CHAPTER 3

MAIN RESULTS

In this chapter, we present some criterion for exponential stability for uncertain switched system that include both stable and unstable modes based on Lyapunov stability theory, dwell-time approach, Newton-Leibniz formula and linear matrix inequality (LMI) techniques. In section 3.1, we derive new sufficient condition for exponential stability of linear switched system without the uncertainties. In section 3.2, we derive new sufficient condition for robust exponential stability of linear switched system. In section 3.3, we derive new sufficient condition for robust exponential stability of nonlinear switched system. Some numerical simulations are given to illustrate the effectiveness of our theoretical results.

Consider a class of uncertain switched system with time-varying delay of the form

$$\begin{cases} \dot{x}(t) = [A_\sigma + \Delta A_\sigma(t)]x(t) + [B_\sigma + \Delta B_\sigma(t)]x(t-h(t)) \\ \qquad\quad + f_\sigma(t, x(t), x(t-h(t))),\ t > 0, \\ x(t) = \phi(t),\ t \in [-h_M, 0], \end{cases} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector. $\sigma(\cdot) : \mathbb{R}^n \to S = \{1, 2, ..., N\}$ is the switching function. Let $i \in S = S_u \cup S_s$ such that $S_u = \{1, 2, ..., r\}$ and $S_s = \{r+1, r+2, ..., N\}$ be the set of the unstable and stable modes, respectively. N denotes the number of subsystems. $A_i, B_i \in \mathbb{R}^{n\times n}$ are given constant matrices and $\phi(t) \in C([-h_M, 0], \mathbb{R}^n)$ is the initial function with the norm $\| \phi \| = \sup_{s\in[-h_M,0]} \| \phi(s) \|$. $\Delta A_i(t), \Delta B_i(t)$ are uncertain matrices satisfy the following conditions:

$$\Delta A_i(t) = E_{1i}F_{1i}(t)H_{1i},\ \Delta B_i(t) = E_{2i}F_{2i}(t)H_{2i}, \tag{3.2}$$

where $E_{ji}, H_{ji}, j = 1, 2, i = 1, 2, ..., N$ are given constant matrices with appropriate dimensions. $F_{ji}(t)$ are unknown, real matrices satisfying:

$$F_{ji}^T(t)F_{ji}(t) \le I,\ j = 1, 2,\ i = 1, 2, ..., N,\ \forall t \ge 0, \tag{3.3}$$

where I is the identity matrix of appropriate dimension.

The nonlinear perturbation $f_i(t, x(t), x(t-h(t))),\ i = 1, 2, ..., N$ satisfies the following condition:

$$\| f_i(t, x(t), x(t-h(t))) \| \leq \gamma_i \| x(t) \| + \delta_i \| x(t-h(t)) \| . \tag{3.4}$$

The time-varying delay function $h(t)$ is assumed to satisfy one of the following conditions:

(i) when $\Delta A_i(t) = 0$ and $\Delta B_i(t) = 0$ and $f_i(t, x(t), x(t-h(t))) = 0$

$$0 \leq h_m \leq h(t) \leq h_M,\ \dot{h}(t) \leq \mu,\ t \geq 0, \tag{3.5}$$

(ii) when $\Delta A_i(t) \neq 0$ or $\Delta B_i(t) \neq 0$ or $f_i(t, x(t), x(t-h(t))) \neq 0$

$$0 \leq h_m \leq h(t) \leq h_M,\ \dot{h}(t) \leq \mu < 1,\ t \geq 0, \tag{3.6}$$

where h_m, h_M and μ are given constants.

Define

$$\lambda_i(t) = \begin{cases} 1, & \sigma = i, \\ 0, & \text{otherwise.} \end{cases} \tag{3.7}$$

Then, we have $\sum_{i=1}^{N} \lambda_i(t) = 1$, and system (3.1) can be written as

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{N} \lambda_i(t)\{[A_i + \Delta A_i(t)]x(t) + [B_i + \Delta B_i(t)]x(t-h(t)) \\ \qquad + f_i(t, x(t), x(t-h(t)))\},\ t > 0, \\ x(t) = \phi(t),\ t \in [-h_M, 0]. \end{cases} \tag{3.8}$$

For simplicity of later presentation, we use the following notations:

$\lambda^+ = \max\limits_i\{\xi_i, \forall i \in S_u\}$, ξ_i denotes the growth rates of the unstable modes.

$\lambda^- = \min\limits_i\{\zeta_i, \forall i \in S_s\}$, ζ_i denotes the decay rates of the stable modes.

$T^+(t_0, t)$ denotes the total activation times of the unstable modes over (t_0, t).

$T^-(t_0, t)$ denotes the total activation times of the stable modes over (t_0, t).

$N(t)$ denotes the number of times the system is switched on (t_0, t).

$l(t)$ denotes the number of times the unstable subsystems are activated on (t_0, t).

$N(t) - l(t)$ denotes the number of times the stable subsystems are activated on (t_0, t).

$$\psi = \frac{\max\limits_i\{\lambda_M(P_i)\}}{\min\limits_j\{\lambda_m(P_j)\}}.$$

$$\alpha_1 = \min\limits_i\{\lambda_m(P_i)\},$$

$$\begin{aligned}
\alpha_2 = {} & \max_i\{\lambda_M(P_i)\} + h_M \max_i\{\lambda_M(Q_i)\} + \frac{h_M^2}{2}\max_i\{\lambda_M(R_i)\} \\
& + h_M^2 \max_i\{\lambda_M(\begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix})\} \\
& + 2h_M^2 \max_i\{\lambda_M(A_i^T T_i A_i), \lambda_M(A_i^T T_i B_i), \lambda_M(B_i^T T_i A_i), \lambda_M(B_i^T T_i B_i)\},
\end{aligned}$$

$$\begin{aligned}
\alpha_3 = {} & \max_i\{\lambda_M(P_i)\} + h_M \max_i\{\lambda_M(Q_i)\} + \frac{h_M^2}{2}\max_i\{\lambda_M(R_i)\} \\
& + h_M^2 \max_i\{\lambda_M(\begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix})\}.
\end{aligned}$$

$$\Omega_{1,i} = \begin{bmatrix} \Phi_{11,i} & \Phi_{12,i} \\ * & \Phi_{13,i} \end{bmatrix},$$

$$\Phi_{11,i} = A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} + h_M A_i^T T_i A_i,$$

$$\Phi_{12,i} = B_i^T P_i + h_M S_{12,i} + h_M A_i^T T_i B_i,$$

$$\Phi_{13,i} = -(1-\mu)e^{-2\beta h_M} Q_i + h_M S_{22,i} + h_M B_i^T T_i B_i.$$

$$\Omega_{2,i} = \begin{bmatrix} \Phi_{21,i} & \Phi_{22,i} \\ * & \Phi_{23,i} \end{bmatrix},$$

$$\Phi_{21,i} = A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} + h_M A_i^T T_i A_i + h_M X_{11,i} + Y_i + Y_i^T,$$

$$\Phi_{22,i} = B_i^T P_i + h_M S_{12,i} + h_M A_i^T T_i B_i + h_M X_{12,i} - Y_i + Z_i^T,$$

$$\Phi_{23,i} = -(1-\mu)e^{-2\beta h_M} Q_i + h_M S_{22,i} + h_M B_i^T T_i B_i + h_M X_{22,i} - Z_i - Z_i^T.$$

$$\Omega_{3,i} = \begin{bmatrix} X_{11,i} & X_{12,i} & Y_i \\ * & X_{22,i} & Z_i \\ * & * & \frac{T_i}{2} \end{bmatrix}.$$

$$\Xi_i = \begin{bmatrix} \Phi_{31,i} & \Phi_{32,i} \\ * & \Phi_{33,i} \end{bmatrix},$$

$$\Phi_{31,i} = A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} + \varepsilon_{1i}^{-1} H_{1i}^T H_{1i} + \varepsilon_{1i} P_i E_{1i}^T E_{1i} P_i + \varepsilon_{2i} P_i E_{2i}^T E_{2i} P_i,$$

$$\Phi_{32,i} = B_i^T P_i + h_M S_{12,i},$$

$$\Phi_{33,i} = -(1-\mu) e^{-2\beta h_M} Q_i + h_M S_{22,i} + \varepsilon_{2i}^{-1} H_{2i}^T H_{2i}.$$

$$\Theta_i = \begin{bmatrix} \Phi_{41,i} & \Phi_{42,i} \\ * & \Upsilon_{43,i} \end{bmatrix},$$

$$\Phi_{41,i} = A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} + \varepsilon_{3i}^{-1} \gamma_i I + \varepsilon_{3i} P_i P_i + \varepsilon_{4i}^{-1} H_{4i}^T H_{4i}$$
$$+ \varepsilon_{4i} P_i E_{4i}^T E_{4i} P_i + \varepsilon_{6i} P_i E_{5i}^T E_{5i} P_i,$$

$$\Phi_{42,i} = B_i^T P_i + h_M S_{12,i},$$

$$\Phi_{43,i} = -(1-\mu) e^{-2\beta h_M} Q_i + h_M S_{22,i} + \varepsilon_{3i}^{-1} \delta_i I + \varepsilon_{5i}^{-1} H_{5i}^T H_{5i}.$$

3.1 Exponential Stability of Linear Switched System with Time-Varying Delay

In this section, we deal with the problem for exponential stability of the zero solution of system (3.1) without the uncertainties and nonlinear perturbation $(\Delta A_i(t) = \Delta B_i(t) = 0,\ f_i(t, x(t), x(t-h(t))) = 0)$.

