Chapter 2
Basic Concepts and Preliminaries

The aim of this chapter is to give some definition and properties of the distribution, the special function, the Fourier transform, partial differential equations and the elementary solution of the partial differential operators which will be used in the later chapters.

2.1 Distribution

In this section, we give some definition and properties of the distribution which will be used in the later chapters.

Definition 2.1.1. Let $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$. The *support of* f is defined to be the closure of the set $S = \{x \in \Omega : f(x) \neq 0\}$. And support of f is denote by Supp$f$.

Definition 2.1.2. A set $\Omega \subset \mathbb{R}^n$ is *compact* if every sequence in Ω has a convergent subsequence whose limit is an element of Ω.

Definition 2.1.3. Let $\Omega \subset \mathbb{R}^n$, define $\mathcal{D} = C_0^\infty(\Omega)$ is the set of all infinitely differentiable functions on Ω with compact support, $\varphi \in \mathcal{D}$ is called test function.

Definition 2.1.4. A sequence of testing function $\varphi_i(x)_{i=1}^\infty$ is said to *converge* to $\varphi(x)$ in $\mathcal{D}$ if all $\varphi_i(x)$ are zero outside a certain region in $\mathbb{R}^n$ and if for every nonnegative integers $m_1, m_2, \ldots, m_n$ the sequence $\left\{\frac{\partial^{m_1+m_2+\cdots+m_n}\varphi_i(x)}{\partial x_1^{m_1}\partial x_2^{m_2}\ldots\partial x_n^{m_n}}\right\}_{i=1}^\infty$ converges uniformly to $\frac{\partial^{m_1+m_2+\cdots+m_n}\varphi(x)}{\partial x_1^{m_1}\partial x_2^{m_2}\ldots\partial x_n^{m_n}}$ on $\mathbb{R}^n$.

Proposition 2.1.5. ([5]) *$\mathcal{D}$ is closed under convergent, that is, the limit of every sequence that converge in $\mathcal{D}$ is also in $\mathcal{D}$.*

Definition 2.1.6. A *distribution* is a mapping $f : \mathcal{D} \to \mathbb{C}$ such that

(1) $\langle f, \varphi \rangle$ is a well defined complex number for every $\varphi \in \mathcal{D}$,

(2) for any $\varphi_1, \varphi_2 \in \mathcal{D}$ and any scalars a_1, a_2,

$$\langle f, a_1\varphi_1 + a_2\varphi_2 \rangle = a_1\langle f, \varphi_1 \rangle + a_2\langle f, \varphi_2 \rangle,$$

(3) for any sequence $\{\varphi_n\}$ in $\mathcal{D}$ such that $\lim_{n\to\infty} \varphi_n = \varphi$ then $\lim_{n\to\infty} \langle f, \varphi_n \rangle = \langle f, \varphi \rangle$.

We note that each continuous(or even locally integrable) function $f(x)$ generates a distribution

$$\langle f, \varphi \rangle = \int f(x)\varphi(x)dx.$$

Definition 2.1.7. A *regular distribution* is a distribution which is generated by a locally integrable function.

Definition 2.1.8. A *singular distribution* is a distribution which is not generated by a locally integrable function.

Definition 2.1.9. The *Dirac-delta* distribution with singularity $\xi \in \mathbb{R}^n$, denoted by $\delta(x-\xi)$, which is defined by

$$\langle \delta(x-\xi), \phi \rangle = \phi(\xi).$$

Definition 2.1.10. Let $x = (x_1, x_2, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$, $\mathcal{S}$ is defined to be the set of all real value functions $\varphi(x)$ that are infinitely smooth and are such that, all nonnegative integer m and $k = (k_1, k_2, \ldots, k_n)$,

$$\|x\|^m |D^k \varphi(x)| \leq C_{mk},$$

for some a constant C_{mk} and denote D^k by $D^k = \frac{\partial^{k_1+k_2+\ldots+k_n}}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_n^{k_n}}$.

Definition 2.1.11. A *tempered distribution* is a mapping $f : \mathcal{S} \to \mathbb{C}$ such that

(1) $\langle f, \varphi \rangle$ is a well defined complex number for every $\varphi \in \mathcal{S}$,

(2) for any $\varphi_1, \varphi_2 \in \mathcal{S}$ and any scalars a_1, a_2,

$$< f, a_1\varphi_1 + a_2\varphi_2 >= a_1 < f, \varphi_1 > +a_2 < f, \varphi_2 >,$$

(3) for any sequence $\{\varphi_n\}$ in $\mathcal{S}$ such that $\lim_{n\to\infty} \varphi_n = \varphi$ then $\lim_{n\to\infty} \langle f, \varphi_n \rangle = \langle f, \varphi \rangle$.

Definition 2.1.12. A space $\mathcal{C}$ of distributions is said to be a *convolution algebra* if it possesses the following properties:

(1) $\mathcal{C}$ is a linear space.

(2) $\mathcal{C}$ is closed under convolution.

(3) Convolution is associative for any three distributions in $\mathcal{C}$.

Definition 2.1.13. Let f be a distribution. The *derivative* $\dfrac{\partial f}{\partial x_k}$ as the distribution given by

$$\langle \frac{\partial f}{\partial x_k}, \phi \rangle = -\langle f, \frac{\partial \phi}{\partial x_k} \rangle,$$

and more generally $D^k f$ denoted by

$$\langle D^k f, \phi \rangle = (-1)^{|k|} \langle f, D^k \phi \rangle,$$

where $|k| = k_1 + k_2 + \cdots + k_n$.

Proposition 2.1.14. ([23]) *Let x be an n-dimensional real variable and y an m-dimensional real variable. Also, let $\varphi(x, y)$ be a testing function in $\mathcal{D}$ define over $\mathbb{R}^{n+m}$. If $f(x)$ is a distribution defined over $\mathbb{R}^n$, then $\theta(y) = \langle f(x), \varphi(x, y) \rangle$ is a testing function of y in $\mathcal{D}$.*

Proposition 2.1.15. ([7]) *Let f be a distribution in m dimensions and g be a distribution in n dimensions. Then the functional h defined by*

$$\langle h(x,y), \varphi(x,y)\rangle = \langle f(x), \langle g(y), \varphi(x,y)\rangle\rangle$$

is a distribution in $m+n$ dimensions.

Definition 2.1.16. The distribution h in Proposition (2.1.15) is called the *tensor (or direct) product* of $f(x)$ and $g(y)$ and is denoted by $h(x,y) = f(x) \times g(y)$, that is,

$$\langle f(x) \times g(y), \varphi(x,y)\rangle = \langle f(x), \langle g(y), \varphi(x,y)\rangle\rangle. \tag{2.1.1}$$

Definition 2.1.17. The *support of a distribution* f is defined as the complement of the largest open set on which f is zero.

