Chapter 1
Introduction

The theory of distributions was initiated by the Russian mathematician S.L. Sobelev in 1936. The concept of distributions was independently developed by the French mathematician L. Schwartz in the 1950s. Since Schwartz was the one who developed the theory almost to its present form. Distributions are often called Schwartz distributions. Schwartz published his theory of distribution that we call generalized function and he established the properties of generalized function. His theory has many useful on many areas of mathematics, particularly on partial differential equations. Generalized function theory has been used in many field of science and engineering. Distributions are often called Schwartz distributions.
It is well know that for the heat equation

$$\frac{\partial}{\partial t}u(x,t) = c^2\triangle u(x,t) \tag{1.0.1}$$

with the initial conditions

$$u(x,0) = f(x)$$

where

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \tag{1.0.2}$$

is the Laplace operator and $u(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times [0,\infty)$, $\mathbb{R}^n$ is the n-dimensional Euclidean space. We obtain the solution

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}}\int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4c^2t}\right) f(y)dy \tag{1.0.3}$$

as the solution of (1.0.1). Now, (1.0.3) can be written $u(x,t) = E(x,t) * f(x)$ where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4c^2t}\right). \tag{1.0.4}$$

$E(x,t)$ is called *the heat kernel*, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$, [see9, p.208-209].

In [16], K. Nonlaopon and A. Kananthai extended (1.0.1) to the equation

$$\frac{\partial}{\partial t}u(x,t) = c^2\Box u(x,t) \tag{1.0.5}$$

with the initial condition

$$u(x,0) = f(x) \tag{1.0.6}$$

where

$$\Box = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown function, $f(x)$ is a given generalized function and c is a positive constant. They obtain

$$u(x,t) = E(x,t) * f(x) \tag{1.0.7}$$

as a solution of (1.0.5) which satisfis (1.0.6) where $E(x,t)$ is the kernel of (1.0.5) and is defined by

$$E(x,t) = \frac{i^q}{(4c^2\pi t)^{n/2}} \exp\left(-\frac{\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2}{4c^2 t}\right). \tag{1.0.8}$$

where $i = \sqrt{-1}$ and $\sum_{i=1}^{p} x_i^2 > \sum_{j=p+1}^{p+q} x_j^2$. Moreover, they obtain $E(x,t) \to \delta$ as $t \to 0$, where δ is the Dirac-delta distribution.

In[14], A. Kananthai studied the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \diamondsuit^k u(x,t) \tag{1.0.9}$$

with the initial condition

$$u(x,0) = f(x),$$

where the operator $\diamondsuit$ was first introduced by A. Kananthai [10, pp.27-37] and is named the Diamond operator which is defined by

$$\diamondsuit^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k. \tag{1.0.10}$$

$p+q=n$ is the dimension of space $\mathbb{R}^n$, $u(x,t)$ is an unknown function, $f(x)$ is a given generalized function and c is a positive constant. They obtain $u(x,t) = E(x,t) * f(x)$ as a solution of (1.0.9) where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right)^k + i(\xi, x)\right] d\xi \tag{1.0.11}$$

and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t > 0$. The function $E(x,t)$ is called the diamond heat kernel or elementary solution (1.0.9).

It is well known that for the 1-dimensional wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t), \tag{1.0.12}$$

we obtain $u(x,t) = f(x+ct) + g(x-ct)$ as a solution of the equation where f and g are continuous. Also for the n-dimensional wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) + c^2(-\triangle) u(x,t) = 0, \tag{1.0.13}$$

with the initial condition

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t} u(x,0) = g(x)$$

where f and g are given continuous functions. By solving the Cauchy problem for such equation, the Fourier transform has been applied and the solution is given by

$$\widehat{u}(\xi,t)=\widehat{f}(\xi)\cos\left(2\pi|\xi|\right)t+\widehat{g}(\xi)\frac{\sin\left(2\pi|\xi|\right)t}{2\pi|\xi|}$$

where $|\xi|^2=\xi_1^2+\xi_2^2+\cdots+\xi_n^2$ [see 6, p177]. By using the inverse Fourier transform, we obtain $u(x,t)$ in the convolution form, that is

$$u(x,t)=f(x)*\Psi_t(x)+g(x)*\Phi_t(x) \tag{1.0.14}$$

where Φ_t is an inverse Fourier transform of $\widehat{\Phi}_t(\xi)=\dfrac{\sin\left(2\pi|\xi|\right)t}{2\pi|\xi|}$ and Ψ_t is an inverse Fourier transform of $\widehat{\Psi}_t(\xi)=\cos\left(2\pi|\xi|\right)t=\dfrac{\partial}{\partial t}\widehat{\Phi}(\xi)$.

