CHAPTER 4

SECOND DEGREE APPROXIMATION

In this chapter, we extend the system of linear matrices inequalities LMI's in case $r = 2$ for approximating optimal solution of copositive programming by using the second order sum of square decomposition.

4.1 Second Order Sum of Squares Decompositions

By directly calculating coefficients $A_m(M)$ of the homogeneous polynomial $P^{(2)}(x)$, we obtain the characterizations of the cone K_n^2. Further, we obtain a necessary and sufficient condition for a matrix M belongs in K_n^2.

Consider the second order sum of square decomposition of the form

$$P^{(r)}(\mathbf{x}) = P(\mathbf{x})(\sum_{k=1}^{n} x_k^2)^r = \sum_{i,j=1}^{n} M_{ij}x_i^2x_j^2(\sum_{k=1}^{n} x_k^2)^r$$

in case $r = 2$ such that

$$P^{(2)}(\mathbf{x}) = P(\mathbf{x})(\sum_{k=1}^{n} x_k^2)^2 = \sum_{i,j=1}^{n} M_{ij}x_i^2x_j^2(\sum_{k=1}^{n} x_k^2)^2.$$

With slightly more effort, we can derive similar system of LMI's for the cones K_n^r if $r \geq 2$. However, d then increases so rapidly with n (recall that $d = O(n^{r=2})$) that the resulting problems become too large for current SDP solvers. In summary, we have the following theorem.

Theorem 4.1.1. (De Klerk and Pasechnik [5]). *Let $M \notin S_n^+ + N_n$ be strictly copositive. Then there are integers $r_K(M)$ and $r_C(M)$ with $1 \leqslant r_K(M) \leqslant r_C(M) < +\infty$, such that*

$$S_n^+ + N_n = K_n^0 \subset K_n^1 \subset \cdots \subset K_n^r \ni M$$

for all $r \geqslant r_K(M)$ while $M \notin K_n^{r_K(M)-1}$, and similarly

$$N_n = C_n^0 \subset C_n^1 \subset \cdots \subset C_n^r \ni M$$

for all $r \geqslant r_C(M)$ while $M \notin C_n^{r_C(M)-1}$.

4.2 Second Degree Approximation of the Cone of Copositive Matrices

In this section, we derive the system of linear matrix inequalities LMI's for the cones K_n^2 by using the similar technique for the system of LMI's for the cone K_n^1 for approximating the copositive programming.

Theorem 4.2.1. *$M \in K_n^2$ if and only if there are n symmetric $n \times n$ matrices $M^{(ij)} \in S_n$ for $i = 1, ..., n$ and $j = 1, ..., n$ such that the following system of linear inequalities has a solution:*

$$\begin{array}{rcll} M - M^{(ii)} & \in & S_n^+ & , i = 1, ..., n, \\ M_{ii}^{(ii)} & \geqslant & 0 & , i = 1, ..., n, \\ 2M_{ii}^{(ij)} + 2M_{ij}^{(ii)} & \geqslant & 0 & i \neq j \\ M_{ii}^{(jj)} + M_{jj}^{(ii)} + 4M_{ij}^{(ij)} & \geqslant & 0 & i \neq j \\ 2(M_{ii}^{(jk)} + 2M_{ij}^{(ik)} + 2M_{ik}^{(ij)} + M_{jk}^{(ii)}) & \geqslant & 0 & i \neq j, j \neq k, i \neq k \\ 4(M_{ij}^{(kl)} + M_{ik}^{(jl)} + M_{il}^{(jk)} + M_{jk}^{(il)} + M_{jl}^{(ik)} + M_{kl}^{(ij)}) & \geqslant & 0 & i < j < k < l \end{array} \tag{4.1}$$

where $M^{(ij)} \in S_n$ for $i = 1, ..., n$ and $j = 1, ..., n$.