Theorem 3.1.1. *The zero solution of system (3.1) with* $\Delta A_i(t) = \Delta B_i(t) = 0$ *and* $f_i(t, x(t), x(t-h(t))) = 0$ *is exponentially stable if there exist symmetric positive definite matrices* $P_i, Q_i, R_i, \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix}, T_i$ *and appropriate dimension matrices* Y_i, Z_i *such that the following assumptions hold:*

*A*1. (i) *For* $i \in S_u$,

$$\Omega_{1,i} > 0. \tag{3.9}$$

(ii) *For* $i \in S_s$,

$$\Omega_{2,i} < 0 \ \text{ and } \ \Omega_{3,i} \geq 0. \tag{3.10}$$

*A*2. *Assume that, for any* t_0 *the switching law guarantees that*

$$\inf_{t \geq t_0} \frac{T^-(t_0,t)}{T^+(t_0,t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*} \tag{3.11}$$

where $\lambda^* \in (0, \lambda^-)$. *Furthermore, there exists* $0 < \nu < \lambda^*$ *such that*

(i) *If the subsystem* $i \in S_u$ *is activated in time intervals* $[t_{i_k-1}, t_{i_k}), k = 1, 2, ...,$ *then*

$$ln\ \psi - \nu(t_{i_k} - t_{i_k-1}) \leq 0,\ k = 1, 2, ... \tag{3.12}$$

(ii) *If the subsystem* $j \in S_s$ *is activated in time intervals* $[t_{j_k-1}, t_{j_k}), k = 1, 2, ...,$ *then*

$$ln\ \psi + \zeta_j h_M - \nu(t_{j_k} - t_{j_k-1}) \leq 0,\ k = 1, 2, ... \tag{3.13}$$

Proof. Consider the following Lyapunov functional:

$V_i(x_t) = V_{1,i}(x(t)) + V_{2,i}(x_t) + V_{3,i}(x_t) + V_{4,i}(x_t) + V_{5,i}(x_t)$

where $x_t \in C([-h_M, 0], \mathbb{R}^n),\ x_t(s) = x(t+s),\ s \in [-h_M, 0]$ and

$$V_{1,i}(x(t)) = x^T(t)P_i x(t),$$
$$V_{2,i}(x_t) = \int_{t-h(t)}^{t} e^{2\beta(s-t)}x^T(s)Q_i x(s)ds,$$
$$V_{3,i}(x_t) = \int_{-h(t)}^{0}\int_{t+s}^{t} e^{2\beta(\xi-t)}x^T(\xi)R_i x(\xi)d\xi ds,$$
$$V_{4,i}(x_t) = \int_{-h(t)}^{0}\int_{t+s}^{t} e^{2\beta(\xi-t)}\begin{bmatrix} x(\xi) \\ x(\xi-h(\xi)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix}\begin{bmatrix} x(\xi) \\ x(\xi-h(\xi)) \end{bmatrix} d\xi ds,$$
$$V_{5,i}(x_t) = \int_{-h(t)}^{0}\int_{t+s}^{t} \dot{x}^T(\xi)T_i\dot{x}(\xi)d\xi ds.$$

It is easy to verify that (See Appendix A.1.)

$$\alpha_1 \parallel x(t) \parallel^2 \leq V_i(x_t) \leq \alpha_2 \parallel x_t \parallel^2,\ t \geq 0. \tag{3.14}$$

We have

$$\begin{aligned} V_{1,i}(x(t)) &\leq \max_i\{\lambda_M(P_i)\} \parallel x(t) \parallel^2 \\ &= \frac{\max_i\{\lambda_M(P_i)\}}{\min_j\{\lambda_m(P_j)\}} \min_j\{\lambda_m(P_j)\}x^T(t)x(t) \\ &\leq \frac{\max_i\{\lambda_M(P_i)\}}{\min_j\{\lambda_m(P_j)\}} x^T(t)P_j x(t) \\ &= \frac{\max_i\{\lambda_M(P_i)\}}{\min_j\{\lambda_m(P_j)\}} V_{1,j}(x(t)). \end{aligned}$$

Let $\psi = \dfrac{\max_i\{\lambda_M(P_i)\}}{\min_j\{\lambda_m(P_j)\}}$. Obviously $\psi \geq 1$ and thus

$$V_i(x_t) \leq \psi V_j(x_t),\ \forall i, j \in S. \tag{3.15}$$

Taking derivative of $V_{1,i}(x(t))$ along trajectories of any subsystem ith we have

$$\begin{aligned} \dot{V}_{1,i}(x(t)) &= \dot{x}^T(t)P_i x(t) + x^T(t)P_i\dot{x}(t) \\ &= \sum_{i=1}^{N}\lambda_i(t)[x^T(t)A_i^T P_i x(t) + x^T(t-h(t))B_i^T P_i x(t) \\ &\quad + x^T(t)P_i A_i x(t) + x^T(t)P_i B_i x(t-h(t))]. \end{aligned}$$

Next, by taking derivative of $V_{2,i}(x_t), V_{3,i}(x_t), V_{4,i}(x_t)$ and $V_{5,i}(x_t)$, respectively, along the system trajectories yields

$$\begin{aligned} \dot{V}_{2,i}(x_t) &= x^T(t)Q_i x(t) - (1-\dot{h}(t))e^{-2\beta h(t)}x^T(t-h(t))Q_i x(t-h(t)) - 2\beta V_{2,i}(x_t) \\ &\leq x^T(t)Q_i x(t) - (1-\mu)e^{-2\beta h(t)}x^T(t-h(t))Q_i x(t-h(t)) - 2\beta V_{2,i}(x_t), \end{aligned}$$
$$\dot{V}_{3,i}(x_t) = \int_{-h(t)}^{0}[x^T(t)R_i x(t) - e^{2\beta s}x^T(t+s)R_i x(t+s)]ds - 2\beta V_{3,i}(x_t)$$

$$\leq h_M x^T(t)R_i x(t) - \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s)R_i x(s)ds - 2\beta V_{3,i}(x_t),$$

$$\dot{V}_{4,i}(x_t) = \int_{-h(t)}^{0} [\begin{bmatrix} x(\xi) \\ x(\xi - h(\xi)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(\xi) \\ x(\xi - h(\xi)) \end{bmatrix}$$

$$- e^{2\beta s} \begin{bmatrix} x(t+s) \\ x(t+s-h(t+s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(t+s) \\ x(t+s-h(t+s)) \end{bmatrix}]ds$$

$$- 2\beta V_{4,i}(x_t)$$

$$\leq h_M \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}$$

$$- \int_{t-h(t)}^{t} e^{2\beta(s-t)} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix} ds$$

$$- 2\beta V_{4,i}(x_t),$$

$$\dot{V}_{5,i}(x_t) = \int_{-h(t)}^{0} [\dot{x}^T(t)T_i\dot{x}(t) - \dot{x}^T(t+s)T_i\dot{x}(t+s)]ds$$

$$\leq h_M \dot{x}^T(t)T_i\dot{x}(t) - \int_{t-h(t)}^{t} \dot{x}^T(s)T_i\dot{x}(s)ds$$

$$= h_M \dot{x}^T(t)T_i\dot{x}(t) - \frac{1}{2}\int_{t-h(t)}^{t} \dot{x}^T(s)T_i\dot{x}(s)ds - \frac{1}{2}\int_{t-h(t)}^{t} \dot{x}^T(s)T_i\dot{x}(s)ds.$$

Then, the derivative of $V_i(x_t)$ along the trajectories of the state $x(t)$ is given by

$$\dot{V}_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Omega_{1,i}^{\star} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - 2\beta V_{2,i}(x_t)$$

$$- \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s)R_i x(s)ds - 2\beta V_{3,i}(x_t)$$

$$- \int_{t-h(t)}^{t} e^{2\beta(s-t)} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix} ds$$

$$-2\beta V_{4,i}(x_t) + h_M \dot{x}(t)^T T_i \dot{x}(t)$$

$$-\frac{1}{2}\int_{t-h(t)}^{t} \dot{x}^T(s)T_i\dot{x}(s)ds - \frac{1}{2}\int_{t-h(t)}^{t} \dot{x}^T(s)T_i\dot{x}(s)ds, \quad (3.16)$$

where

$$\Omega_{1,i}^{\star} = \begin{bmatrix} A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} & B_i^T P_i + h_M S_{12,i} \\ * & -(1-\mu)e^{-2\beta h_M} Q_i + h_M S_{22,i} \end{bmatrix}.$$

Since

$$\int_{-h(t)}^{0}\int_{t+s}^{t} e^{2\beta(\xi-t)}x^T(\xi)R_i x(\xi)d\xi ds \le \int_{-h(t)}^{0}\int_{t-h(t)}^{t} e^{2\beta(\xi-t)}x^T(\xi)R_i x(\xi)d\xi ds$$
$$\le h_M\int_{t-h(t)}^{t} e^{2\beta(s-t)}x^T(s)R_i x(s)ds.$$

Thus, we have

$$-\int_{t-h(t)}^{t} e^{2\beta(s-t)}x^T(s)R_i x(s)ds \le -\frac{1}{h_M}\int_{-h(t)}^{0}\int_{t+s}^{t} e^{2\beta(\xi-t)}x^T(\xi)R_i x(\xi)d\xi ds$$
$$= -\frac{1}{h_M}V_{3,i}(x_t). \tag{3.17}$$

Similarly, we have

$$-\int_{t-h(t)}^{t} e^{2\beta(s-t)}\begin{bmatrix} x(s) \\ x(s-h(s))\end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i}\end{bmatrix}\begin{bmatrix} x(s) \\ x(s-h(s))\end{bmatrix} ds$$
$$\le -\frac{1}{h_M}V_{4,i}(x_t), \tag{3.18}$$

$$-\frac{1}{2}\int_{t-h(t)}^{t}\dot{x}(s)T_i\dot{x}(s)ds \le -\frac{1}{2h_M}V_{5,i}(x_t). \tag{3.19}$$

From (3.16), (3.17), (3.18) and (3.19), we obtain

$$\dot{V}_i(x_t) \le \sum_{i=1}^{N}\lambda_i(t)\begin{bmatrix} x(t) \\ x(t-h(t))\end{bmatrix}^T \Omega_{1,i}\begin{bmatrix} x(t) \\ x(t-h(t))\end{bmatrix} - 2\beta V_{2,i}(x_t)$$
$$-(2\beta+\tfrac{1}{h_M})(V_{3,i}(x_t)+V_{4,i}(x_t)) - \tfrac{1}{2h_M}V_{5,i}(x_t)$$
$$-\frac{1}{2}\int_{t-h(t)}^{t}\dot{x}(s)T_i\dot{x}(s)ds. \tag{3.20}$$

For $i \in S_u$, we have

$$\dot{V}_i(x_t) \le \sum_{i=1}^{N}\lambda_i(t)\begin{bmatrix} x(t) \\ x(t-h(t))\end{bmatrix}^T \Omega_{1,i}\begin{bmatrix} x(t) \\ x(t-h(t))\end{bmatrix}.$$

By $A1-(i)$ and Lemma 2.4.3, there exists $\xi_i > 0$ such that

$$V_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \parallel V_i(x_{t_0}) \parallel e^{\xi_i(t-t_0)},\ t \geq t_0. \tag{3.21}$$

where $\xi_i = \dfrac{2\max\limits_i\{\lambda_M(\Omega_{1,i})\}}{\min\limits_i\{\lambda_m(P_i)\}}$. (See Appendix A.2.)

