CHAPTER 2

PRELIMINARIES

In this chapter, we give some basic definitions, notations, lemmas and results which will be used in the later chapters.

2.1 Notations

The following notations that will be used in this thesis:

$\mathbb{R}^n$ – the n dimensional Euclidean space,

$\mathbb{R}^{n\times n}$ – the set of all $n \times n$ real matrices,

$\|x\|$ – the Euclidean norm of vector x,

I – the identity matrix,

A^T – the transpose of matrix A,

$A > 0, A \geq 0, A < 0, A \leq 0$ – means that A is symmetric positive definite, positive semi-definite, negative definite and negative semi-definite,

$\lambda(A)$ – the set of all eigenvalues of matrix A,

$\lambda_M(A)$ – maximum eigenvalue of matrix A,

$\lambda_m(A)$ – minimum eigenvalue of matrix A,

$\| x \| = \sqrt{\sum_{i=1}^{n} x_i^2}$, for any $x \in \mathbb{R}^n$,

$C([-h_M, 0], \mathbb{R}^n), h_M > 0$ – denotes the space of continuous functions mapping the interval $[-h_M, 0]$ into $\mathbb{R}^n$,

$x_t \in C([-h_M, 0], \mathbb{R}^n)$ defined $x_t(s) = x(t+s), s \in [-h_M, 0]$ and

$\|x_t\| = \sup_{s\in[-h_M,0]} \|x(t+s)\|$,

$\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ – $*$ represents the symmetric form of matrix, namely $* = B^T$.

2.2 Types of Matrix and Function

Definition 2.2.1 (Symmetric Matrix) A real $n \times n$ matrix A is called *symmetric* if

$$A^T = A.$$

Definition 2.2.2 (Positive Definite Matrix) A real $n \times n$ matrix A is called *positive definite* if

$$x^T Ax > 0$$

for all nonzero vectors $x \in \mathbb{R}^n$. It is called *positive semidefinite* if

$$x^T Ax \geq 0.$$

Definition 2.2.3 (Negative Definite Matrix) A real $n \times n$ matrix A is called *negative definite* if

$$x^T Ax < 0$$

for all nonzero vectors $x \in \mathbb{R}^n$. It is called *negative semidefinite* if

$$x^T Ax \leq 0.$$

The follows result are well known.

Lemma 2.2.1 A symmetric matrix is positive semidefinite (definite) matrix if all of its eigenvalues are nonnegative (positive).

Lemma 2.2.2 A symmetric matrix is negative semidefinite (definite) matrix if all of its eigenvalues are nonpositive (negative).

Definition 2.2.4 (Positive Definite Function) A function $f(x) \in \mathbb{R}^n$ is called *positive definite* if $f(\bar{0}) = 0, f(x) > 0$ for all $x \in \mathbb{R}^n$. It is called *positive semidefinite* if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Definition 2.2.5 (Negative Definite Function) A function $f(x) \in \mathbb{R}^n$ is called *negative definite* if $f(\bar{0}) = 0, f(x) < 0$ for all $x \in \mathbb{R}^n$. It is called *negative semidefinite* if $f(x) \leq 0$ for all $x \in \mathbb{R}^n$.

2.3 Lyapunov Function

Consider the system described by

$$\dot{x} = f(t, x),\ \ x(t_0) = x_0, \tag{2.1}$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n,\ x_i = x_i(t),\ f = (f_1, f_2, ..., f_n) \in \mathbb{R}^n$ and $f_i = f_i(t, x_1, x_2, ..., x_n)$ for $i = 1, 2, ..., n$.

Definition 2.3.1 (Lyapunov Function) Let D be a domain $\mathbb{R}^n$ such that $\bar{0} \in D$ $V : D \subseteq \mathbb{R}^n \to \mathbb{R}$, we say that $V(x)$ is a Lyapunov function of system (2.1) if the following conditions hold:

(1) $V(x)$ is continuous on $D \subseteq \mathbb{R}^n$.

(2) $V(x)$ is positive definite such that $V(\bar{0}) = 0$ and $V(x) > 0$ for all $x \neq \bar{0}$.

(3) the derivative of V with respect to (2.1) is negative semidefinite (i.e., $\dot{V}(0) = 0$, and $\dot{V}(x) \leq 0$ for all $x \neq \bar{0}$).

2.4 Stability

Definition 2.4.1 A point $\bar{x}$ is called an *equilibrium point* of equation (2.1) if $f(t, \bar{x}) = 0$ for all $t \geq t_0$. For all purposes of the stability theory we can assume that $\bar{0}$ is an equilibrium of (2.1).

Definition 2.4.2 (Stable) The equilibrium point $\bar{x}$ of equation (2.1) is called *stable* if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon, t_0) > 0$ such that $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \epsilon$ for all $t \geq t_0 \geq 0$.

Definition 2.4.3 (Unstable) The equilibrium point $\bar{x}$ of equation (2.1) is called *unstable* if it is not stable.

Definition 2.4.4 (Asymptotically Stable) The equilibrium point $\bar{x}$ of equation (2.1) is called *asymptotically stable* (denoted A.S.) if it is stable and $\|x(t)\| \to 0$ as $t \to \infty$.

The following Theorems and Lemmas will be used throughout this thesis.

Theorem 2.4.1 [4] The equilibrium point $\bar{x}$ of equation (2.1) is stable if there exists a Lyapunov function for system (2.1). Moreover, if there exists a Lyapunov function whose derivative is negative definite, then the equilibrium point $\bar{x}$ is A.S.

Lemma 2.4.1 [15] Given a positive definite matrix $Q \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, then

$$\lambda_m(Q)x^T x \leq x^T Q x \leq \lambda_M(Q)x^T x.$$

Lemma 2.4.2 [6] For any $x, y \in \mathbb{R}^n$, matrices W, E, F, H with $W > 0, F^T F \leq I$, and scalar $\varepsilon > 0$, one has

(1.) $EFH + H^T F^T E^T \leq \varepsilon^{-1} E E^T + \varepsilon H^T H$,

(2.) $2x^T y \leq x^T W^{-1} x + y^T W y$.

Lemma 2.4.3 [1] Let $u : [t_0, \infty] \rightarrow \mathbb{R}$ satisfy the following delay differential inequality:

$$\dot{u}(t) \leq \alpha u(t) + \beta \sup_{\theta\in[t-\tau,t]} u(\theta),\ t \geq t_0.$$

Assume that $\alpha + \beta > 0$. Then, there exist positive constant ξ and k such that

$$u(t) \leq k e^{\xi(t-t_0)},\ t \geq t_0,$$

where $\xi = \alpha + \beta$ and $k = \sup_{\theta\in[t_0-\tau,t_0]} u(\theta)$.

Lemma 2.4.4 [1] Let the following differential inequality

$$\dot{u} \leq -\alpha u(t) + \beta \sup_{\theta\in[t-\tau,t]} u(\theta),\ t \geq t_0,$$

hold. If $\alpha > \beta > 0$, then there exist positive k and ζ such that

$$u(t) \leq k e^{-\zeta(t-t_0)},\ t \geq t_0,$$

where $\zeta = -\alpha + \beta$ and $k = \sup_{\theta\in[t_0-\tau,t_0]} u(\theta)$.

Lemma 2.4.5 [5] (**Schur Complement**) Given constant symmetric Q, S and $R \in \mathbb{R}^{n\times n}$ where $R > 0, Q = Q^T$ and $R = R^T$ we have

$$\begin{bmatrix} Q & S \\ S^T & -R \end{bmatrix} < 0 \Leftrightarrow Q + SR^{-1}S^T < 0.$$