Proof. By(3.5) for $r = 2$,we have

$$P^{(2)}(x) = \sum_{i,j=1}^{n} M_{ij} x_i^2 x_j^2 \Big(\sum_{k=1}^{n} x_k^2 \Big)^2 = \tilde{x}^T \tilde{M} \tilde{x}$$

where $\tilde{x} = [x^k]_{k \in I^n(4)} \in \mathbb{R}^d$ and $d = \begin{pmatrix} n+3 \\ 4 \end{pmatrix}$

Assume that $M \in K_n^2$,By the Theorem 3.1.2 and (3.9) there exists a $\tilde{M} \in S_d^+$ satisfying (3.8) such that the left-hand side of (3.8) in case $n = 2m$; are satisfied as follows

(1) $\tilde{M}_{iiii,iiii}$, if $n = 8e_i$

(2) $\tilde{M}_{iiij,iiij} + 2\tilde{M}_{iiii,iijj}$, if $n = 6e_i + 2e_j$

(3) $\tilde{M}_{iijj,iijj} + 2(\tilde{M}_{iiii,jjjj} + \tilde{M}_{iiij,ijjj})$, if $n = 4e_i + 4e_j$

(4) $\tilde{M}_{iijk,iijk} + 2(\tilde{M}_{iiii,jjkk} + \tilde{M}_{iiij,ijkk} + \tilde{M}_{iiik,ijjk}) + \tilde{M}_{iijj,iikk}$, if $n = 4e_i + 2e_j + 2e_k$

(5) $\tilde{M}_{ijkl,ijkl} + 2(\tilde{M}_{iijj,kkll} + \tilde{M}_{iikk,jjll} + \tilde{M}_{iill,jjkk} + \tilde{M}_{iijk,jkll} + \tilde{M}_{iijl,jkkl}$
$+\tilde{M}_{iikl,jjkl} + \tilde{M}_{ijjk,ikll} + \tilde{M}_{ijjl,ikkl} + \tilde{M}_{ijll,ijkk})$, if $n = 2e_i + 2e_j + 2e_k + 2e_l$

Similarly for case $n = 2m$ of (3.9)

(6) $A_{iiii}(M) = M_{ii}$

(7) $A_{iiij}(M) = 2(M_{ii} + M_{ij})$

(8) $A_{iijj}(M) = M_{ii} + M_{jj} + 4M_{ij}$

(9) $A_{iijk}(M) = 2(M_{ii} + 2M_{ij} + 2M_{ik} + M_{jk})$

(10) $A_{ijkl}(M) = 4(M_{ij} + M_{ik} + M_{il} + M_{jk} + M_{jl} + M_{kl})$

Putting $S^{(ij)}_{kl} = \tilde{M}_{ijkk,ijll}$ for all $(ijkl)$ and setting $M^{(ij)} = M - \dfrac{S^{(ij)}}{2}$.

Now consider $S^{(ii)} = M - M^{(ii)}$ since $S^{(ii)} \in S^+_n$,then $M - M^{(ii)} \in S^+_n$ for $i = 1, ..., n$

$$
\begin{aligned}
M^{(ii)}_{ii} &= M_{ii} - \frac{S^{(ii)}_{ii}}{2} \\
&= A_{iiii}(M) - \frac{\tilde{M}_{iiii,iiii}}{2} \\
&= \tilde{M}_{iiii,iiii} - \frac{\tilde{M}_{iiii,iiii}}{2} \\
&= \frac{\tilde{M}_{iiii,iiii}}{2} \\
&\geqslant 0
\end{aligned}
$$

$$\begin{aligned}
2M_{ii}^{(ij)} + 2M_{ij}^{(ii)} &= 2(M_{ii} - \frac{S_{ii}^{(ij)}}{2}) + 2(M_{ij} - \frac{S_{ij}^{(ii)}}{2}) \\
&= 2M_{ii} + 2M_{ij} - S_{ii}^{(ij)} - S_{ij}^{(ii)} \\
&= A_{iiij}(M) - \tilde{M}_{iiij,iiij} - \tilde{M}_{iiii,iijj} \\
&= \tilde{M}_{iiij,iiij} + 2\tilde{M}_{iiii,iijj} - \tilde{M}_{iiij,iiij} - \tilde{M}_{iiii,iijj} \\
&= \tilde{M}_{iiii,iijj} \\
&\geqslant 0
\end{aligned}$$