For $i \in S_s$, we have that for

$X_i = \begin{bmatrix} X_{11,i} & X_{12,i} \\ * & X_{22,i} \end{bmatrix} \geq 0$, the following holds:

$$h_M \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T X_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}$$
$$- \int_{t-h(t)}^{t} e^{2\beta(s-t)} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T X_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} ds \geq 0. \tag{3.22}$$

Using the Newton-Leibniz formula [17], we can write

$$x(t-h(t)) = x(t) - \int_{t-h(t)}^{t} \dot{x}(s)ds. \tag{3.23}$$

Then, for any appropriate dimension matrices Y_i and Z_i, we have

$$2[x^T(t)Y_i + x^T(t-h(t))Z_i][x(t) - \int_{t-h(t)}^{t} \dot{x}(s)ds - x(t-h(t))] = 0,$$

we obtain

$$2x^T(t)Y_i x(t) - 2x^T(t)Y_i \int_{t-h(t)}^{t} \dot{x}(s)ds - 2x^T(t)Y_i x(t-h(t)) + 2x^T(t-h(t))Z_i x(t)$$

$$-2x^T(t-h(t))Z_i \int_{t-h(t)}^{t} \dot{x}(s)ds - 2x^T(t-h(t))Z_i x(t-h(t)) = 0. \tag{3.24}$$

From (3.20) with (3.22) and (3.24), we have

$$\dot{V}_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Omega_{2,i} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - 2\beta V_{2,i}(x_t)$$
$$- (2\beta + \frac{1}{h_M})(V_{3,i}(x_t) + V_{4,i}(x_t)) - \frac{1}{2h_M} V_{5,i}(x_t)$$
$$- \int_{t-h(t)}^{t} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(s) \end{bmatrix}^T \Omega_{3,i} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(s) \end{bmatrix} ds. \tag{3.25}$$

By $A1-(ii)$ and Lemma 2.4.4, there exist $\zeta_i > 0$ such that

$$V_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \parallel V_i(x_{t_0}) \parallel e^{-\zeta_i(t-t_0)},\ t \geq t_0. \tag{3.26}$$

where $\zeta_i = \min\{\frac{\min\limits_i\{\lambda_m(-\Omega_{2,i})\}}{\max\limits_i\{\lambda_M(P_i)\}}, 2\beta, \frac{1}{2h_M}\}$. (See Appendix A.3.)

Let $N(t)$ denotes the number of times the system is switched on (t_0, t) such that $\lim\limits_{t\to+\infty} N(t) = +\infty$. Suppose that $\sigma(t_0) = i_0, \sigma(t_1) = i_1, ...$ and $\sigma(t) = i$.

Let $l(t)$ denotes the number of times the unstable subsystems are activated on (t_0, t) and $N(t) - l(t)$ denotes the number of times the stable subsystems are activated on (t_0, t). Suppose that $t_0 < t_1 < t_2 < ...$ and $\lim\limits_{n\to+\infty} t_n = +\infty$.

From (3.21) and (3.26), suppose that the $j\ th$ subsystem of unstable mode is actived on the interval $[t_l, t_{l+1})$,

- if the $i\ th$ subsystem of unstable mode is active on the interval $[t_{l-1}, t_l)$, then

$$V_j(x_t) \leq \psi \parallel V_i(x_{t_{l-1}}) \parallel e^{\xi_i(t_l - t_{l-1})} e^{\xi_j(t-t_l)},\ t \in [t_l, t_{l+1}).$$

- if the $i\ th$ subsystem of stable mode is active on the interval $[t_{l-1}, t_l)$, then

$$V_j(x_t) \leq \psi \parallel V_i(x_{t_{l-1}}) \parallel e^{-\zeta_i(t_l - t_{l-1})} e^{\xi_j(t-t_l)},\ t \in [t_l, t_{l+1}).$$

Suppose that the $j\ th$ subsystem of stable mode is actived on the interval $[t_l, t_{l+1})$,

- if the $i\ th$ subsystem of unstable mode is active on the interval $[t_{l-1}, t_l)$, then

$$V_j(x_t) \leq \psi \parallel V_i(x_{t_{l-1}}) \parallel e^{\xi_i(t_l - t_{l-1})} e^{-\zeta_j(t-t_l)},\ t \in [t_l, t_{l+1}).$$

- if the $i\ th$ subsystem of stable mode is active on the interval $[t_{l-1}, t_l)$, then

$$V_j(x_t) \leq \psi \parallel V_i(x_{t_{l-1}}) \parallel e^{-\zeta_i(t_l - t_{l-1})} e^{-\zeta_j(t-t_l)},\ t \in [t_l, t_{l+1}).$$

In general, we get

$$\begin{aligned} V_i(x_t) &\leq \prod_{m=1}^{l(t)} \psi e^{\xi_{i_m}(t_m - t_{m-1})} \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\zeta_{i_n} h_M} e^{-\zeta_{i_n}(t_n - t_{n-1})} \times \parallel V_{i_0}(x_{t_0}) \parallel e^{-\zeta_i(t - t_{N(t)-1})} \\ &\leq \prod_{m=1}^{l(t)} \psi e^{\lambda^+(t_m - t_{m-1})} \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\zeta_{i_n} h_M} e^{-\lambda^-(t_n - t_{n-1})} \\ &\quad \times \parallel V_{i_0}(x_{t_0}) \parallel e^{-\lambda^-(t - t_{N(t)-1})},\ t \geq t_0. \end{aligned}$$

Using (3.11), we have

$$V_i(x_t) \leq \prod_{m=1}^{l(t)} \psi \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\zeta_{i_n} h_M} \times \parallel V_{i_0}(x_{t_0}) \parallel e^{-\lambda^*(t-t_0)},\ t \geq t_0.$$

By (3.12) and (3.13), we get

$$V_i(x_t) \leq \| V_{i_0}(x_{t_0}) \| e^{-(\lambda^*-\nu)(t-t_0)}, \ t \geq t_0.$$

Thus, by (3.14), we have

$$\| x(t) \| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \| x_{t_0} \| e^{-\frac{1}{2}(\lambda^*-\nu)(t-t_0)}, \ t \geq t_0,$$

which concludes the proof of the Theorem 3.1.1. □

3.2 Robust Exponential Stability of Linear Switched System with Time-Varying Delay

In this section, we deal with the problem for robust exponential stability of the zero solution of system (3.1) without nonlinear perturbation ($f_i(t, x(t), x(t - h(t))) = 0$).

Theorem 3.2.1. *The zero solution of system (3.1) with $f_i(t, x(t), x(t - h(t))) = 0$ is robust exponentially stable if there exist positive real numbers $\varepsilon_{1i}, \varepsilon_{2i}$, positive definite matrices P_i, Q_i, R_i and $\begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix}$ such that the following assumptions hold:*

A1. (*i*) *For $i \in S_u$,*

$$\Xi_i > 0. \tag{3.27}$$

(*ii*) *For $i \in S_s$,*

$$\Xi_i < 0. \tag{3.28}$$

A2. Assume that, for any t_0 the switching law guarantees that

$$\inf_{t \geq t_0} \frac{T^-(t_0, t)}{T^+(t_0, t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*} \tag{3.29}$$

where $\lambda^* \in (0, \lambda^-)$. Furthermore, there exists $0 < \nu < \lambda^*$ such that

(*i*) *If the subsystem $i \in S_u$ is activated in time intervals $[t_{i_k - 1}, t_{i_k}), k = 1, 2, ...$, then*

$$ln\ \psi - \nu(t_{i_k} - t_{i_k - 1}) \leq 0,\ k = 1, 2, ... \tag{3.30}$$

(*ii*) *If the subsystem $j \in S_s$ is activated in time intervals $[t_{j_k - 1}, t_{j_k}), k = 1, 2, ...$, then*

$$ln\ \psi + \zeta_j h_M - \nu(t_{j_k} - t_{j_k - 1}) \leq 0,\ k = 1, 2, ... \tag{3.31}$$

Proof. Consider the following Lyapunov functional:

$V_i(x_t) = V_{1,i}(x(t)) + V_{2,i}(x_t) + V_{3,i}(x_t) + V_{4,i}(x_t)$

where $x_t \in C([-h_M, 0], \mathbb{R}^n), x_t(s) = x(t+s), s \in [-h_M, 0]$ and

$$V_{1,i}(x(t)) = x^T(t)P_i x(t),$$

$$V_{2,i}(x_t) = \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s) Q_i x(s) ds,$$

$$V_{3,i}(x_t) = \int_{-h(t)}^{0} \int_{t+s}^{t} e^{2\beta(\xi-t)} x^T(\xi) R_i x(\xi) d\xi ds,$$

$$V_{4,i}(x_t) = \int_{-h(t)}^{0} \int_{t+s}^{t} e^{2\beta(\xi-t)} \begin{bmatrix} x(\xi) \\ x(\xi - h(\xi)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(\xi) \\ x(\xi - h(\xi)) \end{bmatrix} d\xi ds.$$