In,[19] W. Satsanit, A. Kananthai studied the equation

$$\frac{\partial^2}{\partial t^2}u(x,t)+c^2\left(\Box\right)^k u(x,t)=0 \tag{1.0.15}$$

with $u(x,0)=f(x)$ and $\dfrac{\partial}{\partial t}u(x,0)=g(x)$ where c is a positive constant, k is a nonnegative integer, f and g are continuous and absolutely integrable function. The equation (1.0.15) is motivated by the heat equation of the form

$$\frac{\partial}{\partial t}u(x,t)=-c^2\left(\Box\right)^k u(x,t)$$

[see 16]. They obtain

$$u(x,t)=f(x)*\Psi_t(x)+g(x)*\Phi_t(x) \tag{1.0.16}$$

as a solution of (1.0.15) where Φ_t is an inverse Fourier transform of $\widehat{\Phi}_t(\xi)=$ $\dfrac{\sin c\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k}$ and Ψ_t is an inverse Fourier transform of $\widehat{\Psi}_t(\xi)=\cos c\left(\sqrt{s^2-r^2}\right)^k t$ $=\dfrac{\partial}{\partial t}\widehat{\Phi}_t(\xi)$ where $r^2=\xi_1^2+\xi_2^2+\cdots+\xi_p^2$ and $s^2=\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2$.

We also study the asymptotic form of $u(x,t)$ in (1.0.15) by using ϵ approximation and obtain $u(x,t)=O(\epsilon^{-n/k})$. Moreover, if we put $k=1$ and $p=0$ in (1.0.15) then (1.0.16) reduces to the solution of the $n-$ dimensional wave equation and also if $k=2, n=1$ and $p=0$ in (1.0.15) then (1.0.16) reduces to the solution of beam equation.

In 1988, S.E. Trione [23] has shown that the $n-$ dimensional ultra-hyperbolic equation

$$\Box^k u(x)=\delta(x) \tag{1.0.17}$$

where $\Box^k$ is the ultra-hyperbolic operator iterated $k-$ times is defined by

$$\Box^k=\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial x_{p+1}^2}-\frac{\partial^2}{\partial x_{p+2}^2}-\cdots-\frac{\partial^2}{\partial x_{p+q}^2}\right)^k, \tag{1.0.18}$$

and $x \in \mathbb{R}^n$, then $u(x) = R^H_\alpha(V)$ is an elementary solution of the $\Box^k$ operator where $R^H_{2k}(x)$ is defined by

$$R^H_\alpha(V) = \begin{cases} \frac{V^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{1.0.19}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})}. \tag{1.0.20}$$

The function $R^H_\alpha(V)$ is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Y. Nozaki [17].

By putting $p = 1$ in (1.0.19) and (1.0.20) and remembering the Legendre's duplication of $\Gamma(z)$.
$\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2})$ then the formula (1.0.19) reduces to

$$M^H_\alpha(V) = \begin{cases} \frac{V^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{1.0.21}$$

Here $V = x_1^2 - x_2^2 - \ldots - x_n^2$ and $H_n(\beta) = \pi^{(n-2)/2}2^{\beta-1}\Gamma(\frac{\beta-n+2}{2})\Gamma(\frac{\beta}{2})$.

Next, A. Kananthai has shown that the $n-$ dimensional Laplacian equation $\triangle^k u(x) = \delta(x)$ where $\triangle^k$ is the Laplace operator iterated $k-$ times defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right)^k \tag{1.0.22}$$

and $x \in \mathbb{R}^n$, then $u(x) = (-1)^k R^e_{2k}(x)$ is an elementary solution of the $\triangle^k$ operator where $R^e_{2k}(x)$ is defined by

$$R^e_\alpha(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)} \tag{1.0.23}$$

where

$$W_n(\alpha) = \frac{\pi^{\frac{n}{2}}2^\alpha\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \tag{1.0.24}$$

α is a complex parameter and n is the dimension of $\mathbb{R}^n$.