$$\begin{aligned}
M_{ii}^{(jj)} + M_{jj}^{(ii)} + 4M_{ij}^{(ij)} &= M_{ii} - \frac{S_{ii}^{(jj)}}{2} + M_{jj} - \frac{S_{jj}^{(ii)}}{2} + 4(M_{ij} - \frac{S_{ij}^{(ij)}}{2}) \\
&= M_{ii} + M_{jj} + 4M_{ij} - \frac{S_{ii}^{(jj)}}{2} - \frac{S_{jj}^{(ii)}}{2} - 2S_{ij}^{(ij)} \\
&= A_{iijj}(M) - \frac{\tilde{M}_{iijj,iijj}}{2} - \frac{\tilde{M}_{iijj,iijj}}{2} - 2\tilde{M}_{iiij,ijjj} \\
&= \tilde{M}_{iijj,iijj} + 2\tilde{M}_{iiii,jjjj} + 2\tilde{M}_{iiij,ijjj} - \frac{\tilde{M}_{iijj,iijj}}{2} \\
&\quad - \frac{\tilde{M}_{iijj,iijj}}{2} - 2\tilde{M}_{iiij,ijjj} \\
&= \tilde{M}_{iiii,jjjj} \\
&\geqslant 0
\end{aligned}$$

$$\begin{aligned}
&2(M_{ii}^{(jk)} + 2M_{ij}^{(ik)} + 2M_{ik}^{(ij)} + M_{jk}^{(ii)}) \\
&\quad = 2\big(M_{ii} - \frac{S_{ii}^{(jk)}}{2} + 2(M_{ij} - \frac{S_{ij}^{(ik)}}{2}) + 2(M_{ik} - \frac{S_{ik}^{(ij)}}{2}) + M_{jk} - \frac{S_{jk}^{(ii)}}{2}\big) \\
&\quad = 2(M_{ii} + 2M_{ij} + 2M_{ik} + M_{jk}) - S_{ii}^{(jk)} - 2S_{ij}^{(ik)} - 2S_{ik}^{(ij)} - S_{jk}^{(ii)} \\
&\quad = A_{iijk}(M) - \tilde{M}_{iijk,iijk} - 2\tilde{M}_{iiik,ijjk} - 2\tilde{M}_{iiij,ijkk} - \tilde{M}_{iijj,iikk} \\
&\quad = \tilde{M}_{iijk,iijk} + 2\tilde{M}_{iiii,jjkk} + 2\tilde{M}_{iiij,ijkk} + 2\tilde{M}_{iiik,ijjk} + 2\tilde{M}_{iijj,iikk} \\
&\quad\quad - \tilde{M}_{iijk,iijk} - 2\tilde{M}_{iiik,ijjk} - 2\tilde{M}_{iiij,ijkk} - \tilde{M}_{iijj,iikk} \\
&\quad = 2\tilde{M}_{iiii,jjkk} + \tilde{M}_{iijj,iikk} \\
&\quad \geqslant 0
\end{aligned}$$