It is easy to verify that

$$\alpha_1 \parallel x(t) \parallel^2 \leq V_i(x_t) \leq \alpha_3 \parallel x_t \parallel^2, \ t \geq 0. \tag{3.32}$$

Similar to (3.15), we have

$$V_i(x_t) \leq \psi V_j(x_t), \ \forall i, j \in S. \tag{3.33}$$

Taking derivative of $V_{1,i}(x(t))$ along trajectories of any subsystem ith we have

$$\begin{aligned}\dot{V}_{1,i}(x(t)) &= \dot{x}^T(t)P_i x(t) + x^T(t)P_i \dot{x}(t) \\ &= \sum_{i=1}^{N} \lambda_i(t)[x^T(t)A_i^T P_i x(t) + x^T(t)\Delta A_i^T(t) P_i x(t) + x^T(t-h(t))B_i^T P_i x(t) \\ &\quad + x^T(t-h(t))\Delta B_i^T(t) P_i x(t) + x^T(t)P_i A_i x(t) + x^T(t) P_i \Delta A_i(t) x(t) \\ &\quad + x^T(t) P_i B_i x(t-h(t)) + x^T(t) P_i \Delta B_i(t) x(t-h(t))].\end{aligned}$$

Applying Lemma 2.4.2 and from (3.2) and (3.3), we get

$$2x^T(t)\Delta A_i^T(t)P_i x(t) \leq \varepsilon_{1i}^{-1} x^T(t) H_{1i}^T H_{1i} x(t) + \varepsilon_{1i} x^T(t) P_i E_{1i}^T E_{1i} P_i x(t),$$

$$2x^T(t-h(t))\Delta B_i^T(t)P_i x(t) \leq \varepsilon_{2i}^{-1} x^T(t-h(t)) H_{2i}^T H_{2i} x(t-h(t)) + \varepsilon_{2i} x^T(t) P_i E_{2i}^T E_{2i} P_i x(t).$$

Next, by taking derivative of $V_{2,i}(x_t), V_{3,i}(x_t)$ and $V_{4,i}(x_t)$, respectively, along the system trajectories yields

$$\dot{V}_{2,i}(x_t) \leq x^T(t) Q_i x(t) - (1-\mu) e^{-2\beta h(t)} x^T(t-h(t)) Q_i x(t-h(t)) - 2\beta V_{2,i}(x_t),$$

$$\dot{V}_{3,i}(x_t) \leq h_M x^T(t) R_i x(t) - \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s) R_i x(s) ds - 2\beta V_{3,i}(x_t),$$

$$\begin{aligned}\dot{V}_{4,i}(x_t) \leq h_M &\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} \\ &- \int_{t-h(t)}^{t} e^{2\beta(s-t)} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix} ds \\ &- 2\beta V_{4,i}(x_t).\end{aligned}$$

Then, the derivative of $V_i(x_t)$ along the trajectories of the state $x(t)$ is given by

$$\dot{V}_i(x_t) \leq \sum_{i=1}^{N}\lambda_i(t)\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Xi_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - 2\beta V_{2,i}(x_t)$$
$$- \int_{t-h(t)}^{t} e^{2\beta(s-t)}x^T(s)R_i x(s)ds - 2\beta V_{3,i}(x_t)$$

$$- \int_{t-h(t)}^{t} e^{2\beta(s-t)}\begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix}\begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix} ds - 2\beta V_{4,i}(x_t). \quad (3.34)$$

For $i \in S_u$, we have

$$\dot{V}_i(x_t) \leq \sum_{i=1}^{N}\lambda_i(t)\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Xi_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}.$$

Similar to theorem 3.1.1, we get

$$V_i(x_t) \leq \sum_{i=1}^{N}\lambda_i(t) \parallel V_i(x_{t_0}) \parallel e^{\xi_i(t-t_0)},\ t \geq t_0, \quad (3.35)$$

where $\xi_i = \frac{2\max\limits_i\{\lambda_M(\Xi_i)\}}{\min\limits_i\{\lambda_m(P_i)\}}$.

For $i \in S_s$, from (3.17), (3.18) and (3.34), we have

$$\dot{V}_i(x_t) \leq \sum_{i=1}^{N}\lambda_i(t)\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Xi_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - 2\beta V_{2,i}(x_t)$$

$$-(2\beta + \frac{1}{h_M})(V_{3,i}(x_t) + V_{4,i}(x_t)) \quad (3.36)$$

Similar to theorem 3.1.1, we get

$$V_i(x_t) \leq \sum_{i=1}^{N}\lambda_i(t) \parallel V_i(x_{t_0}) \parallel e^{-\zeta_i(t-t_0)},\ t \geq t_0. \quad (3.37)$$

where $\zeta_i = \min\{\frac{\min\limits_i\{\lambda_m(-\Xi_i)\}}{\max\limits_i\{\lambda_M(P_i)\}}, 2\beta\}$.

In general, with the same argument as in the proof of theorem 3.1.1, we get

$$V_i(x_t) \leq \prod_{m=1}^{l(t)} \psi e^{\lambda^+(t_m - t_{m-1})} \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\zeta_{i_n}h_M} e^{-\lambda^-(t_n - t_{n-1})}$$
$$\times \parallel V_{i_0}(x_{t_0}) \parallel e^{-\lambda^-(t - t_{N(t)-1})},\ t \geq t_0.$$

Using (3.29), we have

$$V_i(x_t) \leq \prod_{m=1}^{l(t)} \psi \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\zeta_{i_n} h_M} \times \parallel V_{i_0}(x_{t_0}) \parallel e^{-\lambda^*(t-t_0)},\ t \geq t_0.$$

By (3.30) and (3.31), we get

$$V_i(x_t) \leq \parallel V_{i_0}(x_{t_0}) \parallel e^{-(\lambda^*-\nu)(t-t_0)},\ t \geq t_0.$$

Thus, by (3.32), we have

$$\parallel x(t) \parallel \leq \sqrt{\tfrac{\alpha_3}{\alpha_1}} \parallel x_{t_0} \parallel e^{-\frac{1}{2}(\lambda^*-\nu)(t-t_0)},\ t \geq t_0,$$

which concludes the proof of the Theorem 3.2.1. □

3.3 Robust Exponential Stability of Nonlinear Switched System with Time-Varying Delay

In this section, we deal with the problem for robust exponential stability of the zero solution of system (3.1).

Theorem 3.3.1. *The zero solution of system (3.1) is robust exponentially stable if there exist positive real numbers $\varepsilon_{3i}, \varepsilon_{4i}, \varepsilon_{5i}$, positive definite matrices P_i, Q_i, R_i and $\begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix}$ such that the following assumptions hold:*

A1. (i) *For $i \in S_u$,*

$$\Theta_i > 0. \tag{3.38}$$

(ii) *For $i \in S_s$,*

$$\Theta_i < 0. \tag{3.39}$$

A2. Assume that, for any t_0 the switching law guarantees that

$$\inf_{t \geq t_0} \frac{T^-(t_0,t)}{T^+(t_0,t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*} \tag{3.40}$$

where $\lambda^* \in (0, \lambda^-)$. Furthermore, there exists $0 < \nu < \lambda^*$ such that

(i) *If the subsystem $i \in S_u$ is activated in time intervals $[t_{i_k - 1}, t_{i_k}), k = 1, 2, ...,$ then*

$$ln\ \psi - \nu(t_{i_k} - t_{i_k - 1}) \leq 0,\ \ k = 1, 2, ... \tag{3.41}$$

(ii) *If the subsystem $j \in S_s$ is activated in time intervals $[t_{j_k - 1}, t_{j_k}), k = 1, 2, ...,$ then*

$$ln\ \psi + \zeta_j h_M - \nu(t_{j_k} - t_{j_k - 1}) \leq 0,\ \ k = 1, 2, ... \tag{3.42}$$

Proof. Consider the following Lyapunov functional:

$V_i(x_t) = V_{1,i}(x(t)) + V_{2,i}(x_t) + V_{3,i}(x_t) + V_{4,i}(x_t)$

where $x_t \in C([-h_M, 0], \mathbb{R}^n), x_t(s) = x(t+s), s \in [-h_M, 0]$ and

$$V_{1,i}(x(t)) = x^T(t)P_ix(t),$$
$$V_{2,i}(x_t) = \int_{t-h(t)}^{t} e^{2\beta(s-t)}x^T(s)Q_ix(s)ds,$$
$$V_{3,i}(x_t) = \int_{-h(t)}^{0}\int_{t+s}^{t} e^{2\beta(\xi-t)}x^T(\xi)R_ix(\xi)d\xi ds,$$
$$V_{4,i}(x_t) = \int_{-h(t)}^{0}\int_{t+s}^{t} e^{2\beta(\xi-t)} \begin{bmatrix} x(\xi) \\ x(\xi-h(\xi)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(\xi) \\ x(\xi-h(\xi)) \end{bmatrix} d\xi ds.$$

It is easy to verify that

$$\alpha_1 \parallel x(t) \parallel^2 \leq V_i(x_t) \leq \alpha_3 \parallel x_t \parallel^2,\ t \geq 0. \tag{3.43}$$