Proposition 2.1.18. ([7]) *Let f and g be distributions in n dimensions. Then the function h defined by*

$$\langle h, \varphi\rangle = \langle f(x) \times g(y), \varphi(x+y)\rangle \tag{2.1.2}$$

is a distribution provided that it satisfies either of the following conditions:

(1) *Either f or g has bounded support, or*

(2) *In one dimension the supports of both f and g are bounded on the same side (for instance, $f = 0$ for $x < a$, and $g = 0$ for $y < b$).*

Definition 2.1.19. The distribution h in Proposition (2.1.15) is called the *convolution* of f and g and is denoted by $h = f * g$, that is,

$$\langle f * g, \varphi\rangle = \langle f(x) \times g(y), \varphi(x+y)\rangle. \tag{2.1.3}$$

Now we shall give some helpful properties of convolutions.

Proposition 2.1.20. ([7],[23]) *Let f, g and h be distributions.*

(1) *For δ is the Dirac-delta function, we have*

$$f * \delta = f. \tag{2.1.4}$$

(2) *If f and g satisfy at least one of the (1) and (2) of proposition 2.1.12, then*

$$f * g = g * f. \tag{2.1.5}$$

(3) *If $P(D)$ is a linear partial differential operator with constant coefficients and f and g satisfy at least one of the (1) and (2) of proposition 2.1.12, then*

$$P(D)f * g = P(D)(f * g) = f * P(D)g. \tag{2.1.6}$$

2.2 The Special Functions and Fourier Transform

In this section, we shall present the definitions of the special function. In addition, we shall give some properties of the gamma function.

Definition 2.2.1. The *gamma function* is denoted by Γ and is defined by

$$\Gamma(z) = \int_0^\infty e^t t^{z-1} dt, \tag{2.2.1}$$

where z is a complex number with $\mathrm{Re}\, z > 0$.

A result that yields an immediate analytic continuation from the left half plane is the following properties.

Proposition 2.2.2. ([1]) *Let z be a complex number. Then*

$$(1)\ \Gamma(z) = \frac{\Gamma(z+1)}{z},\ \ z \neq 0, -1, -2, \ldots, \tag{2.2.2}$$

$$(2)\ \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},\ \ z \neq 0, \pm 1, \pm 2, \ldots. \tag{2.2.3}$$

Proposition 2.2.3. ([1]) *(Legendre's duplication formula) Let z be a complex number. Then*

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z),\ \ z \neq 0, -1, -2, \ldots. \tag{2.2.4}$$

Definition 2.2.4. Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Denoted by

$$V = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2 \tag{2.2.5}$$

the nondegenerated quadratic form and $p + q = n$ is the dimension of the space $\mathbb{R}^n$.

Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ and $\overline{\Gamma}_+$ denotes it closure. For any complex number α, define the function

$$R_\alpha^H(V) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.2.6}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})}. \tag{2.2.7}$$

The function $R_\alpha^H(V)$ is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Y. Nozaki [17].

It is well known that $R_\alpha^H(u)$ is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of α if $Re(\alpha) < n$. Let supp $R_\alpha^H(V)$ denote the support of $R_\alpha^H(u)$ and suppose supp $R_\alpha^H(V) \subset \bar{\Gamma}_+$, that is supp $R_\alpha^H(u)$ is compact.

From S.E.Trione [22], R_{2k}^H is an elementary solution of the operator $\Box^k$ that is

$$\Box^k R_{2k}^H(u) = \delta(x) \tag{2.2.8}$$

Definition 2.2.5. Let $x = (x_1, x_2, ..., x_n)$ be a point of $\mathbb{R}^n$ and $|x| = (x_1^2 + x_2^2 + ... + x_n^2)^{\frac{1}{2}}$ the function $R^e_\alpha(x)$ denoted the elliptic kernel of Marcel Riesz and is defined by

$$R^e_\alpha(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)} \tag{2.2.9}$$

where

$$W_n(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \tag{2.2.10}$$

α is a complex parameter and n is the dimension of $\mathbb{R}^n$.

It can be shown that $R^e_{-2k}(x) = (-1)^k \triangle^k \delta(x)$ where $\triangle^k$ is defined by (**??**). It follows that $R^e_0(x) = \delta(x)$, [2].

Moreover, we obtain $(-1)^k R^e_{2k}(x)$ is an elementary solution of the operator $\triangle^k$ that is

$$\triangle^k((-1)^k R^e_{2k}(x) = \delta(x) \tag{2.2.11}$$

[See 10, Lemma 2.4]

Lemma 2.2.6. (The convolution of $R^H_\alpha(x)$ and $R^e_\alpha(x)$)

Let $R^e_\alpha(x)$ and $R^H_\alpha(x)$ be defined by (2.2.9) and (2.2.6) respectively, then we obtain the following formulas:

(1)$R^e_\alpha(x) * R^e_\alpha(x) = R^e_{\alpha+\beta}(x)$ where α and β are complex parameters

(2)$R^H_\alpha(x) * R^H_\alpha(x) = R^H_{\alpha+\beta}(x)$ for α and β are both integers and except only the case both α and β are both integers.

Proof. Proof of the first formula, [5]

Proof of the second formula, for the case α and β are both even integers [See 5] , and for the case α is odd and β is even or α is even and β is odd , we know from Trione [21]

$$\Box^k R^H_\alpha(x) = R^H_{\alpha-2k}(x) \tag{2.2.12}$$

and

$$\Box^k R^H_{2k}(x) = \delta(x) \quad , \quad k = 0, 1, 2, 3, \cdots \tag{2.2.13}$$

where $\Box^k$ is the ultra-hyperbolic operator iterated k-times defined by

$$\Box^k = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k .$$

Now let m be an odd integer, we have

$$\Box^k R^H_m(x) = R^H_{m-2k}(x)$$

and

$$R^H_{2k}(x) * \Box^k R^H_m(x) = R^H_{2k}(x) * R^H_{m-2k}(x)$$

or

$$\left(\Box^k R^H_{2k}(x)\right) * R^H_m(x) = R^H_{2k}(x) * R^H_{m-2k}(x)$$
$$\delta * R^H_m(x) = R^H_{2k}(x) * R^H_{m-2k}(x).$$

Thus

$$R^H_m(x) = R^H_{2k}(x) * R^H_{m-2k}(x)$$

Since m is odd, hence $m-2k$ is odd and $2k$ is a positive even. Put $\alpha = 2k$, $\beta = m-2k$, we obtain

$$R^H_\alpha(x) * R^H_\beta(x) = R^H_{\alpha+\beta}(x)$$

for α is a nonnegative even and β is odd.