$$
\begin{aligned}
&4(M_{ij}^{(kl)} + M_{ik}^{(jl)} + M_{il}^{(jk)} + M_{jk}^{(il)} + M_{jl}^{(ik)} + M_{kl}^{(ij)}) \\
&= 4\big(M_{ij} - \frac{S_{ij}^{(kl)}}{2} + M_{ik} - \frac{S_{ik}^{(jl)}}{2} + M_{il} - \frac{S_{il}^{(jk)}}{2} + M_{jk} - \frac{S_{jk}^{(il)}}{2} \\
&\quad + M_{jl} - \frac{S_{jl}^{(ik)}}{2} + M_{kl} - \frac{S_{kl}^{(ij)}}{2}\big) \\
&= 4(M_{ij} + M_{ik} + M_{il} + M_{jk} + M_{jl} + M_{kl}) - 2\big(S_{ij}^{(kl)} + S_{ik}^{(jl)} \\
&\quad + S_{il}^{(jk)} + S_{jk}^{(il)} + S_{jl}^{(ik)} + S_{kl}^{(ij)}\big) \\
&= A_{ijkl}(M) + 2(\tilde{M}_{iikl,jjkl} + \tilde{M}_{iijl,jkkl} + \tilde{M}_{iijk,jkll} + \tilde{M}_{ijjl,ikkl} \\
&\quad + \tilde{M}_{ijjk,ikll} + \tilde{M}_{ijkk,ijll}) \\
&= \tilde{M}_{ijkl,ijkl} + 2\tilde{M}_{iijj,kkll} + 2\tilde{M}_{iikk,jjll} + 2\tilde{M}_{iill,jjkk} + 2\tilde{M}_{iijk,jkll} \\
&\quad + 2\tilde{M}_{iijl,jkkl} + 2\tilde{M}_{iikl,jjkl} + 2\tilde{M}_{ijjk,ikll} + 2\tilde{M}_{ijjl,ikkl} + 2\tilde{M}_{ijll,ijkk} \\
&\quad - 2\tilde{M}_{iikl,jjkl} - 2\tilde{M}_{iijl,jkkl} - 2\tilde{M}_{iijk,jkll} - 2\tilde{M}_{ijjl,ikkl} - 2\tilde{M}_{ijjk,ikll} \\
&\quad - 2\tilde{M}_{ijkk,ijll} \\
&= \tilde{M}_{ijkl,ijkl} + 2\tilde{M}_{iijj,kkll} + 2\tilde{M}_{iikk,jjll} + 2\tilde{M}_{iill,jjkk} \\
&\geqslant 0.
\end{aligned}
$$

In these cases, we use the degree of freedom of choosing entry of $\tilde{M}$ so that $\tilde{M}$ remains feasible and also the fact that every principle major of positive definite is positive definite.

Thus we have got a solution to the system of LMI's (4.1). Conversely, Assuming that a solution to (4.1) is given. Then we have,

$$
\begin{aligned}
P^{(2)}(x) &= \big(\sum_{i=1}^{n} x_i^2\big)^2 (x \circ x)^T M (x \circ x) \\
&= \big(\sum_{i=1}^{n} x_i^2\big)^2 (x \circ x)^T (M - M^{(ij)})(x \circ x) + \big(\sum_{i=1}^{n} x_i^2\big)^2 (x \circ x)^T (M^{(ij)})(x \circ x)
\end{aligned}
$$

Since $M - M^{(ij)} \in S_n^+$ for every i, the first sum is a s.o.s. Let now consider the second sum,

$$
\begin{aligned}
&\Big(\sum_{i=1}^{n} x_i^2\Big)^2 (x\circ x)^T (M^{(ij)})(x\circ x)\\
&\qquad = \sum_{i,j,k,l} M_{kl}^{(ij)} x_i^2 x_j^2 x_k^2 x_l^2\\
&\qquad = \sum_{i} M_{ii}^{(ii)} x_i^8 + \sum_{i\neq j} (2M_{ii}^{(ij)} + 2M_{ij}^{(ii)}) x_i^6 x_j^2\\
&\qquad\quad + \sum_{i\neq j} (M_{ii}^{(jj)} + M_{jj}^{(ii)} + 4M_{ij}^{(ij)}) x_i^4 x_j^4\\
&\qquad\quad + \sum_{i\neq j, j\neq k, i\neq k} 2(M_{ii}^{(jk)} + 2M_{ij}^{(ik)} + 2M_{ik}^{(ij)} + M_{jk}^{(ii)}) x_i^4 x_j^2 x_k^2\\
&\qquad\quad + \sum_{i<j<k<l} 4(M_{ij}^{(kl)} + M_{ik}^{(jl)} + M_{il}^{(jk)} + M_{jk}^{(il)} + M_{jl}^{(ik)} + M_{kl}^{(ij)}) x_i^2 x_j^2 x_k^2 x_l^2\\
&\qquad = \sum_{i} \Big(\sqrt{M_{ii}^{(ii)}}\, x_i^4\Big)^2 + \sum_{i\neq j} \Big(\sqrt{2M_{ii}^{(ij)} + 2M_{ij}^{(ii)}}\, x_i^3 x_j\Big)^2\\
&\qquad\quad + \sum_{i\neq j} \Big(\sqrt{M_{ii}^{(jj)} + M_{jj}^{(ii)} + 4M_{ij}^{(ij)}}\, x_i^2 x_j^2\Big)^2\\
&\qquad\quad + \sum_{i\neq j, j\neq k, i\neq k} \Big(\sqrt{2(M_{ii}^{(jk)} + 2M_{ij}^{(ik)} + 2M_{ik}^{(ij)} + M_{jk}^{(ii)})}\, x_i^2 x_j x_k\Big)^2\\
&\qquad\quad + \sum_{i<j<k<l} \Big(\sqrt{4(M_{ij}^{(kl)} + M_{ik}^{(jl)} + M_{il}^{(jk)} + M_{jk}^{(il)} + M_{jl}^{(ik)} + M_{kl}^{(ij)})}\, x_i x_j x_k x_l\Big)^2
\end{aligned}
$$