Similar to (3.15), we have

$$V_i(x_t) \leq \psi V_j(x_t),\ \forall i, j \in S. \tag{3.44}$$

Taking derivative of $V_{1,i}(x(t))$ along trajectories of any subsystem ith we have

$$\begin{aligned}\dot{V}_{1,i}(x(t)) &= \dot{x}^T(t)P_ix(t) + x^T(t)P_i\dot{x}(t)\\
&= \sum_{i=1}^{N}\lambda_i(t)[x^T(t)A_i^TP_ix(t) + x^T(t)\Delta A_i^T(t)P_ix(t) + x^T(t-h(t))B_i^TP_ix(t)\\
&\quad + x^T(t-h(t))\Delta B_i^T(t)P_ix(t) + f_i^T(t, x(t), x(t-h(t)))P_ix(t) + x^T(t)P_iA_ix(t)\\
&\quad + x^T(t)P_i\Delta A_i(t)x(t) + x^T(t)P_iB_ix(t-h(t)) + x^T(t)P_i\Delta B_i(t)x(t-h(t))\\
&\quad + x^T(t)P_if_i(t, x(t), x(t-h(t)))].\end{aligned}$$

from lemma 2.4.2, we have

$$\begin{aligned}2f_i^T(t, x(t), x(t-h(t)))P_ix(t) &\leq f_i^T(t, x(t), x(t-h(t)))W_i^{-1}f_i(t, x(t), x(t-h(t)))\\
&\quad + x^T(t)P_iW_iP_ix(t).\end{aligned}$$

By choosing $W_i = \varepsilon_{3i}I_i$ and from (3.4), we have

$$\begin{aligned}2f_i^T(t, x(t), x(t-h(t)))P_ix(t) &\leq \varepsilon_{3i}^{-1}f_i^T(t, x(t), x(t-h(t)))f_i(t, x(t), x(t-h(t)))\\
&\quad + \varepsilon_{3i}x^T(t)P_iP_ix(t)\\
&\leq \varepsilon_{3i}^{-1}[\gamma_ix^T(t)x(t) + \delta_ix^T(t-h(t))x(t-h(t))]\\
&\quad + \varepsilon_{3i}x^T(t)P_iP_ix(t).\end{aligned}$$

Applying Lemma 2.4.2 and from (3.2) and (3.3), we get

$$2x^T(t)\Delta A_i^T(t)P_ix(t) \leq \varepsilon_{4i}^{-1}x^T(t)H_{4i}^TH_{4i}x(t) + \varepsilon_{4i}x^T(t)P_iE_{4i}^TE_{4i}P_ix(t),$$

$$2x^T(t-h(t))\Delta B_i^T(t)P_ix(t) \leq \varepsilon_{5i}^{-1}x^T(t-h(t))H_{5i}^TH_{5i}x(t-h(t)) + \varepsilon_{5i}x^T(t)P_iE_{5i}^TE_{5i}P_ix(t).$$

Next, by taking derivative of $V_{2,i}(x_t), V_{3,i}(x_t)$ and $V_{4,i}(x_t)$, respectively, along the system trajectories yields

$$\dot{V}_{2,i}(x_t) \leq x^T(t)Q_ix(t) - (1-\mu)e^{-2\beta h(t)}x^T(t-h(t))Q_ix(t-h(t)) - 2\beta V_{2,i}(x_t),$$

$$\dot{V}_{3,i}(x_t) \leq h_M x^T(t)R_ix(t) - \int_{t-h(t)}^{t} e^{2\beta(s-t)}x^T(s)R_ix(s)ds - 2\beta V_{3,i}(x_t),$$

$$\dot{V}_{4,i}(x_t) \leq h_M \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}$$
$$- \int_{t-h(t)}^{t} e^{2\beta(s-t)} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix} ds$$
$$- 2\beta V_{4,i}(x_t).$$

Then, the derivative of $V_i(x_t)$ along the trajectories of the state $x(t)$ is given by

$$\dot{V}_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Theta_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - 2\beta V_{2,i}(x_t)$$
$$- \int_{t-h(t)}^{t} e^{2\beta(s-t)}x^T(s)R_ix(s)ds - 2\beta V_{3,i}(x_t)$$
$$- \int_{t-h(t)}^{t} e^{2\beta(s-t)} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix} ds$$
$$-2\beta V_{4,i}(x_t). \tag{3.45}$$

For $i \in S_u$, we have

$$\dot{V}_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Theta_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}.$$

Similar to theorem 3.1.1, we get

$$V_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \parallel V_i(x_{t_0}) \parallel e^{\xi_i(t-t_0)}, \; t \geq t_0. \tag{3.46}$$

where $\xi_i = \dfrac{2\max_i\{\lambda_M(\Theta_i)\}}{\min_i\{\lambda_m(P_i)\}}$.

For $i \in S_s$, from (3.17), (3.18) and (3.45), we have

$$\dot{V}_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Theta_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - 2\beta V_{2,i}(x_t)$$
$$-(2\beta + \frac{1}{h_M})(V_{3,i}(x_t) + V_{4,i}(x_t)). \tag{3.47}$$

Similar to theorem 3.1.1, we get

$$V_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \parallel V_i(x_{t_0}) \parallel e^{-\zeta_i(t-t_0)},\ t \geq t_0. \tag{3.48}$$

where $\zeta_i = \min\{\dfrac{\min\limits_i\{\lambda_m(-\Theta_i)\}}{\max\limits_i\{\lambda_M(P_i)\}}, 2\beta\}$.

In general, with the same argument as in the proof of theorem 3.1.1, we get

$$V_i(x_t) \leq \prod_{m=1}^{l(t)} \psi e^{\lambda^+(t_m-t_{m-1})} \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\zeta_{i_n} h_M} e^{-\lambda^-(t_n-t_{n-1})} \times \parallel V_{i_0}(x_{t_0}) \parallel e^{-\lambda^-(t-t_{N(t)-1})},\ t \geq t_0.$$

Using (3.40), we have

$$V_i(x_t) \leq \prod_{m=1}^{l(t)} \psi \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\zeta_{i_n} h_M} \times \parallel V_{i_0}(x_{t_0}) \parallel e^{-\lambda^*(t-t_0)},\ t \geq t_0.$$

By (3.41) and (3.42), we get

$$V_i(x_t) \leq \parallel V_{i_0}(x_{t_0}) \parallel e^{-(\lambda^*-\nu)(t-t_0)},\ t \geq t_0.$$

Thus, by (3.43), we have

$$\parallel x(t) \parallel \leq \sqrt{\tfrac{\alpha_3}{\alpha_1}} \parallel x_{t_0} \parallel e^{-\frac{1}{2}(\lambda^*-\nu)(t-t_0)},\ t \geq t_0,$$

which concludes the proof of the Theorem 3.3.1. □

Remark 3.1 The matrix inequalities $(3.27), (3.28), (3.38)$ and (3.39) are not linear matrix inequalities, we can not solve the solution of these inequalities by using Matlab's LMI control toolbox, so we gives an equivalent version of the inequalities $(3.27), (3.28), (3.38)$ and (3.39) as in the following propositions.

Proposition 3.1 The inequality (3.27) in theorem 3.2.1 is equivalent to the following LMI:

$$\Lambda_i = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} \\ * & \Lambda_{22} & 0 & 0 \\ * & * & \Lambda_{33} & 0 \\ * & * & * & \Lambda_{44} \end{bmatrix} > 0 \tag{3.49}$$

where

$\Lambda_{11} = A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} + \varepsilon_{1i}^{-1} H_{1i}^T H_{1i},$

$\Lambda_{12} = B_i^T P_i + h_M S_{12,i},$

$\Lambda_{13} = P_i E_{1i}^T,$

$\Lambda_{14} = P_i E_{2i}^T,$

$\Lambda_{22} = -(1-\mu)e^{-2\beta h_M} Q_i + h_M S_{22,i} + \varepsilon_{2i}^{-1} H_{2i}^T H_{2i},$

$\Lambda_{33} = \frac{1}{\varepsilon_{1i}} I,$

$\Lambda_{44} = \frac{1}{\varepsilon_{2i}} I.$

Proof. From Lemma 2.4.5, we obtain $\Lambda_i > 0$ equivalent to

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ * & \Lambda_{22} & 0 \\ * & * & \Lambda_{33} \end{bmatrix} + \begin{bmatrix} \Lambda_{14} \\ 0 \\ 0 \end{bmatrix} \Lambda_{44}^{-1} \begin{bmatrix} \Lambda_{14}^T & 0 & 0 \end{bmatrix} > 0,$$

we obtain

$$\begin{bmatrix} \Lambda_{11} + \Lambda_{14}\Lambda_{44}^{-1}\Lambda_{14}^T & \Lambda_{12} & \Lambda_{13} \\ * & \Lambda_{22} & 0 \\ * & * & \Lambda_{33} \end{bmatrix} > 0.$$

We will use this argument again, we obtain that the inequality (3.27) is equivalent to $\Lambda_i > 0$.

Proposition 3.2 The inequality (3.28) in theorem 3.2.1 is equivalent to the following LMI:

$$\Lambda_i^\star = \begin{bmatrix} \Lambda_{11}^\star & \Lambda_{12}^\star & \Lambda_{13}^\star & \Lambda_{14}^\star \\ * & \Lambda_{22}^\star & 0 & 0 \\ * & * & \Lambda_{33}^\star & 0 \\ * & * & * & \Lambda_{44}^\star \end{bmatrix} < 0 \tag{3.50}$$

where

$\Lambda_{11}^\star = A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} + \varepsilon_{1i}^{-1} H_{1i}^T H_{1i}$,

$\Lambda_{12}^\star = B_i^T P_i + h_M S_{12,i}$,

$\Lambda_{13}^\star = P_i E_{1i}^T$,

$\Lambda_{14}^\star = P_i E_{2i}^T$,

$\Lambda_{22}^\star = -(1-\mu)e^{-2\beta h_M} Q_i + h_M S_{22,i} + \varepsilon_{2i}^{-1} H_{2i}^T H_{2i}$,

$\Lambda_{33}^\star = -\frac{1}{\varepsilon_{1i}} I$,

$\Lambda_{44}^\star = -\frac{1}{\varepsilon_{2i}} I$.