In 1997[5], A. Kananthai first introduced the Diamond operator $\diamondsuit^k$ iterated $k-$ times, defined by

$$\diamondsuit^k = \left(\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right)^k, \ p+q=n$$

is the dimension of the $n-$ dimensional Euclidean space $\mathbb{R}^n$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer. The operator $\diamondsuit^k$ can be expressed in the form

$$\diamondsuit^k = \triangle^k\Box^k = \Box^k\triangle^k, \tag{1.0.25}$$

where $\Box^k$ is the ultra-hyperbolic operator iterated $k-$ times and $\triangle^k$ is the Laplacian operator iterated $k-$ times defined by (1.0.18) and (1.0.21) respectively. He has shown that the solution of the convolution form $u(x) = (-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is a unique elementary solution of $\Diamond^k$, where $R^e_{2k}(x)$ and $R^H_{2k}(x)$ are defined by (1.0.23) and (1.0.19) respectively, that is

$$\Diamond^k \left((-1)^k R^e_{2k}(x) * R^H_{2k}(x)\right) = \delta. \tag{1.0.26}$$

In 2009, W. Satsanit [18], first introduced $\otimes^k$ operator and $\circledast^k$ operator where $\otimes^k$ is the operator iterated $k-$ times defined by

$$\otimes^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]^k \tag{1.0.27}$$

or the $\otimes^k$ operator can be express in the following form

$$\begin{aligned}
\otimes^k &= \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]^k \\
&= \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 + \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)\cdot \right. \\
&\qquad \left. \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right) + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\
&= \Box^k\left(\triangle^2 - \frac{1}{4}(\triangle+\Box)(\triangle-\Box)\right)^k \\
&= \left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\Box^3\right)^k
\end{aligned} \tag{1.0.28}$$

Similarly the $\circledast^k$ operator is defined by

$$\circledast^k = \left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^k. \tag{1.0.29}$$

where $\Diamond$, $\triangle$ and $\Box$ are defined by (1.0.10), (1.0.22) and (1.0.18) with $k = 1$ respectively.

In 1988, S.E. Trione [21] studied the elementary solution of the ultra-hyperbolic Klein-Gordon operator iterated $k-$ times is defined by

$$(\Box + m^2)^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} + m^2\right)^k, \tag{1.0.30}$$

and she obtained is the elementary solution $W^H_{2k}(v,m)$ defined by

$$W^H_{2k}(v,m) = \sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(k+r)}{r!\Gamma(k)}\left(m^2\right)^r R^H_{2k+2r}(v), \tag{1.0.31}$$

where $R^H_{\alpha+2r}(v)$ is defined by (1.0.19).

Next, A. Kananthai [12] studied the operator

$$\left(\diamondsuit + m^4\right)^k G(x) = \delta(x), \tag{1.0.32}$$

he obtain the function

$$G(x) = \left[W^H_{2k}(v,m) * W^e_{2k}(s,m)\right] * \left(S^{*k}(x)\right)^{*-1} \tag{1.0.33}$$

is an Green function for the operator $(\diamondsuit + m^4)^k$ iterated $k-$ times, and $W^e_{2k}(s,m)$ is defined by

$$W^e_{2k}(s,m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r!\Gamma(k)} \left(m^2\right)^r (-1)^{k+r} R^e_{2k+2r}(s), \tag{1.0.34}$$

and $R^e_{2k+2r}(s)$ is defined by (1.0.23) with $\alpha = 2k+2r$, m is a nonnegative real number and

$$S(x) = \delta - m^2 \left(W^H_2(v,m) * W^e_2(s,m)\right) * \left(R^H_{-2}(v) + R^e_{-2}(s)\right) \tag{1.0.35}$$

$S^{*k}(x)$ denotes the convolution of $S(x)$ itself $k-$ times, $\left(S^{*k}(x)\right)$ is the inverse of $S^{*k}(x)$in the convolution algebra.

In[16], A. Kananthai, G. Sritanratana studied non-linear equation

$$\diamondsuit^k u(x) = f\left(x, \triangle^{k-1}\Box^k u(x)\right), \tag{1.0.36}$$

where $\diamondsuit^k$ is defined by (1.0.10), $\triangle$ is the Laplacian operator, $\Box$ is the ultra-hyperbolic operator,$p+q=n$, $x \in \mathbb{R}^n$ and $u(x)$ is an unknown function, f is first derivative for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω, n is even with $n \geq 4$ and if f is bounded on Ω, then

$$u(x) = (-1)^{k-1} R^e_{2(k-1)}(s) * R^H_{2k}(v) * W(x)$$

is a solution (1.0.36) with the boundary condition

$$u(x) = R^H_{2k}(v) * (-1)^{k-2} \left(R^e 2(k-2)\right)^{(m)}$$

for $x \in \partial\Omega$ and $m = \frac{(n-4)}{2}$.