For the case α is a negative even and β is odd , by (2.2.8) we have

$$\Box^k R^H_0(x) = R^H_{-2k}(x)$$

or

$$\Box^k \delta = R^H_{-2k}(x).$$

Where $R^H_0(x) = \delta$. Now for m is odd,

$$R^H_{-2k}(x) * \Box^k R^H_m(x) = R^H_{-2k}(x) * R^H_{m-2k}(x)$$

or

$$\left(\Box^k R^H_{2k}(x)\right) * R^H_m(x) = R^H_{-2k}(x) * R^H_{m-2k}(x)$$
$$\delta * R^H_m(x) = R^H_{-2k}(x) * R^H_{m-2k}(x).$$

Thus

$$R^H_{m-2(2k)}(x) = R^H_{-2k}(x) * R^H_{m-2k}(x).$$

Put $\alpha = -2k$ and $\beta = m-2k$, now α is a negative even and β is odd. Then we obtain

$$R^H_\alpha(x) * R^H_\beta(x) = R^H_{\alpha+\beta}(x).$$

That completes the proof.

Lemma 2.2.7. The function $R^H_{-2k}(x)$ and $R^e_{-2k}(x)$ are the inverse of the convolution algebra of R^{2k} and R^e_{2k} respectively, that is

$$R^H_{-2k}(x) * R^H_{2k}(x) = R^H_{-2k+2k}(x) = R_0(x) = \delta(x)$$

and

$$(-1)^k R^e_{-2k}(x) * (-1)^k R^e_{2k}(x) = (-1)^{2k} R^e_{-2k+2k}(x) = S_0(x) = \delta(x)$$

Proof. [See 21, p.118, p.158], [See 1, p.123] and [See 5, p.10].

Definition 2.2.8. Let $x = (x_1, x_2, ..., x_n)$ be a point of $\mathbb{R}^n$ and the function $W^H_\alpha(u,m)$ is defined by

$$W^H_\alpha(u,m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma\left(\frac{\alpha}{2}+r\right)}{r!\Gamma\left(\frac{\alpha}{2}\right)} \left(m^2\right)^r R^H_{\alpha+2r}(u) \qquad (2.2.14)$$

where $R^H_{\alpha+2r}(u)$ is defined by (2.2.6) and m is a nonnegative real number.

Lemma 2.2.9. (The existence of the convolution $W^H_{2k}(u,m) * W^e_{2k}(v,m)$) The convolution $W^H_{2k}(u,m) * W^e_{2k}(v,m)$ exists and is a tempered distribution where $W^H_{2k}(u,m)$ and $W^e_{2k}(v,m)$ are defined by (2.3.11) and (2.3.12) with $\alpha = 2k$.

Proof. From (2.3.11) and (2.3.12), we have

$$\begin{aligned} W^H_{2k}(u,m) * W^e_{2k}(v,m) &= \left(\sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r!\Gamma(k)} \left(m^2\right)^r R^H_{2k+2r}(u)\right) \\ &\quad * \left(\sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r!\Gamma(k)} \left(m^2\right)^r R^e_{2k+2r}(v)\right) \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s \Gamma(k+s)}{s!\Gamma(k)} \left(m^2\right)^s \frac{(-1)^r \Gamma(k+r)}{r!\Gamma(k)} \left(m^2\right)^r \\ &\quad \times 2(-1)^{k+r}\left[R^H_{2k+2s}(u) * R^e_{2k+2r}(v)\right]. \end{aligned}$$

A.Kananthai [12] has shown $R^H_{2k+2s}(u) * R^e_{2k+2r}(v)$ exists and is a tempered distribution. It follows that $W^H_{2k}(u,m) * W^e_{2k}(v,m)$ exists and also is a tempered distribution.

Lemma 2.2.10. Given P is a hyper-function then

$$P\delta^k(p) + k\delta^{(k-1)}(p) = 0$$

where $\delta^{(k)}$ is the Dirac-delta distribution with k derivatives.

Proof. [7].

Lemma 2.2.11. Given the equation

$$\Box u(x) = f(x, u(x)), \tag{2.2.15}$$

where f is defined and has continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of R^n and $\partial\Omega$ is the boundary of Ω. Assume that f is bounded, that is $|f(x,u)| \leq N$ and the boundary condition $u(x) = 0$ for $x \in \partial\Omega$. Then we obtain $u(x)$ as a unique solution of (2.2.15)

Proof. We can prove the existence of the solution $u(x)$ of (2.2.15) by the method of iterations and the Schuder's estimates.The details of the proof are given by Courant and Hilbert , [4].

Definition 2.2.12. Let $f \in L_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)dx \tag{2.2.16}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the inner product in $\mathbb{R}^n$ and $dx = dx_1 dx_2 \ldots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(x)dx. \tag{2.2.17}$$

If f is a distribution with compact supports by [23, Theorem 7.4-3] Eq.(2.1) can be written as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \left\langle f(x), e^{-i(\xi,x)} \right\rangle. \tag{2.2.18}$$

Definition 2.2.13. Let $t > 0$ and p is a real number

$$f(t) = O(t^p) \text{ as } t \to 0 \Leftrightarrow t^{-p}|f(t)| \text{ is bounded as } t \to 0$$
$$\text{and } f(t) = o(t^p) \text{ as } t \to 0 \Leftrightarrow t^{-p}|f(t)| \to 0 \text{ as } t \to 0$$

Lemma 2.2.14. Given the function

$$f(x) = \exp\left[-\sqrt{-\left(\sum_{i=1}^{p} x_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q} x_j^2\right)^3}\,\right]$$

where $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $\quad p+q=n,\ \sum_{i=1}^{p} x_i^2 < \sum_{j=p+1}^{p+q} x_j^2$. Then

$$|\int_{\mathbb{R}^n} f(x)dx| \leq \frac{\Omega_p\Omega_q}{18}\cdot\frac{\Gamma(\frac{n}{3})\Gamma(\frac{p}{6})\Gamma(\frac{6-n}{6})}{\Gamma(\frac{6-q}{6})},$$

where Γ denotes the Gamma function. That is $\int_{\mathbb{R}^n} f(x)dx$ is bounded.

Proof.

