 0 &&, i \neq j, \\ M_{jk}^{(i)} + M_{ik}^{(j)} + M_{ij}^{(k)} &\geq 0 &&, i < j < k. \end{aligned}$$