Furthermore, W. Satsanit first introduced the operator $\boxtimes^k$ iterated $k-$ times which is defined by

$$\boxtimes^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^6 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^6\right)^k, \tag{1.0.37}$$

where $p+q=n$ is the dimension of the $n-$ dimensional Euclidean space $\mathbb{R}^n$ and k is a nonnegative integer. Actually (1.0.37) can be rewritten in the following form

$$\boxtimes^k = \otimes^k \circledast^k, \tag{1.0.38}$$

where the operators $\otimes^k$ and $\circledast^k$ are defined by (1.0.28) and (1.0.29) respectively.

The thesis is organized as follow: In chapter 2, we give some useful definitions and properties of the special function, partial differential equations, distributions and elementary solution.

In chapter 3, we study the nonlinear equation of the form

$$\frac{\partial}{\partial t}u(x,t) - c^2(-\otimes)^k u(x,t) = f(x,t,u(x,t))$$

, where $\otimes^k$ is the operator iterated k-times, defined by

$$\otimes^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]^k$$

, where $p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown for $(x,t)=(x_1,x_2,\ldots,x_n,t) \in \mathbb{R}^n \times (0,\infty)$, k is a positive integer and c is a positive constant, f is the given function in nonlinear form depending on x,t and $u(x,t)$. On suitable conditions for f, p, q, k and the spectrum, we obtain the unique solution $u(x,t)$ of such equation.Moreover,if we put $p=0, k=1$, we obtain the solution of non-linear heat equation.

In chapter 4, we study the generalized wave equation of the form

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2(\otimes)^k u(x,t) = 0$$

with the initial conditions

$$u(x,0) = f(x), \quad \frac{\partial}{\partial t}u(x,0) = g(x)$$

where $u(x,t) \in \mathbb{R}^n \times [0,\infty)$, $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\otimes^k$ is the operator iterated $k-$times defined by

$$\otimes^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]^k$$

c is a positive constant, k is a nonnegative integer, f and g are continuous and absolutely integrable functions. We obtain $u(x,t)$ as a solution for such equation. Moreover, by ϵ-approximation we also obtain the asymptotic solution $u(x,t) = O(\epsilon^{-n/3k})$. In particular, if we put $k=1$ and $p=0$, the $u(x,t)$ reduces to the solution of the wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2(\triangle)^3 u(x,t) = 0.$$

which is related to the triharmonic wave equation.

In chapter 5, we propose to use the idea of A. Kananthai [7] and A.H. Zemanian [23], to find the Green function of the $(\otimes + m^6)^k$ operator iterated $k-$times, that is we consider the equation

$$\left(\otimes + m^6\right)^k G(x) = \delta(x)$$

and m is positive real number, $\delta(x)$ is the Dirac-delta distribution. At first we find the green function of the operator $(\otimes + m^6)^k$ and after that we apply such a green function to solve the solution of the equation $(\otimes + m^6)^k G(x) = f(x)$ where f is a generalized function and $G(x)$ is an unknown for $x \in \mathbb{R}^n$.

In chapter 6, we study the $\boxtimes^k$ operator iterated $k-$ times and is defined by

$$\boxtimes^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^6 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^6 \right)^k,$$

where $p+q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x)$ is an unknown function for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $f(x)$ is the generalized function, k is a positive integer. Firstly, we study the solution of the equation $\boxtimes^k u(x) = f(x)$. It is found that the solution $u(x)$ depends on the condition of p and q and a solution is related to the solution of the Laplace equation and the wave equation. Finally, we study the solution of the nonlinear equation $\boxtimes^k u(x) = f(x, \Box^{k-1} L^k \circledast^k u(x)$. It is found that the existence of the solution $u(x)$ of such an equation depends on the condition of f and $\Box^{k-1} L^k \circledast^k u(x)$. Moreover a solution $u(x)$ related the inhomogeneous equation depends on the condition of p, q and k.