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \exp\left[-\sqrt{-\left(\sum_{i=1}^{p} x_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q} x_j^2\right)^3}\,\right] dx.$$

Let us transform to bipolar coordinates defined by

$$x_1 = r\omega_1,\quad x_2 = r\omega_2, \ldots,\quad x_p = r\omega_p$$
$$dx_1 = rd\omega_1,\quad dx_2 = rd\omega_2, \ldots,\quad dx_p = rd\omega_p$$

and

$$x_{p+1} = s\omega_{p+1},\quad x_{p+2} = s\omega_{p+2}, \ldots,\quad x_{p+q} = s\omega_{p+q}$$
$$dx_{p+1} = sd\omega_{p+1},\quad dx_{p+2} = sd\omega_{p+2}, \ldots,\quad dx_{p+q} = sd\omega_{p+q}$$

where $\omega_1^2 + \omega_2^2 + \ldots + \omega_p^2 = 1$ and $\omega_{p+1}^2 + \omega_{p+2}^2 + \ldots + \omega_{p+q}^2 = 1$.
Thus

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \exp\left[-\sqrt{s^6 - r^6}\right] r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$$

where $dx = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively,

$$|\int_{\mathbb{R}^n} f(x)dx| \leq \int_{\mathbb{R}^n} \exp\left[-\sqrt{s^6 - r^6}\right] r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q.$$

By computing directly, we obtain

$$\int_{\mathbb{R}^n} f(x)dx = \Omega_p\Omega_q \int_0^\infty \int_0^s \exp\left[-\sqrt{s^6 - r^6}\right] r^{p-1}s^{q-1}drds$$

where $\Omega_p = \dfrac{2\pi^{p/2}}{\Gamma(p/2)}$ and $\Omega_q = \dfrac{2\pi^{q/2}}{\Gamma(q/2)}$. Thus

$$|\int_{\mathbb{R}^n} f(x)dx| \leq \Omega_p\Omega_q \int_0^\infty \int_0^s \exp\left[-\sqrt{s^6 - r^6}\right] r^{p-1}s^{q-1}drds.$$

Put $r^3 = s^3 \sin\theta,\ 3r^2dr = s^3\cos\theta d\theta$ and $0 \le \theta \le \dfrac{\pi}{2}$,

$$
\begin{aligned}
|\int_{\mathbb{R}^n} f(x)dx| &\le \frac{\Omega_p\Omega_q}{3}\int_0^\infty\int_0^s e^{-\sqrt{s^6 - s^6\sin^2\theta}} s^{p-1}(\sin\theta)^{\frac{p-3}{3}} s^q \cos\theta d\theta ds \\
&= \frac{\Omega_p\Omega_q}{3}\int_0^\infty\int_0^s e^{-s^3\cos\theta} s^{p+q-1}(\sin\theta)^{\frac{p-3}{3}} \cos\theta d\theta ds.
\end{aligned}
$$

Put $y = s^3\cos\theta,\ ds = \dfrac{dy}{3s^2\cos\theta}$,

$$
\begin{aligned}
|\int_{\mathbb{R}^n} f(x)dx| &\le \frac{\Omega_p\Omega_q}{9}\int_0^{\pi/2}\int_0^\infty e^{-y}\left(\frac{y}{\cos\theta}\right)^{\frac{n-3}{3}}(\sin\theta)^{\frac{p-3}{3}}\cos\theta d\theta\frac{dy}{\cos\theta} \\
&= \frac{\Omega_p\Omega_q}{9}\int_0^{\pi/2}\int_0^\infty e^{-y}y^{\frac{n-3}{3}}(\cos\theta)^{\frac{3-n}{3}}(\sin\theta)^{\frac{p-3}{3}}dyd\theta \\
&= \frac{\Omega_p\Omega_q}{9}\Gamma\left(\frac{n}{3}\right)\int_0^{\pi/2}(\cos\theta)^{\frac{3-n}{3}}(\sin\theta)^{\frac{p-3}{3}}d\theta \\
&= \frac{\Omega_p\Omega_q}{18}\Gamma\left(\frac{n}{3}\right)\beta\left(\frac{p}{6},\frac{6-n}{6}\right) \\
|\int_{\mathbb{R}^n} f(x)dx| &\le \frac{\Omega_p\Omega_q}{18}\frac{\Gamma(\frac{n}{3})\Gamma(\frac{p}{6})\Gamma(\frac{6-n}{6})}{\Gamma(\frac{6-q}{6})}.
\end{aligned}
$$

That is $\int_{\mathbb{R}^n} f(x)dx$ is bounded.

Definition 2.2.15. *The spectrum of the kernel $E(x,t)$ defined by (2.3.8) is the bounded support of the Fourier transform $\widehat{E(\xi,t)}$ for any fixed $t > 0$.*

Definition 2.2.16. *Let $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ be a point in $\mathbb{R}^n$ and write*

$$u = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \cdots - \xi_{p+q}^2, \quad p + q = n.$$

Denote by $\Gamma_+ = \{\xi \in \mathbb{R}^n : \xi_1 > 0 \text{ and } u > 0\}$ the set of an interior of the forward cone and denote by $\overline{\Gamma}_+$ the closure of Γ_+. Let Ω be the spectrum of $E(x,t)$ for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $\widehat{E(\xi,t)}$ be the Fourier transform of $E(x,t)$ and define

$$
\widehat{E(\xi,t)} = \begin{cases} \dfrac{1}{(2\pi)^{n/2}}\exp\left[c^2t\left(\left(\displaystyle\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\displaystyle\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right], & \text{for } \xi \in \Gamma_+ \\ 0, & \text{for } \xi \notin \Gamma_+ \end{cases} \tag{2.2.19}
$$

Lemma 2.2.17. (The Fourier transform of $(\otimes)^k\delta$)

$$\mathcal{F}(\otimes)^k\delta = \frac{(-1)^{3k}}{(2\pi)^{n/2}}\left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^3 - \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^3\right]^k$$

where $\mathcal{F}$ is the Fourier transform defined by Eq.(2.2.18) and if the norm of ξ is given by $\|\xi\| = (\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2)^{1/2}$ then

$$\mathcal{F}(\otimes)^k\delta \le \frac{M}{(2\pi)^{n/2}}\|\xi\|^{6k},$$

which M is the positive constant and $\mathcal{F}(\otimes)^k$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution. Moreover, by Eq.(2.2.17)

$$(\otimes)^k\delta = \mathcal{F}^{-1}\frac{1}{(2\pi)^{n/2}}\left[\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3-\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3\right]^k$$