Proof. From Lemma 2.4.5, we obtain $\Lambda_i^\star < 0$ equivalent to

$$\begin{bmatrix} \Lambda_{11}^\star & \Lambda_{12}^\star & \Lambda_{13}^\star \\ * & \Lambda_{22}^\star & 0 \\ * & * & \Lambda_{33}^\star \end{bmatrix} + \begin{bmatrix} \Lambda_{14}^\star \\ 0 \\ 0 \end{bmatrix} \Lambda_{44}^{\star -1} \begin{bmatrix} \Lambda_{14}^{\star T} & 0 & 0 \end{bmatrix} < 0,$$

we obtain

$$\begin{bmatrix} \Lambda_{11}^\star + \Lambda_{14}^\star \Lambda_{44}^{\star -1} \Lambda_{14}^{\star T} & \Lambda_{12}^\star & \Lambda_{13}^\star \\ * & \Lambda_{22}^\star & 0 \\ * & * & \Lambda_{33}^\star \end{bmatrix} < 0.$$

We will use this argument again, we obtain that the inequality (3.28) is equivalent to $\Lambda_i^\star < 0$.

Proposition 3.3 The inequality (3.38) in theorem 3.3.1 is equivalent to the following inequality:

$$\Theta_i = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & \Theta_{15} \\ * & \Theta_{22} & 0 & 0 & 0 \\ * & * & \Theta_{33} & 0 & 0 \\ * & * & * & \Theta_{44} & 0 \\ * & * & * & * & \Theta_{55} \end{bmatrix} > 0 \qquad (3.51)$$

where

$\Theta_{11} = A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} + \varepsilon_{4i}^{-1}\gamma_i I + \varepsilon_{4i}^{-1} H_{4i}^T H_{4i}$,

$\Theta_{12} = B_i^T P_i + h_M S_{12,i}$,

$\Theta_{13} = P_i$,

$\Theta_{14} = P_i E_{4i}^T$,

$\Theta_{15} = P_i E_{5i}^T$,

$\Theta_{22} = -(1-\mu)e^{-2\beta h_M} Q_i + h_M S_{22,i} + \varepsilon_{3i}^{-1}\delta_i I + \varepsilon_{5i}^{-1} H_{5i}^T H_{5i}$.

$\Theta_{33} = \frac{1}{\varepsilon_{3i}} I$,

$\Theta_{44} = \frac{1}{\varepsilon_{4i}} I$.

$\Theta_{55} = \frac{1}{\varepsilon_{5i}} I$.

Proof. From Lemma 2.4.5, we obtain $\Theta_i > 0$ equivalent to

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} \\ * & \Theta_{22} & 0 & 0 \\ * & * & \Theta_{33} & 0 \\ * & * & * & \Theta_{44} \end{bmatrix} + \begin{bmatrix} \Theta_{15} \\ 0 \\ 0 \\ 0 \end{bmatrix} \Theta_{55}^{-1} \begin{bmatrix} \Theta_{15}^T & 0 & 0 & 0 \end{bmatrix} > 0,$$

we obtain

$$\begin{bmatrix} \Theta_{11} + \Theta_{15}\Theta_{55}^{-1}\Theta_{15}^T & \Theta_{12} & \Theta_{13} & \Theta_{14} \\ * & \Theta_{22} & 0 & 0 \\ * & * & \Theta_{33} & 0 \\ * & * & * & \Theta_{44} \end{bmatrix} > 0.$$

We will use this argument twice, we obtain that the inequality (3.38) is equivalent to $\Theta_i > 0$.

Proposition 3.4 The inequality (3.39) in theorem 3.3.1 is equivalent to the following LMI:

$$\Theta_i^\star = \begin{bmatrix} \Theta_{11}^\star & \Theta_{12}^\star & \Theta_{13}^\star & \Theta_{14}^\star & \Theta_{15}^\star \\ * & \Theta_{22}^\star & 0 & 0 & 0 \\ * & * & \Theta_{33}^\star & 0 & 0 \\ * & * & * & \Theta_{44}^\star & 0 \\ * & * & * & * & \Theta_{55}^\star \end{bmatrix} < 0 \tag{3.52}$$

where

$\Theta_{11}^\star = A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} + \varepsilon_{3i}^{-1}\gamma_i I + \varepsilon_{4i}^{-1} H_{4i}^T H_{4i}$,

$\Theta_{12}^\star = B_i^T P_i + h_M S_{12,i}$,

$\Theta_{13}^\star = P_i$,

$\Theta_{14}^\star = P_i E_{4i}^T$,

$\Theta_{15}^\star = P_i E_{5i}^T$,

$\Theta_{22}^\star = -(1-\mu)e^{-2\beta h_M} Q_i + h_M S_{22,i} + \varepsilon_{3i}^{-1}\delta_i I + \varepsilon_{5i}^{-1} H_{5i}^T H_{5i}$.

$\Theta_{33}^\star = -\frac{1}{\varepsilon_{3i}} I$,

$\Theta_{44}^\star = -\frac{1}{\varepsilon_{4i}} I$.

$\Theta_{55}^\star = -\frac{1}{\varepsilon_{5i}} I$.

Proof. From Lemma 2.4.5, we obtain $\Theta_i < 0$ equivalent to

$$\begin{bmatrix} \Theta_{11}^\star & \Theta_{12}^\star & \Theta_{13}^\star & \Theta_{14}^\star \\ * & \Theta_{22}^\star & 0 & 0 \\ * & * & \Theta_{33}^\star & 0 \\ * & * & * & \Theta_{44}^\star \end{bmatrix} + \begin{bmatrix} \Theta_{15}^\star \\ 0 \\ 0 \\ 0 \end{bmatrix} \Theta_{55}^{\star -1} \begin{bmatrix} \Theta_{15}^{\star T} & 0 & 0 & 0 \end{bmatrix} < 0,$$

we obtain

$$\begin{bmatrix} \Theta_{11}^\star + \Theta_{15}^\star \Theta_{55}^{\star -1} \Theta_{15}^{\star T} & \Theta_{12}^\star & \Theta_{13}^\star & \Theta_{14}^\star \\ * & \Theta_{22}^\star & 0 & 0 \\ * & * & \Theta_{33}^\star & 0 \\ * & * & * & \Theta_{44}^\star \end{bmatrix} < 0.$$

We will use this argument twice, we obtain that the inequality (3.39) is equivalent to $\Theta_i^\star < 0$.

Example 3.1.1 Consider the linear switched system (3.1) without matric uncertainties with time-varying delay $h(t) = 0.51sin^2t$. Let $N = 2$, $S_u = \{1\}, S_s = \{2\}$ where

$$A_1 = \begin{bmatrix} 0.0024 & 0.00001 \\ 0.000005 & 0.0023 \end{bmatrix}, B_1 = \begin{bmatrix} 0.0022 & 0.00015 \\ 0.00012 & -0.0420 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.2150 & 0.0110 \\ 0.0120 & 0.0001 \end{bmatrix}, B_2 = \begin{bmatrix} -0.0240 & 0.0001 \\ 0.0001 & -0.0250 \end{bmatrix}.$$

We have $h_M = 0.51,\ \mu = 1.02, \lambda(A_1+B_1) = 0.0046, -0.0399, \lambda(A_2) = -0.2156, 0.0007$. Let $\beta = 0.5$.

Since one of the eigenvalue of $A_1 + B_1$ is negative and one of eigenvalue of A_2 is positive, we can't use results in [1] to calculate stability of switched system (3.1).

By using the Matlab LMI toolbox, we have the solution matrices of (3.9) for unstable subsystem and (3.10) for stable subsystem as the following:

For unstable subsystem, we get

$$P_1 = \begin{bmatrix} 41.6819 & 0.0001 \\ 0.0001 & 41.5691 \end{bmatrix}, Q_1 = \begin{bmatrix} 24.7813 & -0.0002 \\ -0.0002 & 24.7848 \end{bmatrix}, R_1 = \begin{bmatrix} 33.1027 & -0.0001 \\ -0.0001 & 33.1044 \end{bmatrix},$$

$$S_{11,1} = \begin{bmatrix} 33.1027 & -0.0001 \\ -0.0001 & 33.1044 \end{bmatrix}, S_{12,1} = \begin{bmatrix} -0.0372 & -0.0023 \\ -0.0023 & 0.7075 \end{bmatrix}, S_{22,1} = \begin{bmatrix} 50.0412 & 0.0001 \\ 0.0001 & 50.0115 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 41.7637 & -0.0001 \\ -0.0001 & 41.7920 \end{bmatrix}.$$

For stable subsystem, we get

$$P_2 = \begin{bmatrix} 71.8776 & 2.3932 \\ 2.3932 & 110.8889 \end{bmatrix}, Q_2 = \begin{bmatrix} 7.2590 & -0.3265 \\ -0.3265 & 0.8745 \end{bmatrix}, R_2 = \begin{bmatrix} 10.4001 & -0.4667 \\ -0.4667 & 1.2806 \end{bmatrix},$$

$$S_{11,2} = \begin{bmatrix} 12.7990 & -0.4854 \\ -0.4854 & 3.5031 \end{bmatrix}, S_{12,2} = \begin{bmatrix} -3.1787 & 0.0240 \\ 0.0240 & -2.8307 \end{bmatrix}, S_{22,2} = \begin{bmatrix} 4.6346 & -0.0289 \\ -0.0289 & 4.0835 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 16.9964 & 0.0394 \\ 0.0394 & 17.7152 \end{bmatrix}, X_{11,2} = \begin{bmatrix} 17.2639 & -0.1536 \\ -0.1536 & 14.2310 \end{bmatrix}, X_{12,2} = \begin{bmatrix} -9.6485 & -0.1466 \\ -0.1466 & -12.5573 \end{bmatrix},$$

$$X_{22,2} = \begin{bmatrix} 16.9716 & -0.1635 \\ -0.1635 & 13.8095 \end{bmatrix}, Y_2 = \begin{bmatrix} -3.4666 & -0.1525 \\ -0.1525 & -6.3485 \end{bmatrix}, Z_2 = \begin{bmatrix} 6.8776 & -0.0574 \\ -0.0574 & 5.7924 \end{bmatrix}.$$

By straight forward calculation, the growth rate is $\lambda^+ = \xi = 2.8291$, the decay rate is $\lambda^- = \zeta = 0.0063$, $\lambda(\Omega_{1,1}) = 25.8187, 25.8188, 58.7463, 58.8011$, $\lambda(\Omega_{2,2}) = -10.1108, -3.7678, -2.0403, -0.7032$ and $\lambda(\Omega_{3,2}) = 1.4217, 4.2448, 5.4006, 9.1514, 29.3526, 30.0607$. Taking $\lambda^* = 0.0001$ and $\nu = 0.00001$.