where we have used the non-negativity of the last condition of 4.1 to obtain the last equality. Note that the first two sums of the last expression vanish due to the second and the third condition of 4.1. Thus $P^{(2)}(\mathbf{x})$ is represented as a s.o.s. □

We therefore change the SDP approximations of the copositive cone to arrive at a series of LP approximations of copositive cone. These approximation are weaker than the SDP, but can be solved more easily.

Theorem 4.2.2. *$M \in C_n^2$ if and only if there are n symmetric $n \times n$ matrices $M^{(ij)} \in S_n$ for $i = 1, ..., n$ and $j = 1, ..., n$ such that the following system of linear* *inequalities has a solution:*

$$
\begin{array}{rcll}
M - M^{(ii)} & \in & N_n^+ & , i = 1, ..., n, \\
M_{ii}^{(ii)} & \geqslant & 0 & , i = 1, ..., n, \\
2M_{ii}^{(ij)} + 2M_{ij}^{(ii)} & \geqslant & 0 & i \neq j \\
M_{ii}^{(jj)} + M_{jj}^{(ii)} + 4M_{ij}^{(ij)} & \geqslant & 0 & i \neq j \\
2(M_{ii}^{(jk)} + 2M_{ij}^{(ik)} + 2M_{ik}^{(ij)} + M_{jk}^{(ii)}) & \geqslant & 0 & i \neq j, j \neq k, i \neq k \\
4(M_{ij}^{(kl)} + M_{ik}^{(jl)} + M_{il}^{(jk)} + M_{jk}^{(il)} + M_{jl}^{(ik)} + M_{kl}^{(ij)}) & \geqslant & 0 & i < j < k < l
\end{array}
$$

where $M^{(ij)} \in S_n$ *for* $i = 1, ..., n$ *and* $j = 1, ..., n$.

4.3 Examples

In this section, we give some examples for approximating the problems of the copositive programming in the form:

$$
\begin{array}{ll}
\min & tr(C^T X) \\
s.t. & A \bullet X = b \\
 & X \in \mathbb{K}_n.
\end{array}
$$

We approximated this problem via Matlab program, and we have a optimization toolbox is called "linprog". The linprog is a package for solving a linear programming problem, then we changed this problem into the linear programming problem as the form:

$$
\begin{array}{ll}
\min & c^T x \\
s.t. & Ax = b \\
 & x \geq 0.
\end{array}
$$

And we condider LMI's in the theorem 4.2.2 for checking the condition $X \in C_n$.