Proof. By Eq. (2.3)

$$\begin{aligned}
\mathcal{F}\otimes^k\delta &= \frac{1}{(2\pi)^{n/2}}\left\langle(\otimes)^k\delta, e^{-i(\xi.x)}\right\rangle\\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta,(\otimes)^k e^{-i(\xi.x)}\right\rangle\\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta,(\otimes)^{k-1}(\otimes)e^{-i(\xi.x)}\right\rangle\\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta,(\otimes)^{k-1}\left(\frac{3}{4}\Diamond\triangle+\frac{1}{4}\Box^3\right)e^{-i(\xi.x)}\right\rangle\\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta,(\otimes)^{k-1}\left(\frac{3}{4}\Diamond\triangle\right)e^{-i(\xi.x)}\right\rangle+\frac{1}{(2\pi)^{n/2}}\left\langle\delta,(\otimes)^{k-1}\left(\frac{1}{4}\Box^3\right)e^{-i(\xi.x)}\right\rangle\\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta,(\otimes)^{k-1}\frac{3}{4}(-1)^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^2-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right](-1)\left(\sum_{i=1}^{n}\xi_i^2\right)e^{-i(\xi.x)}\right\rangle\\
&\quad+\frac{1}{(2\pi)^{n/2}}\left\langle\delta,(\otimes)^{k-1}\frac{1}{4}(-1)^3\left[\left(\sum_{i=1}^{p}\xi_i^2\right)-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)\right]^3e^{-i(\xi.x)}\right\rangle\\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta,(\otimes)^{k-1}\frac{3}{4}(-1)^3\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^2-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right]\left(\sum_{i=1}^{n}\xi_i^2\right)e^{-i(\xi.x)}\right\rangle\\
&\quad+\frac{1}{2\pi)^{n/2}}\left\langle\delta,(\otimes)^{k-1}\left(\frac{1}{4}(-1)^3\left[\left(\sum_{i=1}^{p}\xi_i^2\right)-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)\right]^3\right)e^{-i(\xi.x)}\right\rangle\\
&= \frac{(-1)^3}{(2\pi)^{n/2}}\left\langle\delta,(\otimes)^{k-1}\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right]e^{-i(\xi.x)}\right\rangle
\end{aligned}$$

By keeping on operator $(\otimes)$ with $k-1$ times , we obtain

$$\mathcal{F}(\otimes)^k\delta = \frac{(-1)^{3k}}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3-\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right]^k$$

Now,

$$\begin{aligned}
|\mathcal{F}(\otimes)^k\delta| &= \frac{1}{(2\pi)^{n/2}}\left|\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3-\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right|^k\\
&= \frac{1}{(2\pi)^{n/2}}\left|\xi_1^2+\xi_2^2+\xi_3^2+\ldots+\xi_p^2-\xi_{p+1}^2-\xi_{p+2}^2-\ldots-\xi_{p+q}^2\right|^k\cdot\\
&\quad\left|\left(\xi_1^2+\ldots+\xi_p^2\right)^2+\left(\xi_1^2+\ldots+\xi_p^2\right)\left(\xi_{p+1}^2+\ldots+\xi_{p+q}^2\right)+\left(\xi_{p+1}^2+\ldots+\xi_{p+q}^2\right)^2\right|^k
\end{aligned}$$

or

$$
\begin{aligned}
|\mathcal{F}(\otimes)^k\delta| &\le \frac{1}{(2\pi)^{n/2}}\left|\xi_1^2+\ldots+\xi_n^2\right|^k \left|\left(\xi_1^2+\ldots+\xi_n^2\right)^2+\left(\xi_1^2+\ldots+\xi_n^2\right)^2+\left(\xi_1^2+\ldots+\xi_n^2\right)^2\right|^k \\
&\le \frac{M}{(2\pi)^{n/2}}\|\xi\|^{6k}
\end{aligned}
$$

where $\|\xi\| = (\xi_1^2+\xi_2^2+\ldots+\xi_n^2)^{1/2}$, $\xi_i(i=1,2,\ldots,n)\in\mathbb{R}$ and M is a positive constant. Hence we obtain $\mathcal{F}\otimes\delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution.

Since $\mathcal{F}$ is $1-1$ transformation from the space $\mathcal{S}'$ of the tempered distribution to the real space $\mathbb{R}$, then by (2.2.17)

$$
\otimes\delta = \mathcal{F}^{-1}\frac{1}{(2\pi)^{n/2}}\left[\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3-\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3\right].
$$

That completes the proof.

2.3 Partial Differential Equation

A partial differential operator L of order m in N variables

$$
L=\sum_{|\alpha|\le m}A_\alpha D^\alpha, \tag{2.3.1}
$$

where $\alpha=(\alpha_1,\ldots,\alpha_N)$ is a multi-index, the $\alpha_n's$ are non-negative integers, $|\alpha| = \alpha_1,\alpha_2,\ldots,\alpha_N$ and $A_\alpha = A_{\alpha_1,\alpha_2,\ldots,\alpha_N}(x_1,x_2,\ldots,x_N)$ are functions in R^N (possibly constant), and

$$
D^\alpha=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1},\ldots,\left(\frac{\partial}{\partial x_N}\right)^{\alpha_N}=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_N^{\alpha_N}}. \tag{2.3.2}
$$

For example, the most general linear partial differential operator of order 2 in two independent variables is

$$
\begin{aligned}
L &= \sum_{|\alpha|\le m}A_\alpha D^\alpha \\
&= A_{2,0}(x_1,x_2)\frac{\partial^2}{\partial x_1^2}+A_{1,1}(x_1,x_2)\frac{\partial^2}{\partial x_1\partial x_2}+A_{0,2}(x_1,x_2)\frac{\partial^2}{\partial x_2^2} \\
&+ A_{1,0}(x_1,x_2)\frac{\partial}{\partial x_1}+A_{0,1}(x_1,x_2)\frac{\partial}{\partial x_2}+A_{0,0}(x_1,x_2).
\end{aligned} \tag{2.3.3}
$$

When seeking solution of the equation $L(x)=Y$ we may be interested in a solution which is a differential function, or just a function with D^α understood as generalized derivatives, or finally, we may seek a solution which is a distribution. For this reason the solutions of such an equation are classed as follows,

1) Classical Solution. Let f be a function on $\mathbb{R}^N$. Every function u on $\mathbb{R}^N$ which is sufficiently differentiable so that

$$
\sum_{|\alpha|\le m}A_\alpha D^\alpha
$$

is well defined as a function and such that the equation

$$Lu = \sum_{|\alpha| \le m} A_\alpha D^\alpha u = f \tag{2.3.4}$$

is satisfied is called a classical solution of (2.3.4).

2) Weak solution. By a weak solution of (2.3.4) we mean a function on $\mathbb{R}^N$ which need not be sufficiently differentiable to make Lu meaningful in the classical sense. In the case f may be a function or a distribution.

3) Distributional Solution. Let $f \in D'(\mathbb{R}^N)$. Every $u \in D'(\mathbb{R}^N)$ satisfied (2.3.18) is called a distributional solution of (2.3.4).

Note that if f in (2.3.4) is a singular distribution, then the equation cannot have a classical solution. The remarkable fact is that including distributions one can generate new solutions of classical equations (equations where f is a function). Some classical equations may not even have a classical solution, but can have distributional solutions.