Thus, from inequality (3.11), we have $T^- \geq 456.3226\, T^+$. Given $T^+ = 0.1$ then $T^- \geq 45.63226$.

We choose the following switching rule:

(i) for $t \in [0, 0.1) \cup [50, 50.1) \cup [100, 100.1) \cup [150, 150.1) \cup ...$, system $i = 1$ is activated.

(ii) for $t \in [0.1, 50) \cup [50.1, 100) \cup [100.1, 150) \cup [150.1, 200) \cup ...$, system $i = 2$ is activated.

Then, by theorem 3.1.1, the switching system (3.1) is exponentially stable. Moreover, the solution $x(t)$ of the system satisfies

$$\| x(t) \| \leq 11.8915 e^{-0.000045t}, \ t \in [0, \infty).$$

The trajectories of solution of the system switching between the systems $i = 1$ and $i = 2$ are shown in Figure 3.1, Figure 3.2 and Figure 3.3, respectively.

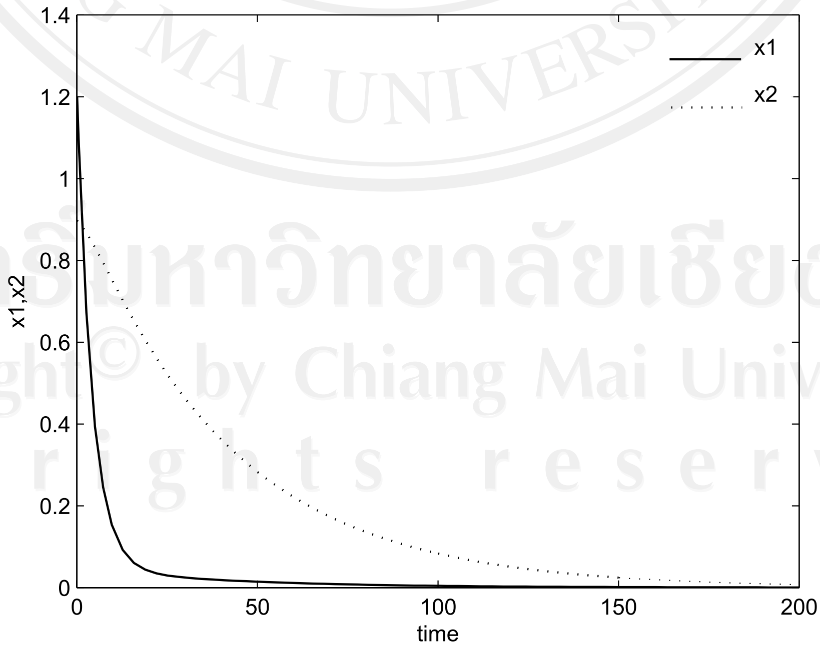

Figure 3.1: The trajectories of solution of the linear switched system.

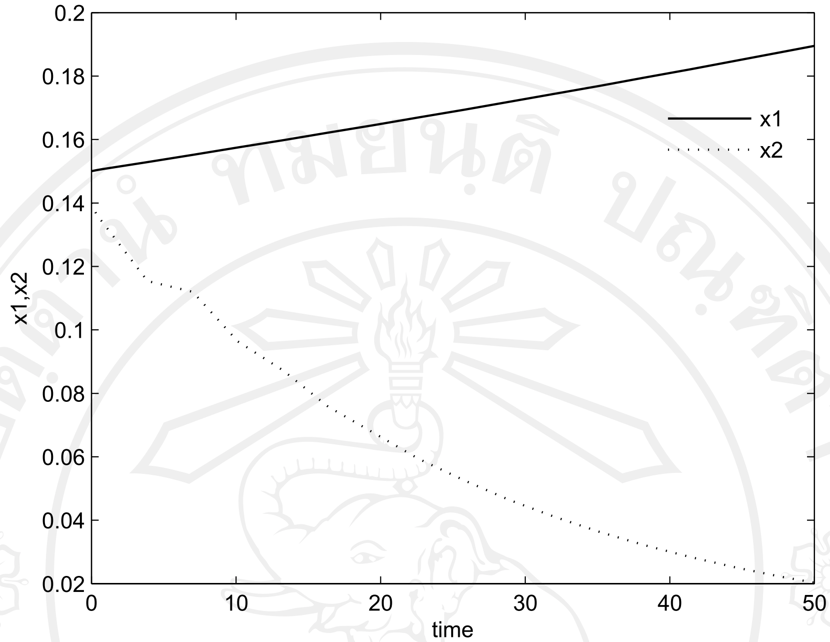

Figure 3.2: The trajectories of solution of system $i = 1$.

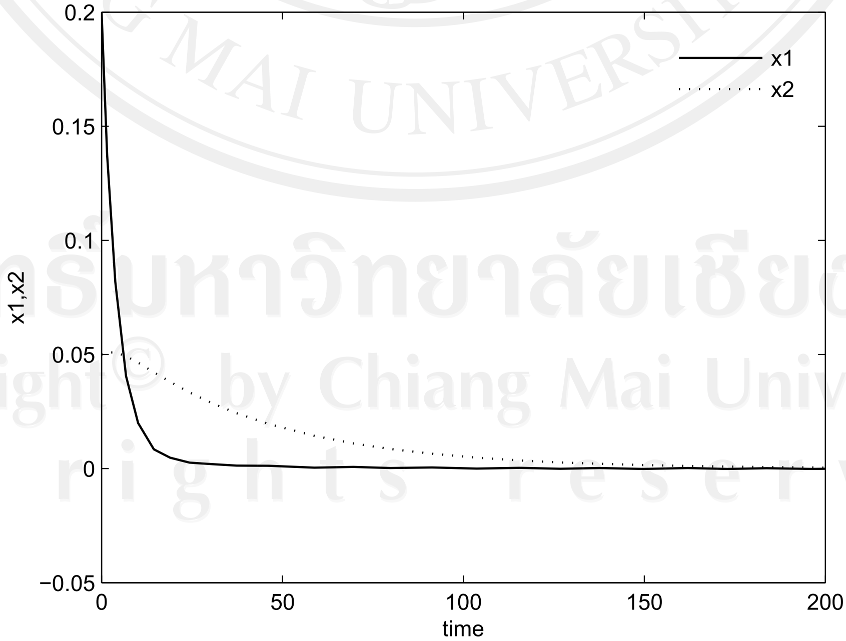

Figure 3.3: The trajectories of solution of system $i = 2$.

Example 3.1.2 Consider the nonlinear switched system (3.1) with time-varying delay $h(t) = 0.25sin^2t$. Let $N = 2$, $S_u = \{1\}, S_s = \{2\}$ where

$$A_1 = \begin{bmatrix} 0.1130 & 0.00013 \\ 0.00015 & -0.0033 \end{bmatrix}, B_1 = \begin{bmatrix} 0.0002 & 0.0012 \\ 0.0014 & -0.5002 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -5.5200 & 1.0002 \\ 1.0003 & -6.5500 \end{bmatrix}, B_2 = \begin{bmatrix} 0.0245 & 0.0001 \\ 0.0001 & 0.0237 \end{bmatrix},$$

$$E_{1i} = E_{2i} = \begin{bmatrix} 0.2000 & 0.0000 \\ 0.0000 & 0.2000 \end{bmatrix}, H_{1i} = H_{2i} = \begin{bmatrix} 0.1000 & 0.0000 \\ 0.0000 & 0.1000 \end{bmatrix}, i = 1, 2,$$

$$F_{1i} = F_{2i} = \begin{bmatrix} sin\ t & 0 \\ 0 & sin\ t \end{bmatrix}, i = 1, 2,$$

$$f_1(t, x(t), x(t-h(t))) = \begin{bmatrix} 0.1x_1(t)sin(x_1(t)) \\ 0.1x_2(t-h(t))cos(x_2(t)) \end{bmatrix},$$

$$f_2(t, x(t), x(t-h(t))) = \begin{bmatrix} 0.5x_1(t)sin(x_1(t)) \\ 0.5x_2(t-h(t))cos(x_2(t)) \end{bmatrix}.$$

From

$$\begin{aligned} \| f_1(t, x(t), x(t-h(t))) \|^2 &= [0.1x_1(t)sin(x_1(t))]^2 + [0.1x_2(t-h(t))cos(x_2(t))]^2 \\ &\leq 0.01x_1^2(t) + 0.01x_2^2(t-h(t)) \\ &\leq 0.01 \| x(t) \|^2 + 0.01 \| x(t-h(t)) \|^2 \\ &\leq 0.01[\| x(t) \| + \| x(t-h(t)) \|]^2, \end{aligned}$$

we obtain

$$\| f_1(t, x(t), x(t-h(t))) \| \leq 0.1 \| x(t) \| + 0.1 \| x(t-h(t)) \| .$$

From

$$\begin{aligned} \| f_2(t, x(t), x(t-h(t))) \|^2 &= [0.5x_1(t)sin(x_1(t))]^2 + [0.5x_2(t-h(t))cos(x_2(t))]^2 \\ &\leq 0.25x_1^2(t) + 0.25x_2^2(t-h(t)) \\ &\leq 0.25 \| x(t) \|^2 + 0.25 \| x(t-h(t)) \|^2 \\ &\leq 0.25[\| x(t) \| + \| x(t-h(t)) \|]^2, \end{aligned}$$

we obtain

$$\| f_2(t, x(t), x(t-h(t))) \| \leq 0.5 \| x(t) \| + 0.5 \| x(t-h(t)) \| .$$

We have $h_M = 0.25, \gamma_1 = 0.1, \delta_1 = 0.1, \gamma_2 = 0.5, \delta_2 = 0.5, \lambda(A_1) = 0.11300016,$ -0.00330016. Let $\beta = 0.5, \mu = 0.5$.