Example 4.3.1 Consider for the case $i, j, k, l = 1, 2, 3$ and given

$$c^T = \begin{bmatrix} 4 & 2 & 10 & 4 & -8 & 4 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{1\times 60}$$

$$A = \begin{bmatrix} 4 & 16 & 4 & 4 & 22 & 4 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -8 & 2 & -18 & 6 & 6 & 10 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{2\times 60}$$

$$b = \begin{bmatrix} 312 \\ 60 \end{bmatrix}$$

Consider the condition, $M - M^{(ii)} \in S_n^+$ ▶ $M - M^{(11)} \geqslant 0$

▶ $M - M^{(22)} \geqslant 0$

▶ $M - M^{(33)} \geqslant 0$

$$M^{(ij)} = \begin{bmatrix} M^{(11)} & M^{(12)} & M^{(13)} \\ M^{(21)} & M^{(22)} & M^{(23)} \\ M^{(31)} & M^{(32)} & M^{(33)} \end{bmatrix}_{9x9}$$

$$M = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \quad M^{(11)} = \begin{bmatrix} M_{11}^{(11)} & M_{12}^{(11)} & M_{13}^{(11)} \\ M_{21}^{(11)} & M_{22}^{(11)} & M_{23}^{(11)} \\ M_{31}^{(11)} & M_{32}^{(11)} & M_{33}^{(11)} \end{bmatrix}_{3x3}$$

since $M^{(11)} \in S_n$ and $M \in S_n^+$, then

$M - M^{(11)} \geqslant 0 \implies$ 1. $x_{11} - M_{11}^{(11)} \geqslant 0$

2. $x_{12} - M_{12}^{(11)} \geqslant 0$

3. $x_{13} - M_{13}^{(11)} \geqslant 0$

4. $x_{22} - M_{22}^{(11)} \geqslant 0$

5. $x_{23} - M_{23}^{(11)} \geqslant 0$

6. $x_{33} - M_{33}^{(11)} \geqslant 0$

Similarly for $M - M^{(22)}$ and $M - M^{(33)}$.

$, M_{ii}^{(ii)} \geqslant 0$ ▶ $M_{11}^{(11)} \geqslant 0$

▶ $M_{22}^{(22)} \geqslant 0$

▶ $M_{33}^{(33)} \geqslant 0$

and $2M_{ii}^{(ij)} + 2M_{ij}^{(ii)} \geqslant 0, i \neq j$ ▶ $2M_{11}^{(12)} + 2M_{12}^{(11)} \geqslant 0$

▶ $2M_{11}^{(13)} + 2M_{13}^{(11)} \geqslant 0$

▶ $2M_{22}^{(21)} + 2M_{21}^{(22)} \geqslant 0$

▶ $2M_{22}^{(23)} + 2M_{23}^{(22)} \geqslant 0$

▶ $2M_{33}^{(31)} + 2M_{31}^{(33)} \geqslant 0$

▶ $2M_{33}^{(32)} + 2M_{32}^{(33)} \geqslant 0$

Similarly for condition 4 and 5.

Thus we got 39 conditions for checking the condition $X \in C_n$. And the approximation result is

▶ Optimization terminated successfully.

$$x = \begin{bmatrix} 0.0000 & 0.0000 & 1.3143 \\ 0.0000 & 0.0000 & 13.9429 \\ 1.3143 & 13.9429 & 0.0000 \end{bmatrix}$$

x is a copositive matrix.

Example 4.3.2 Consider for the case $i, j, k, l = 1, 2, 3, 4$ and given

$$c^T = \begin{bmatrix} 4 & 14 & 24 & 4 & 0 & 6 & 14 & 14 & 16 & 2 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{1\times 170}$$

$$A = \begin{bmatrix} 4 & 10 & -4 & 18 & -10 & 6 & 18 & 14 & 4 & 6 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 18 & 0 & 0 & -8 & 2 & 2 & 20 & -6 & 12 & 10 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{2\times 170}$$

$$b = \begin{bmatrix} 376 \\ 140 \end{bmatrix}$$

We used the same technique in example 4.3.1 for checking $X \in C_n$ and the approximation result is

▶ Optimization terminated successfully.

$$x = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 9.8492 \\ 0.0000 & 0.0000 & 0.0000 & 11.0397 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 9.8492 & 11.0397 & 0.0000 & 0.0000 \end{bmatrix}$$

x is a copositive matrix.