Equations of the form

$$LG = \delta \tag{2.3.5}$$

are of particular interest. Suppose G is a distribution satisfying (2.3.19). Then, for any distribution f with compact support, the convolution $f * G$ is well defined and

$$\begin{aligned} L(f * G) &= \sum_{|\alpha| \le m} A_\alpha D^\alpha (f * G) \\ &= \sum_{|\alpha| \le m} A_\alpha (f * D^\alpha G) \\ &= f * \left(\sum_{|\alpha| \le m} A_\alpha D^\alpha G \right) \\ &= f * \delta = f \end{aligned}$$

Thus, if G is a solution of $LG = \delta$, then $f * G$ is a solution of $Lu = f$. This explains the importance of the equations $Lu = \delta$.

Definition 2.3.1. Consider the linear partial differential equation

$$P(D)u = f, \tag{2.3.6}$$

where f is a distribution, u is an unknown function, and $P(D)$ a linear partial differential operator with constant coefficients. A function $E(x)$ is called elementary solution of equation (2.3.6) if $P(D)E(x) = \delta$, where δ is the Dirac-delta function.

Lemma 2.3.2. Let L be the operator defined by

$$L = \frac{\partial}{\partial t} + c^2(-\circledast)^k \tag{2.3.7}$$

where $\circledast^k$ is the operator iterated k-times defined by

$$\circledast^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^3 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right]^k,$$

$p+q=n$ is the dimension of $\mathbb{R}^n$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $t \in (0, \infty)$, k is a positive integer and c is the positive constant. Then we obtain

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[-c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right)^k + i(\xi, x) \right] d\xi \quad (2.3.8)$$

as the elementary solution of (2.3.7) in the spectrum $\Omega \subset \mathbb{R}^n$ for $t > 0$.

Proof. Let $LE(x,t) = \delta(x,t)$ where $E(x,t)$ is the kernel or the elementary solution of the operator L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t} E(x,t) + c^2(-\circledast)^k E(x,t) = \delta(x)\delta(t)$$

take the Fourier transform defined by (2.2.1) to both sides of the equation

$$\frac{\partial}{\partial t} \widehat{E(\xi,t)} + c^2 \left[\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right]^k \widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}} \delta(t).$$

Thus

$$\widehat{E(\xi,t)} = \frac{H(t)}{(2\pi)^{n/2}} \exp\left[-c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right)^k \right]$$

where $H(t)$ is the Heaviside function. Since $H(t) = 1$ for $t > 0$,

$$\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}} \exp\left[-c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right)^k \right],$$

so we have

$$E(\xi,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{E(\xi,t)} d\xi.$$

By (2.2.3),

$$E(\xi,t) = \frac{1}{(2\pi)^{n/2}} \int_\Omega e^{i(\xi,x)} \widehat{E(\xi,t)} d\xi$$

where Ω is the spectrum of $E(x,t)$. Thus

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[-c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right)^k + i(\xi, x) \right] d\xi.$$

for $t > 0$. □

Lemma 2.3.3. Given the equation $\triangle^k u(x) = \delta(x)$ for $x \in \mathbb{R}^n$, where $\triangle^k$ is defined by (**??**). Then

$$u(x) = (-1)^k R^e_{2k}(x)$$

is an elementary solution of the $\triangle^k$ operator where $R^e_{2k}(x)$ is defined by (**??**), with $\alpha = 2k$.

Proof. [See 10, Lemma 2.4, p.31]. □

Lemma 2.3.4. Given the equation $\square^k u(x) = \delta(x)$ for $x \in \mathbb{R}^n$, where $\square^k$ is defined by (**??**). Then

$$u(x) = R^H_{2k}(x)$$

is an elementary solution of the $\square^k$ operator where $R^H_{2k}(x)$ is defined by (**??**), with $\beta = 2k$

Proof. [See 10, p.11]. □

Lemma 2.3.5. Given the equation

$$\left(\square + m^2\right)^k K(x) = \delta(x) \tag{2.3.9}$$

where $\left(\square + m^2\right)^k$ is the operator iterated $k-$times defined by (1.0.30) then $K(x) = W^H_{2k}(u, m)$ is an elementary solution or Green function of (2.3.9) where $W^H_{2k}(u, m)$ is defined by (1.0.31) with $\alpha = 2k$.

Proof. [See 22, p.21, formula VI3].

From (1.0.30) if $q = 0$ then $\left(\square + m^2\right)^k$ reduces to the Helmholtz operator $\left(\triangle + m^2\right)^k$ where

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}.$$

Thus, by (1.0.30), for $q = 0$ we obtain the equation

$$\left(\triangle + m^2\right)^k K(x) = \delta(x) \tag{2.3.10}$$

with an elementary solution $K(x) = W^H_{2k}(v, m)$ where

$$v = x_1^2 + x_2^2 + \cdots + x_p^2.$$

Now,

$$W^H_{2k}(v, m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma\left(\frac{2k}{2} + r\right)}{r!\Gamma\left(\frac{2k}{2}\right)} \left(m^2\right)^r R^H_{2k+2r}(v). \tag{2.3.11}$$

We have $R^H_{2k}(v) = 2(-1)^k R^e_{2k}()$. Thus, we write

$$W^e_{2k}(v, m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma\left(\frac{2k}{2} + r\right)}{r!\Gamma\left(\frac{2k}{2}\right)} \left(m^2\right)^r 2(-1)^{k+r} R^e_{2k+2r}(v). \tag{2.3.12}$$

In general, if $p = n$

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

and

$$v = x_1^2 + x_2^2 + \cdots + x_n^2.$$

we obtain

$$K(x) = W_{2k}^e(v, m)$$

as an elementary solution of (2.3.10).

Lemma 2.3.6. Given the equation

$$\Box^k u(x) = 0, \tag{2.3.13}$$

where $\Box^k$ is defined by (1.0.18) and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ then $u(x) = (R^H_{2(k-1)}(V))^{(m)}$ is a solution of (2.3.13) where $(R^H_{2(k-1)}(V))^{(m)}$ is defined by (1.0.19) with m - derivatives, $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension with $\beta = 2(k-1)$ and V is defined by (2.2.5).