Since one of the eigenvalue of A_1 is negative, we can't use results in [1] to calculate stability of switched system (3.1).

By using the Matlab LMI toolbox, we have the solution matrices of (3.51) for unstable subsystem and (3.52) for stable subsystem as the following:

For unstable subsystem, we get

$\varepsilon_{31} = 0.8901,\ \varepsilon_{41} = 0.8901,\ \varepsilon_{51} = 0.8901,$

$$P_1 = \begin{bmatrix} 0.2745 & -0.0000 \\ -0.0000 & 0.2818 \end{bmatrix}, Q_1 = \begin{bmatrix} 0.4818 & -0.0000 \\ -0.0000 & 0.5097 \end{bmatrix}, R_1 = \begin{bmatrix} 0.8649 & -0.0000 \\ -0.0000 & 0.8729 \end{bmatrix},$$

$$S_{11,1} = \begin{bmatrix} 0.8649 & -0.0000 \\ -0.0000 & 0.8729 \end{bmatrix}, S_{12,1} = 10^{-4} \times \begin{bmatrix} -0.1291 & -0.8517 \\ -0.8517 & 0.1326 \end{bmatrix},$$

$$S_{22,1} = \begin{bmatrix} 1.0877 & -0.0000 \\ -0.0000 & 1.0902 \end{bmatrix}.$$

For stable subsystem, we get

$\varepsilon_{32} = 2.0180,\ \varepsilon_{42} = 2.0180,\ \varepsilon_{52} = 2.0180,$

$$P_2 = \begin{bmatrix} 0.2741 & 0.0407 \\ 0.0407 & 0.2323 \end{bmatrix}, Q_2 = \begin{bmatrix} 1.3330 & -0.0069 \\ -0.0069 & 1.3330 \end{bmatrix}, R_2 = \begin{bmatrix} 1.0210 & -0.0002 \\ -0.0002 & 1.0210 \end{bmatrix},$$

$$S_{11,2} = \begin{bmatrix} 1.0210 & -0.0002 \\ -0.0002 & 1.0210 \end{bmatrix}, S_{12,2} = \begin{bmatrix} -0.0016 & -0.0002 \\ -0.0002 & -0.0016 \end{bmatrix},$$

$$S_{22,2} = \begin{bmatrix} 0.8236 & -0.0006 \\ -0.0006 & 0.8236 \end{bmatrix}.$$

By straight forward calculation, the growth rate is $\lambda^+ = \xi = 8.5413$, the decay rate is $\lambda^- = \zeta = 0.1967$, $\lambda(\Theta_1) = 0.1976, 0.2079, 1.1443, 1.1723$ and $\lambda(\Theta_2) = -0.7682, -0.6494, -0.0646, -0.0588$. Taking $\lambda^* = 0.0001$ and $\nu = 0.00001$.

Thus, from inequality (3.40), we have $T^- \geq 43.4456\ T^+$. Given $T^+ = 0.1$ then $T^- \geq 4.34456$.

We choose the following switching rule:

(i) for $t \in [0, 0.1) \cup [5.0, 5.1) \cup [10.0, 10.1) \cup [15.0, 15.1) \cup ...$, system $i = 1$ is activated.

(ii) for $t \in [0.1, 5.0) \cup [5.1, 10.0) \cup [10.1, 15.0) \cup [15.1, 20.0) \cup ...$, system $i = 2$ is activated.

Then, by theorem 3.3.1, the switched system (3.1) is exponentially stable. Moreover, the solution $x(t)$ of the system satisfies

$$\| x(t) \| \leq 1.8770e^{-0.000045t}, \ t \in [0, \infty).$$

The trajectories of solution of the system switching between the systems $i = 1$ and $i = 2$ are shown in Figure 3.4, Figure 3.5 and Figure 3.6, respectively.

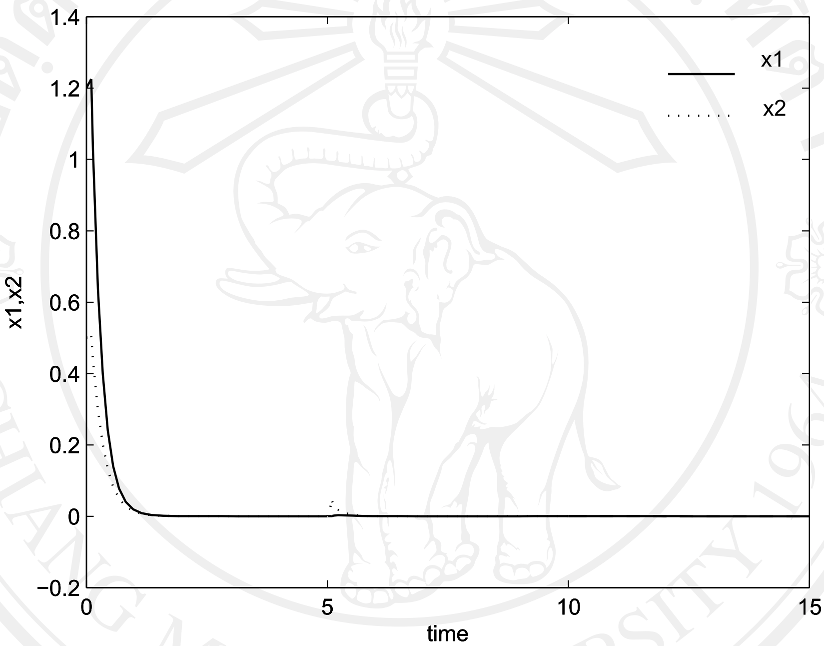

Figure 3.4: The trajectories of solution of the nonlinear switched system.

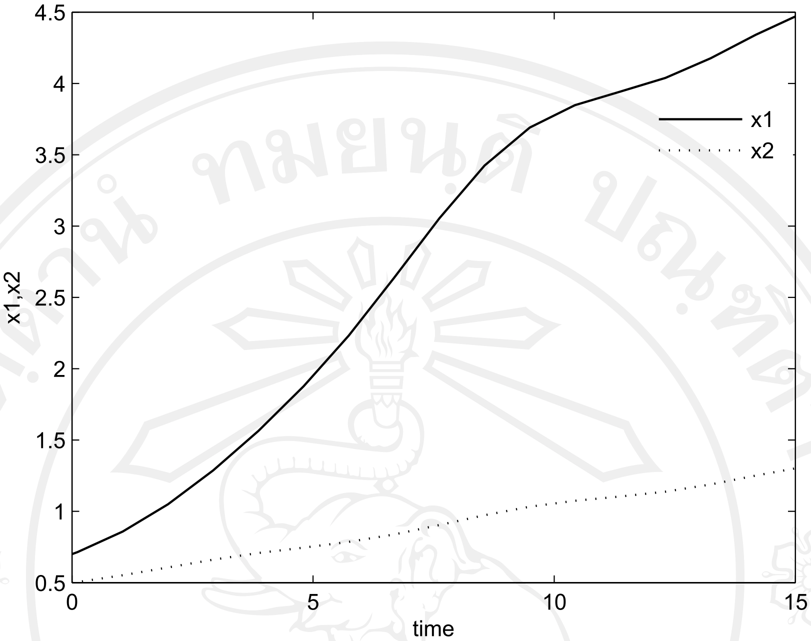

Figure 3.5: The trajectories of solution of system $i = 1$.

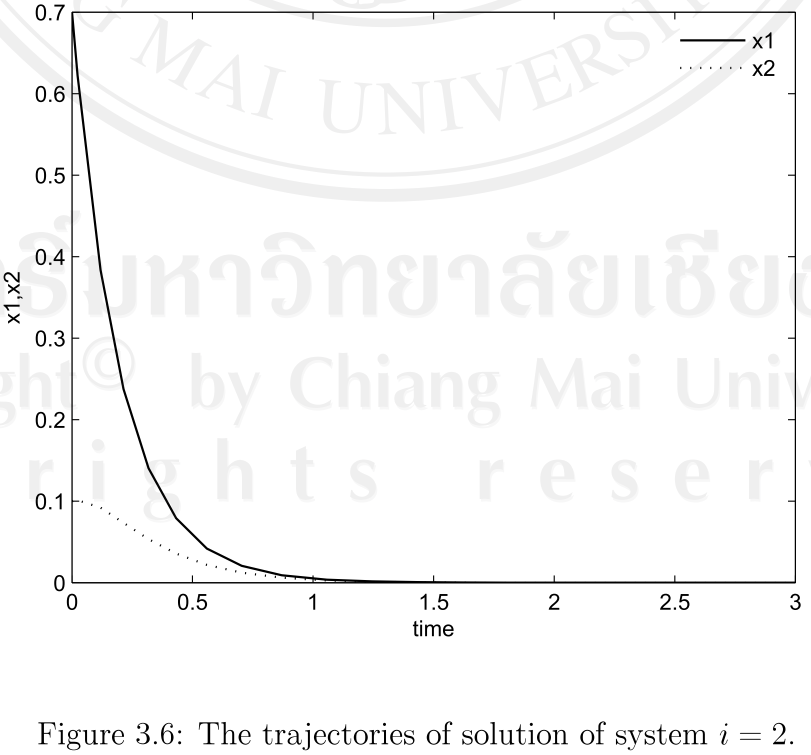

Figure 3.6: The trajectories of solution of system $i = 2$.