Proof. We first to show that the generalized function $\delta^{(m)}(r^2 - s^2)$ where $r^2 = x_1^2 + x_2^2 + \ldots + x_p^2$ and $s^2 = x_{p+1}^2 + x_{p+2}^2 + \ldots + x_{p+q}^2$, $p + q = n$ is a solution of the equation

$$\Box u(x) = 0, \tag{2.3.14}$$

where $\Box$ is defined by (**??**) with $k = 1$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

$$\frac{\partial}{\partial x_i}\delta^{(m)}(r^2 - s^2) = 2x_i\delta^{(m+1)}(r^2 - s^2)$$

$$\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2) = 2\delta^{(m+1)}(r^2 - s^2) + 4x_i^2\delta^{(m+2)}(r^2 - s^2)$$

$$\begin{aligned}
\Box\delta^{(m)}(r^2 - s^2) &= \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2) \\
&= 2p\delta^{(m+1)}(r^2 - s^2) + 4r^2\delta^{(m+2)}(r^2 - s^2) \\
&= 2p\delta^{(m+1)}(r^2 - s^2) + 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2) \\
&+ 4s^2\delta^{(m+2)}(r^2 - s^2) \\
&= 2p\delta^{(m+1)}(r^2 - s^2) - 4(m+2)\delta^{(m+1)}(r^2 - s^2) \\
&+ 4s^2\delta^{(m+2)}(r^2 - s^2) \\
&= (2p - 4(m+2))\delta^{(m+1)}(r^2 - s^2) + 4s^2\delta^{(m+2)}(r^2 - s^2).
\end{aligned}$$

By Lemma 2.2.11 with $P = r^2 - s^2$. Similarly ,

$$\begin{aligned}
\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\delta^{(m)}(r^2 - s^2) &= (-2q + 4(m+2))\delta^{(m+1)}(r^2 - s^2) \\
&\quad + 4r^2\delta^{(m+2)}(r^2 - s^2).
\end{aligned}$$

Thus

$$\begin{aligned}
\Box\delta^{(m)}(r^2 - s^2) &= \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2) - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2) \\
&= (2(p+q) - 8(m+2))\delta^{(m+1)}(r^2 - s^2) - 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2) \\
&= (2n - 8(m+2))\delta^{(m+1)}(r^2 - s^2) + 4(m+2)\delta^{(m+1)}(r^2 - s^2) \\
&= (2n - 4(m+2))\delta^{(m+1)}(r^2 - s^2).
\end{aligned}$$

If $2n-4(m+2)=0$, we have $\Box\delta^{(m)}(r^2-s^2)=0$. That is $u(x)=\delta^{(m)}(r^2-s^2)$ is a solution of (2.3.9) with $m=\frac{n-4}{2}, n\geq 4$ and n is even dimension. We write

$$\Box^k u(x)=\Box(\Box^{k-1}u(x))=0.$$

From the above proof we have $\Box^{k-1}u(x)=\delta^{(m)}(r^2-s^2)$ with $m=\frac{n-4}{2}, n\geq 4$ and n is even dimension. Convolving the above equation by $R^H_{2(k-1)}(V)$, we obtain

$$R^H_{2(k-1)}(V)*\Box^{k-1}u(x)=R^H_{2(k-1)}(V)*\delta^{(m)}(r^2-s^2)$$

$$\Box^{k-1}(R^H_{2(k-1)}(V))*u(x)=(R^H_{2(k-1)}(V))^{(m)},\quad where\quad V=(r^2-s^2)$$

$$\delta*u(x)=u(x)=(R^H_{2(k-1)}(V))^{(m)} \tag{2.3.15}$$

by (2.2.8) and $V=r^2-s^2$ is defined by Definition (2.2.5).

Thus $u(x)=(R^H_{2(k-1)}(V))^{(m)}$ is a solution of (2.3.13) with $m=\frac{n-4}{2}, n\geq 4$ and n is even dimension.

Lemma 2.3.7. Given the equation

$$\triangle^k u(x)=0, \tag{2.3.16}$$

where $\triangle^k$ is defined by (1.0.22) and $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$. We obtain

$$u(x)=(-1)^{k-1}(R^e_{2(k-1)}(x))^{(m)}$$

is a solution of (2.3.16) where $(R^e_{2(k-1)}(x))^{(m)}$ is defined by (2.2.9) with m-derivatives, $m=\frac{n-4}{2}, n\geq 4$ and n is even dimension with $\alpha=2(k-1)$.

Proof. The proof of Lemma 2.3.7 is similar to the proof of Lemma 2.3.6.

Lemma 2.3.8. Given the equation

$$\otimes^k G(x)=\delta(x) \tag{2.3.17}$$

then

$$G(x)=\left(R^H_{6k}(x)*(-1)^{2k}R^e_{4k}(x)\right)*\left(C^{*k}(x)\right)^{*-1} \tag{2.3.18}$$

is the Green function or an elementary solution for the $\otimes^k$ operator iterated $k-$times where $\otimes^k$ is defined by (**??**), and

$$C(x)=\frac{3}{4}R^H_4(x)+\frac{1}{4}(-1)^2R^e_4(x) \tag{2.3.19}$$

$C^{*k}(x)$ denotes the convolution of $C(x)$ itself $k-$times , $\left(C^{*k}(x)\right)^{*-1}$ denotes the inverse of $C^{*k}(x)$ in the convolution algebra, $R^H_{6k}(x)$ is defined by (1.0.19) with $\alpha=6k$ and $R^e_{4k}(x)$ is defined by (1.0.23) with $\alpha=4k$. Moreover $G(x)$ is a tempered distribution.

Proof. From (**??**), we have

$$\otimes^k G(x) = \left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\square^3\right)^k G(x) = \delta(x)$$

or we can write

$$\left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\square^3\right)\left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\square^3\right)^{k-1} G(x) = \delta(x).$$

Convolving both sides of the above equation by $R_6^H(x) * (-1)^2 R_4^e(x)$,

$$\left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\square^3\right) * \left(R_6^H(x) * (-1)^2 R_4^e(x)\right)\left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\square^3\right)^{k-1} G(x)$$
$$= \delta(x) * R_6^H(x) * (-1)^2 R_4^e(x)$$

or

$$\left(\frac{3}{4}\square\left(R_2^H(x)\right) * \triangle^2(-1)^2 R_4^e(x) * R_4^H(x) + \frac{1}{4}\square^3 R_6^H(x) * (-1)^2 R_4^e(x)\right) *$$
$$\left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\square^3\right)^{k-1} G(x) = \delta(x) * R_6^H(x) * (-1)^2 R_4^e(x)$$

By (2.2.8) and (2.2.11), we obtain

$$\left(\frac{3}{4}\delta * \delta * R_4^H(x) + \frac{1}{4}\delta * (-1)^2 R_4^e(x)\right) * \left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\square^3\right)^{k-1} G(x)$$
$$= \delta(x) * R_6^H(x) * (-1)^2 R_4^e(x).$$

Thus

$$\left(\frac{3}{4}R_4^H(x) + \frac{1}{4}(-1)^2 R_4^e(x)\right) * \left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\square^3\right)^{k-1} G(x) = R_6^H(x) * (-1)^2 R_4^e(x)$$

keeping on convolving both sides of the above equation by $R_6^H(x) * (-1)^2 R_4^e(x)$ up to $k-1$ times, we obtain

$$C^{*k}(x) * G(x) = \left(R_6^H(x) * (-1)^2 R_4^e(x)\right)^{*k}$$

the symbol $*k$ denotes the convolution of itself $k-$times. By properties of $R_\alpha(x)$ [See, Lemma 2.2.6], we have

$$\left(R_6^H(x) * (-1)^2 R_4^e(x)\right)^{*k} = R_{6k}^H(x) * (-1)^{2k} R_{4k}^e(x).$$

Thus,

$$C^{*k}(x) * G(x) = R_{6k}^H(x) * (-1)^{2k} R_{4k}^e(x). \tag{2.3.20}$$

Now, consider the function $C^{*k}(x)$, since $R_6^H(x) * (-1)^2 R_4^e(x)$ is a tempered distribution. Thus $C(x)$ defined by (2.3.19) is a tempered distribution, we obtain $C^{*k}(x)$ is a tempered distribution, $R_{6k}^H(x) * (-1)^{2k} R_{4k}^e(x) \in \mathcal{S}'$, the space of tempered distribution.

Choose $\mathcal{S}' \subset \mathcal{D}'_{\mathcal{R}}$ where $\mathcal{D}'_{\mathcal{R}}$ is the right-side distribution which is a subspace of $\mathcal{D}'$ of distribution.

Thus $R^H_{6k}(x)*(-1)^{2k}R^e_{4k}(x) \in \mathcal{D}'_{\mathcal{R}}$. It follow that $R^H_{6k}(x)*(-1)^{2k}R^e_{4k}(x)$ is an element of convolution algebra, since $\mathcal{D}'_{\mathcal{R}}$ is a convolution algebra. Hence Zemanian [23], the equation (2.3.20) has a unique solution

$$G(x) = \left(R^H_{6k}(x) * (-1)^{2k}R^e_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} \tag{2.3.21}$$

where $\left(C^{*k}(x)\right)^{*-1}$ is an inverse of $C^{*k}(x)$ in the convolution algebra, $G(x)$ is called the Green function or an elementary solution of the $\otimes^k$ operator. That completes the proof.

Lemma 2.3.9. Given the equation

$$\circledast^k H(x) = \delta(x) \tag{2.3.22}$$

then

$$H(x) = \left(R^H_{4k}(x) * (-1)^{3k}R^e_{6k}(x)\right) * \left(S^{*k}(x)\right)^{*-1} \tag{2.3.23}$$

is the Green function or an elementary solution for the $\circledast^k$ operator iterated $k-$times where $\circledast^k$ is defined by (1.0.29), and

$$S(x) = \frac{3}{4}(-1)^2R^e_4(x) + \frac{1}{4}R^H_4(x) \tag{2.3.24}$$

$S^{*k}(x)$ denotes the convolution of $S(x)$ itself $k-$times, $\left(S^{*k}(x)\right)^{*-1}$ denotes the inverse of $S^{*k}(x)$ in the convolution algebra, $R^H_{4k}(x)$ is defined by (2.2.6) with $\alpha = 4k$ and $R^e_{4k}(x)$ is defined by (2.2.9) with $\alpha = 6k$. Moreover $H(x)$ is a tempered distribution.

Proof. The proof of Lemma 2.3.9 is similar to the proof of Lemma 2.3.8.

Lemma 2.3.10. Given the equation

$$L^k_1K(x) = \delta(x) \tag{2.3.25}$$

where L^k_1 be the operator iterated $k-$times defined by

$$L^k_1 = (\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2)^k \tag{2.3.26}$$

and $\triangle$ and $\Box$ is defined by (1.0.22) and (1.0.18) with $k = 1$ respectively. Then we obtain $K(x)$ is an elementary solution of the L^k_1 operator where

$$K(x) = \left(R^H_{4k}(x) * (-1)^{2k}R^e_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} \tag{2.3.27}$$

and

$$C(x) = \frac{3}{4}R^H_4(x) + \frac{1}{4}(-1)^2R^e_4(x). \tag{2.3.28}$$

$C^{*k}(x)$ denotes the convolution of $C(x)$ itself $k-$times, $\left(C^{*k}(x)\right)^{*-1}$ denotes the inverse of $C^{*k}(x)$ in the convolution algebra $R^H_{4k}(x)$ is defined by (2.2.6) with $\alpha = 4k$ and $R^e_{4k}(x)$ is defined by (2.2.9) with $\alpha = 4k$. Moreover $K(x)$ is a tempered distribution.

Proof. The proof of Lemma 2.3.10 is similar to the proof of Lemma 2.3.8.

Lemma 2.3.11.

$$L_2^k I(x) = \delta(x) \tag{2.3.29}$$

where L_2^k be the operator defined by $L_2^k = (\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2)^k$, $\triangle$ and $\Box$ is defined by (1.0.22) and (1.0.18) with $k = 1$ respectively. Then we obtain $I(x)$ is an elementary solution of the L_2^k operator where

$$I(x) = \left(R_{4k}^H(x) * (-1)^{2k} R_{4k}^e(x)\right) * \left(S^{*k}(x)\right)^{*-1} \tag{2.3.30}$$

and

$$S(x) = \frac{3}{4}(-1)^2 R_4^e(x) + \frac{1}{4} R_4^H(x)$$

$S^{*k}(x)$ denotes the convolution of S itself $k-$times , $\left(S^{*k}(x)\right)^{*-1}$ denotes the inverse of $S^{*k}(x)$ in the convolution algebra. Moreover $I(x)$ is a tempered distribution.

Proof. The proof of Lemma 2.3.11 is similar to the proof of Lemma 2.3.8.

Lemma 2.3.12. Given the equation

$$\left(\diamondsuit + m^4\right)^k H(x) = \delta(x) \tag{2.3.31}$$

then

$$H(x) = \left[W_{2k}^H(u,m) * W_{2k}^e(v,m)\right] * \left(I^{*k}(x)\right)^{*-1} \tag{2.3.32}$$

is an Green function for the operator $(\diamondsuit + m^4)^k$ iterated $k-$times where $\diamondsuit$ is the Diamond operator defined by (1.0.10), m is nonnegative real number and

$$I(x) = \delta - m^2\left(W_2^H(u,m) * W_2^e(v,m)\right) * \left(R_{-2}^H(u) + R_{-2}^e(v)\right) \tag{2.3.33}$$

$I^{*k}(x)$ denotes the convolution of I itself $k-$times, $\left(I^{*k}(x)\right)^{*-1}$ denotes the inverse of $I^{*k}(x)$ in the convolution algebra. Moreover $I(x)$ is a tempered distribution.

Proof. [See 12].